Homology cylinders and knot theory (joint work with Hiroshi GODA)

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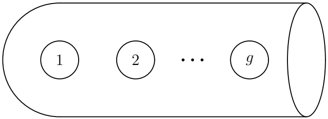
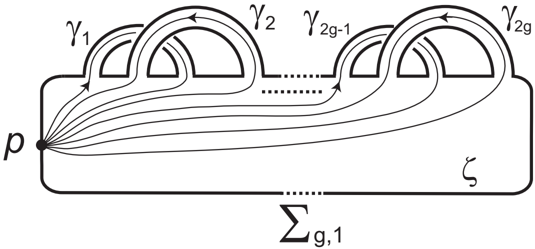
Contents

§1. Introduction

- $\Sigma_{g,1} =$ 

($g \geq 0$, oriented)

with a standard cell decomposition:



Definition (Goussarov, Habiro, Garoufalidis-Levine, Levine)

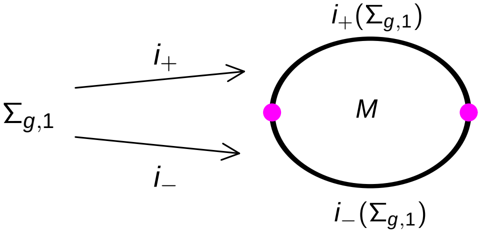
(M, i_+, i_-) : a *homology cylinder* (HC) *over* $\Sigma_{g,1}$

$\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} M \text{ : a compact oriented 3-manifold,} \\ i_+, i_- : \Sigma_{g,1} \hookrightarrow \partial M \text{ two embeddings} \end{cases}$ (markings)

satisfying

1. i_+: orientation-preserving, i_-: orientation-reversing;
2. $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$,
   $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$;
3. $i_+|_{\partial\Sigma_{g,1}} = i_-|_{\partial\Sigma_{g,1}}$;
4. $i_+, i_- : H_*(\Sigma_{g,1}; \mathbb{Z}) \xrightarrow{\cong} H_*(M; \mathbb{Z})$ isomorphisms.

- (M, i_+, i_-) : a homology cylinder (over $\Sigma_{g,1}$)



Definition

$\mathcal{C}_{g,1} := \{(M, i_+, i_-) : \text{HC over } \Sigma_{g,1}\}/(\text{marking compatible diffeo}).$

Stacking

Definition (Stacking operation)

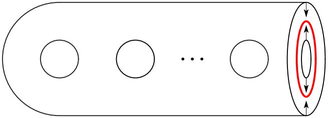
For (M, i_+, i_-), $(N, j_+, j_-) \in \mathcal{C}_{g,1}$,

$$(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-) \quad \in \mathcal{C}_{g,1}$$

$\rightsquigarrow$ $\mathcal{C}_{g,1}$ becomes a monoid.

unit: $(\Sigma_{g,1} \times [0,1], \mathrm{id} \times 1, \mathrm{id} \times 0)$

where corners of $\Sigma_{g,1} \times [0,1]$ are rounded, and



Examples

1. $\mathcal{M}_{g,1}$: the mapping class group of $\Sigma_{g,1}$

   $[\varphi] \in \mathcal{M}_{g,1}$, i.e.

   $$\varphi : \Sigma_{g,1} \xrightarrow{\sim} \Sigma_{g,1}\text{: a diffeo. s.t. } \varphi\big|_{\partial\Sigma_{g,1}} = \mathrm{id}$$

   $$\Longrightarrow (\Sigma_{g,1} \times [0,1], \mathrm{id} \times 1, \varphi \times 0) \in \mathcal{C}_{g,1}.$$

We can check

$\mathcal{M}_{g,1} \hookrightarrow \mathcal{C}_{g,1}$: monoid embedding

$\rightsquigarrow$ $\mathcal{C}_{g,1}$ is an *enlargement* of $\mathcal{M}_{g,1}$.

