Non-commutative Reidemeister torsion and Morse-Novikov theory

Takahiro KITAYAMA

JSPS research fellow (DC)
Graduate School of Mathematical Sciences, the University of Tokyo

0. Introduction

M : compact 3-manifold
$\rho : \pi_1 M \to GL(n, R)$: representation
$\rightsquigarrow\ \Delta_{M,\rho}(t) \in R[t, t^{-1}]/\langle \pm a t^l \rangle_{l \in \mathbb{Z}}^{a \in R^\times}$: twisted Alexander polynomial

[Friedl-Vidussi '08]
$\{\Delta_{M,\rho}\ ;\ \rho : \pi_1 M \to \mathfrak{S}_n \hookrightarrow GL(n, \mathbb{Z})\}$ detects fiberedness of M.

<u>Which representations should we consider?</u>

One approach is to investigate "the twisted Alexander polynomial associated to $\pi_1 M \to \pi_1 M/(\pi_1 M)^{(n+1)}$".

[Cochran '04, Harvey '05]
$\Delta_M^{(n)}(t) \in \mathcal{K}_n(t)_{ab}^\times/\langle a t^l \rangle_{l \in \mathbb{Z}}^{a \in \mathcal{K}_n^\times}$: higher-order Alexander polynomial
$\mathcal{K}_n(t)$: skew field

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Aim of the talk

To give a dynamical and Morse theoretical presentation of $\Delta_M^{(n)}(t)$
(a generalization of a fiberedness obstruction)

Key tools

X : closed Riemannian manifold with $b_1(X) > 0$ and $\chi(X) = 0$

· (Non-commutative) Reidemeister torsion $\tau_\rho(X)$

· Circle-valued Morse theory $f : X \to S^1$

[Friedl '07]

$X = M \;\Rightarrow\; \Delta_M^{(n)}(t) \;\sim\; \tau_{\rho_n}(M).$

[Milnor '68]

$f : X \to S^1$: fiber bundle with monodromy φ

$\Rightarrow \quad \tau_{f_*}(X) \;=\; \zeta_\varphi := \exp\left(\sum_{k=1}^{\infty} \sum_{x \in \mathrm{Fix}\varphi^k} \frac{\mathrm{index}(\varphi^k, x)}{k} t^k\right).$

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$\mathcal{O}$: closed orbits $\rightsquigarrow$ ζ_f : Lefschetz-type zeta function
$\mathrm{Crit}\, f \rightsquigarrow C_*^{Nov}(f) \rightsquigarrow \tau_{ab}^{Nov}(f)$: Novikov torsion

[Hutchings-Lee '99], [Pajitnov '99