Introduction (6)

2. surgery along clovers (Goussarov) or claspers (Habiro)

3. surgery along pure string links (Habegger, Levine)

4. connected sum with a homology 3-sphere X:

$$((\Sigma_{g,1} \times [0,1])\#X, \mathrm{id} \times 1, \mathrm{id} \times 0) \in \mathcal{C}_{g,1}.$$

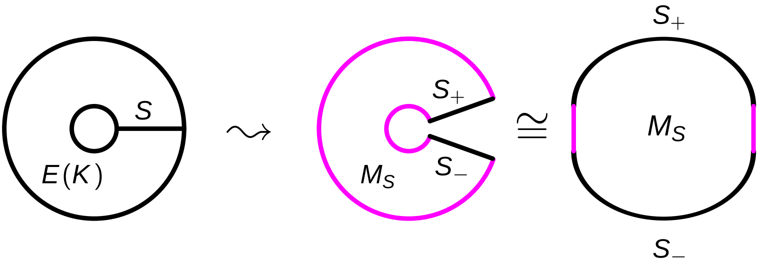
Today we focus on

5. complementary sutured manifolds of Seifert surfaces of a special class of knots.

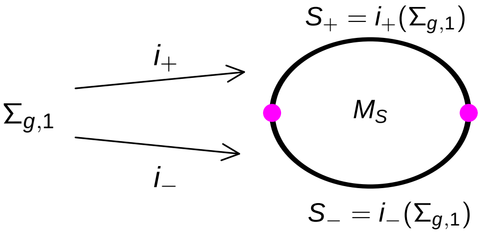
§2. Homologically fibered knots

- $K \subset S^3$: a knot,
  S : a Seifert surface of K of genus g.

  M_S : the cobordism obtained from $E(K)$ by cutting along S
  $=$ the (complementary) sutured manifold for S.



- By fixing an identification $i : \Sigma_{g,1} \xrightarrow{\cong} S$, we obtain a *marked* sutured manifold (M_S, i_+, i_-):



Question When this becomes a homology cylinder?

Homologically fibered knots (3)

Proposition (Crowell-Trotter, . . ., Goda-S.)

K : a knot in S^3,

K has a Seifert surface S of genus g s.t. M_S is a homology product (over a copy of S)

$\Longleftrightarrow$ The following hold:

- S is a minimal genus Seifert surface,
- The Alexander polynomial $\Delta_K(t)$ of K is monic,
- $\deg(\Delta_K(t)) = 2\,\mathrm{genus}(K)$.

Homologically fibered knots (4)

Definition

A knot K in S^3 is said to be *homologically fibered* if

(1) $\Delta_K(t)$ is monic,

(2) $\deg(\Delta_K(t)) = 2\,\mathrm{genus}(K)$.

Remarks

- (Fibered knots) $\subset$ (HFknots [Homologically Fibered knots]) corresponds to $\mathcal{M}_{g,1} \subset \mathcal{C}_{g,1}$.
- We can define *rational* homologically fibered knots ($\mathbb{Q}$-HFknots) by assuming only (2).

Homologically fibered knots (5)

"Uniqueness"

Proposition

K : an HFknot of genus g

S_1, S_2 : minimal genus Seifert surfaces

For any markings of ∂M_{S_1} and ∂M_{S_2}, $\exists N \in \mathcal{C}_{g,1}$ s.t.

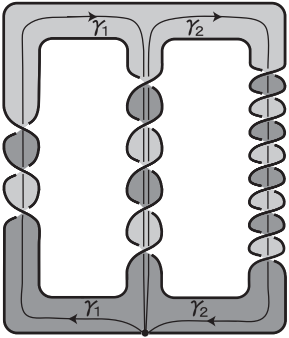
$$M_{S_1} \cdot N = N \cdot M_{S_2} \quad \in \mathcal{C}_{g,1}$$

In particular, any monoid homomorphism

$$\mathcal{C}_{g,1} \twoheadrightarrow A \qquad \text{w/ } A\text{: an abelian group}$$

gives an invariant of HFknots.

Break



Pretzel knot $P(-3, 5, 9)$ is an HFknot.

Easy to see

$P(-2n+1, 2n+1, 2n^2+1)$ is an HFknot for any $k \geq 1$.

§3. Factorization formulas of Alexander invariants

Classical case

- $K \subset S^3$: a knot,
  S : a Seifert surface of K w/ a Seifert matrix A.

Assume that A is invertible over $\mathbb{Q}$ (i.e. K is a $\mathbb{Q}$-HFknot).
Then

$$\begin{aligned}\Delta_K(t) &= \det(A^T - tA)\\ &= \det(A^T)\det(I_{2g(S)} - t(A^T)^{-1}A)\end{aligned}$$

What does this *factorization* mean?

We can check:

- A^T and A represent

$$i_+, i_- : \mathbb{Z}^{2g} \cong H_1(\Sigma_{g,1}) \longrightarrow H_1(M_S) \cong \mathbb{Z}^{2g}$$

under certain bases of $H_1(\Sigma_{g,1})$ and $H_1(M_S)$. In fact,

$$\begin{aligned} \det(A) &= \text{The top (bottom) coeff. of } \Delta_K(t) \\ &= \pm |H_1(M, i_+(\Sigma_{g,1})| \\ &= \tau(C_*(M_S, i_+(\Sigma_{g,1}); \mathbb{Q})) \quad \text{torsion} \end{aligned}$$

- $\sigma(M_S) := (A^T)^{-1} A \in Sp(2g, \mathbb{Q})$.
  (Can regard $\sigma(M_S)$ as an H_1-monodromy of M_S.)

Roughly speaking, our factorization formula says

$$\begin{aligned}\Delta_K(t) &= \det(A^T)\det(I_{2g(S)} - t(A^T)^{-1}A) \\ &= (\text{torsion of } M_S)\cdot(\textit{effect} \text{ of } H_1\text{-monodromy of } M_S).\end{aligned}$$

Remark By Milnor,

$$\frac{\Delta_K(t)}{1-t} = \tau_{\mathbb{Z}}(K),$$

where $\tau_{\mathbb{Z}}(K)$ is the Reidemeister torsion associated with the $\mathbb{Z}$-cover of $E(K)$.

- For an HFknot K,

$$\begin{aligned}\Delta_K(t) &= \det(A^T)\det(I_{2g(S)} - t(A^T)^{-1}A) \\ &= \pm\det(I_{2g(S)} - t(A^T)^{-1}A).\end{aligned}$$

$\rightsquigarrow$ The factorization formula is useless for HFknots!

$\rightsquigarrow$ We will give a generalization by using twisted homology.

Higher-order case (Twisted coefficients)

- K : an HFknot,
- $M_S = (M_S, i_+, i_-) \in \mathcal{C}_{g,1}$: an HC associated with K,
- $\mathcal{K} := \mathrm{Frac}(\mathbb{Z}H_1(M_S)) \cong \mathbb{Q}(t_1, \ldots, t_{2g})$ as twisted coefficients.

Lemma

For $\pm \in \{+,-\}$, $H_*(M_S, i_\pm(\Sigma_{g,1}); \mathcal{K}) = 0$.

cf. classical case: $H_*(M_S, i_\pm(\Sigma_{g,1}); \mathbb{Z}) = 0$.

Definition

- The $\mathcal{K}$-torsion $\tau_{\mathcal{K}}(M_S)$ is

$$\tau_{\mathcal{K}}(M_S) := \tau(C_*(M_S, i_+(\Sigma_{g,1}); \mathcal{K})) \in GL(\mathcal{K})/\sim .$$

- The Magnus matrix $r_{\mathcal{K}}(M_S) \in GL(2g, \mathcal{K})$ is the representation matrix of the right $\mathcal{K}$-isom.:

$$\begin{array}{ccccc} H_1(\Sigma_{g,1}, p; i_-^*\mathcal{K}) & \xrightarrow[i_-]{\cong} & H_1(M_S, p; \mathcal{K}) & \xrightarrow[i_+^{-1}]{\cong} & H_1(\Sigma_{g,1}, p; i_+^*\mathcal{K}) \\ \| & & & & \| \\ \mathcal{K}^{2g} & & \xrightarrow[r_{\mathcal{K}}(M_S)\cdot]{\cong} & & \mathcal{K}^{2g} \end{array}$$

Remark   By substituting $t_i \mapsto 1$, we have

$$\tau_{\mathcal{K}}(M_S) \mapsto \det A = \pm 1, \qquad r_{\mathcal{K}}(M_S) \mapsto \sigma(M_S).$$

- If K is fibered w/ the monodromy $\varphi \in \mathcal{M}_{g,1}$, then

$$r_{\mathcal{K}}(M_S) = \overline{\left(\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i}\right)}_{1 \leq i,j \leq 2g}.$$

Thus, $r_{\mathcal{K}}$ generalizes the magnus representation for $\mathcal{M}_{g,1}$.

Theorem (Fibering obstructions)

K, M_S: as before.

If K is fibered, then

1. all the entries of the Magnus matrix $r_{\mathcal{K}}(M_S)$ are Laurent polynomials in $\mathbb{Q}[t_1^{\pm}, \ldots, t_{2g}^{\pm}] \subset \mathcal{K} = \mathbb{Q}(t_1, \ldots, t_{2g})$,
2. the $\mathcal{K}$-torsion $\tau_{\mathcal{K}}(M_S)$ is trivial.

Higher-order Alexander invariant (torsion)

- $\rho : \pi_1(E(K)) \longrightarrow \dfrac{\pi_1(E(K))}{\pi_1(E(K))''} =: D_2(K)$
  the natural projection on the metabelian quotient,
- $t \in H_1(E(K))$: an oriented meridian loop.

Then we have

$$D_2(K) \cong H_1(M_S) \rtimes H_1(E(K)) = H_1(M_S) \rtimes \langle t \rangle$$

and

$$\mathbb{Z}D_2(K) \hookrightarrow \mathbb{Z}D_2(K)(\mathbb{Z}D_2(K) - \{0\})^{-1} = \mathcal{K}(t;\sigma),$$

where $\mathcal{K}(t;\sigma)$ is the (skew) field of rational functions over $\mathcal{K} = \mathrm{Frac}(\mathbb{Z}H_1(M_S))$ with twisting σ.

Factorization formulas of Alexander invariants (9)

Theorem (Goda-S. Factorization formula)

- $\rho : \pi_1(E(K)) \longrightarrow D_2(K) = H_1(M_S) \rtimes \langle t \rangle$: the natural proj.
- $t \in H_1(E(K))$: an oriented meridian loop.

We can define

$$\tau_{\mathcal{K}(t;\sigma)}(E(K)) := \tau(C_*(E(K); \mathcal{K}(t;\sigma))),$$

the noncommutative higher-order torsion associated with ρ (defined by Cochran, Harvey and Friedl).

Moreover, it factorizes into

$$\tau_{\mathcal{K}(t;\sigma)}(E(K)) = \frac{\tau_{\mathcal{K}}(M_S) \cdot (I_{2g} - t \cdot r_{\mathcal{K}}(M_S))}{1-t} \in GL(\mathcal{K}(t;\sigma))/\sim,$$

Remarks.

1. $$\det(\tau_{\mathcal{K}(t;\sigma)}(E(K))) = \frac{\det(\tau_{\mathcal{K}}(M_S)) \cdot \det(I_{2g} - t \cdot r_{\mathcal{K}}(M_S))}{1-t}$$

   $\rightsquigarrow$ $\det(\tau_{\mathcal{K}}(M_S))$: the "leading coefficient" of $\tau_{\mathcal{K}(t;\sigma)}(E(K))$.

   Know as *decategorification* of sutured Floer homology (Friedl-Juhász-Rasmussen)

2. Similar formulas of the form

   Alexander inv. $=$ (torsion) $\cdot$ (monodromy) :

   1. Formulas of Hutchings-Lee, Goda-Matsuda-Pajitnov and Kitayama using Morse-Novikov theory.
   2. Kirk-Livingston-Wang for string links.

§4. Computations

Facts on fibered knots vs. HFknots

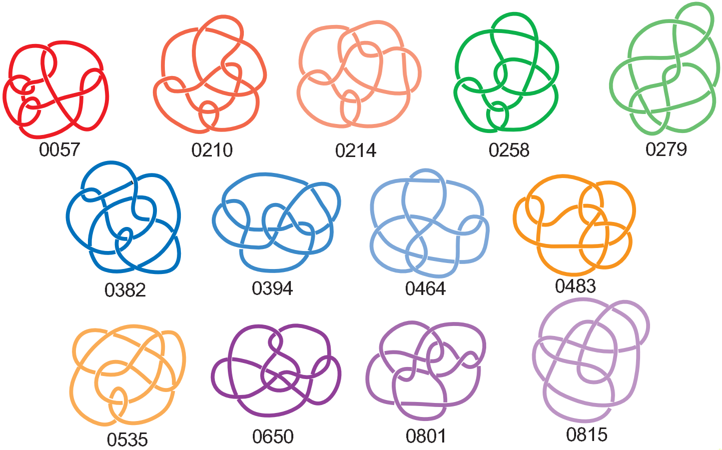
- HFknots with at most 11-crossings are all fibered.
- There are 13 non-fibered HFknots with 12-crossings. In particular, Friedl-Kim showed that these 13 knots are not fibered by using twisted Alexander polynomial associated with finite representations.

We can also use $r_{\mathcal{K}}(M_S)$ and $\tau_{\mathcal{K}}(M_S)$ to detect the non-fiberedness of HFknots.

Recipe

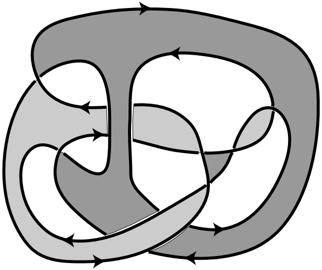
1. Get all the pictures of those 13 knots.
   [By Computer (Database (KnotInfo) on Internet)]
2. For each of them,
   1. Find a minimal genus Seifert surface S.
      [By hand]
   2. Calculate an admissible presentation of $\pi_1(M_S)$.
      [By hand]
   3. Compute $r_{\mathcal{K}}(M_S)$ and $\tau_{\mathcal{K}}(M_S)$.
      [By hand and also by computer program]

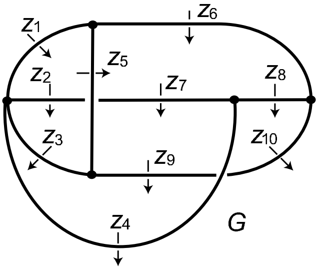
Non-fibered HFknots with 12-crossings



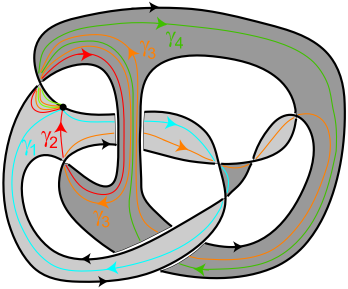
Computations (4)

2. Example of calculation of admissible presentation





12n0057



12n0057

Generators $i_-(\gamma_1), \ldots, i_-(\gamma_4),\ z_1, \ldots, z_{10},\ i_+(\gamma_1), \ldots, i_+(\gamma_4)$

Relations $z_1 z_5 z_6^{-1},\ z_2 z_3 z_4 z_1,\ z_3 z_9^{-1} z_5^{-1},\ z_7 z_4 z_8^{-1},\ z_8 z_{10} z_6,$
$z_2 z_5 z_7^{-1} z_5^{-1},\ z_9 z_4 z_{10}^{-1} z_4^{-1},\ i_-(\gamma_1) z_1^{-1} z_5^{-1},\ i_-(\gamma_2) z_2,$
$i_-(\gamma_3) z_4 z_8 z_7 z_5^{-1},\ i_-(\gamma_4) z_4,\ i_+(\gamma_1) z_5^{-1},\ i_+(\gamma_2) z_9^{-1} z_6^{-1},$
$i_+(\gamma_3) z_6 z_4 z_7 z_5^{-1} z_3^{-1} z_5 z_6^{-1},\ i_+(\gamma_4) z_6 z_7^{-1} z_6^{-1}$

Computations (6)

Computational results for 12n0057

$$r_{\mathcal{K}}(M_S) = \begin{pmatrix} \frac{x3+x1\,x2^2\,(-1+x2\,(-1+x4))-x2\,x3\,x4}{x1\,x2^2\,(-1+x2\,(-1+x4))} & -\frac{(-1+x4)\,(-1+x2\,x4)}{-1+x2\,(-1+x4)} & \frac{x4}{1+x2-x2\,x4} & 0 \\ -\frac{(1+x1\,x2)\,x3}{x1^2\,x2\,(-1+x2\,(-1+x4))} & -\frac{x2\,(1+x1\,x2)\,(-1+x4)}{x1\,(-1+x2\,(-1+x4))} & -\frac{(1+x2)\,(1+x1\,x2^2\,(-1+x4))}{x1\,x2\,(-1+x2\,(-1+x4))} & \frac{1}{x4} \\ \frac{x3}{x1\,(-1+x2\,(-1+x4))} & \frac{x2^2\,(-1+x4)}{-1+x2\,(-1+x4)} & \frac{x2\,(1+x2)\,(-1+x4)}{-1+x2\,(-1+x4)} & 0 \\ \frac{(x1\,x2^2-x3)\,x4}{x1^2\,x2\,(-1+x2\,(-1+x4))} & \frac{x2\,x4\,(x1\,x2+x3-x3\,x4)}{x1\,x3\,(-1+x2\,(-1+x4))} & \frac{(1+x2)\,(x1\,x2^2-x3)\,x4}{x1\,x2\,x3\,(-1+x2\,(-1+x4))} & 1 \end{pmatrix},$$

$$\tau_{\mathcal{K}}(M_S) = x1\,x2^4 + x1\,x2^5 - x1\,x2^5\,x4,$$

where $\mathtt{xj} = i_+(\gamma_j)$.

Each of $r_{\mathcal{K}}(M_S)$ and $\tau_{\mathcal{K}}(M_S)$ shows that 12n0057 is not fibered!

We computed $r_{\mathcal{K}}(M_S)$ and $\tau_{\mathcal{K}}(M_S)$ similarly for the 13 knots and checked that detected the non-fiberedness of all 13 HF-knots.

§5. Abelian quotient of the monoid of homology cylinders

Definition (Irreducible homology cylinders)

$$\mathcal{C}_{g,1}^{\mathrm{irr}} := \{(M, i_+, i_-) \in \mathcal{C}_{g,1} \mid M \text{ is an irreducible 3-mfd.}\}.$$

Question Does there exist non-trivial abelian quotients of $\mathcal{C}_{g,1}^{\mathrm{irr}}$?

Note that $\mathcal{M}_{g,1}$ is a perfect group (i.e. no non-trivial abelian quotients).

Theorem (Goda-S.)

The monoid $\mathcal{C}_{g,1}^{\mathrm{irr}}$ *has an abelian quotient isomorphic to* $(\mathbb{Z}_{\geq 0})^{\infty}$.

Sketch of Proof

We use the rank of *sutured Floer homology*.

An HC $(M, i_+, i_-) \in \mathcal{C}_{g,1}^{\mathrm{irr}}$ can be regarded as a sutured manifold (M, ζ) with $\zeta = i_+(\partial \Sigma_{g,1}) = i_-(\partial \Sigma_{g,1})$.

Moreover, the sutured manifold (M, ζ) is *balanced* in the sense of Juhász. So the sutured Floer homology $SFH(M, \zeta)$ of (M, ζ) is defined.

Consider

$$R : \mathcal{C}_{g,1}^{\mathrm{irr}} \longrightarrow \mathbb{Z}_{\geq 0}$$

defined by

$$R(M, i_+, i_-) = \mathrm{rank}_{\mathbb{Z}}(SFH(M, \zeta)).$$

By deep results of Ni and Juhász, we have

- $R(M, i_+, i_-) \geq 1$ for any $(M, i_+, i_-) \in \mathcal{C}_{g,1}^{\mathrm{irr}}$.
- $R(M, i_+, i_-) = 1 \iff (M, i_+, i_-) \in \mathcal{M}_{g,1} \subset \mathcal{C}_{g,1}^{\mathrm{irr}}$.
- $R(M \cdot N) = R(M) \cdot R(N)$ for $M, N \in \mathcal{C}_{g,1}^{\mathrm{irr}}$.

Therefore we obtain the rank homomorphism

$$R : \mathcal{C}_{g,1}^{\mathrm{irr}} \longrightarrow \mathbb{Z}_{>0}$$

to the multiplicative monoid $\mathbb{Z}_{>0}$.

We further decompose R by using the prime decomposition of integers:

$$R = \bigoplus_{p\,:\ \text{prime}} R_p : \mathcal{C}_{g,1}^{\mathrm{irr}} \longrightarrow \mathbb{Z}_{>0}^{\times} = \bigoplus_{p\,:\ \text{prime}} \mathbb{Z}_{\geq 0}^{(p)},$$

where $\mathbb{Z}_{\geq 0}^{(p)}$ is a copy of $\mathbb{Z}_{\geq 0}$, the monoid of non-negative integers whose product is given by sum.

Let M_n be the HC obtained from an HFknot

$$P_n := P(-2n+1, 2n+1, 2n^2+1).$$

Then

$$\begin{aligned} R(M_n) &= \mathrm{rank}_{\mathbb{Z}}(SFH(M_n, \zeta)) \\ &= \widehat{HFK}(S^3, P_n, 1) \\ &= 2n^2 - 2n + 1. \end{aligned}$$

Easy arithmetic shows our claim. □

Remark

The homomorphism R is not homology cobordism invariant.

Here, $(M, i_+, i_-), (N, i_+, i_-) \in \mathcal{C}_{g,1}$ are *homology cobordant*.

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists\, W$: a cpt oriented smooth 4-mfd s.t.

- $\partial W = M \cup (-N)/(i_+(x) = j_+(x),\, i_-(x) = j_-(x)) \quad x \in \Sigma_{g,1}$;
- the inclusions $M \hookrightarrow W$, $N \hookrightarrow W$ induce isomorphisms on the integral homology.

§6. Problems

- Is there *Categorification* of factorization formulas???

$$\begin{array}{ccc}
\widehat{HFK}(K) & \rightsquigarrow & \Delta_K(t), \\
 & \overset{?}{\rightsquigarrow} & \tau_{\mathcal{K}(t^{\pm};\sigma)}(E(K)), \\
SFH(M_S, K) & \rightsquigarrow & \tau_{\mathcal{K}}(M_S), \\
??? & \rightsquigarrow & r_{\mathcal{K}}(M_S). \\
 & \textit{decategorification} &
\end{array}$$

Problems (2)

- Is the rank homomorphism

$$R = \bigoplus_{p:\text{ prime}} R_p : \mathcal{C}_{g,1}^{\mathrm{irr}} \longrightarrow \mathbb{Z}_{>0}^{\times} = \bigoplus_{p:\text{ prime}} \mathbb{Z}_{\geq 0}^{(p)},$$

a homology cobordism invariant modulo 2 for some p?

Supporting fact:

Theorem (Cha-Friedl-Kim)

If $(M, i_+, i_-), (N, j_+, j_-) \in \mathcal{C}_{g,1}$ are homology cobordant, then $\exists q \in \mathcal{K} = \mathit{Frac}(\mathbb{Z}H_1(\Sigma_{g,1}))$ such that

$$\tau_{\mathcal{K}}(M) = \tau_{\mathcal{K}}(N) \cdot q \cdot \overline{q} \quad \in \mathcal{K}/(\pm H_1(\Sigma_{g,1})).$$

Fin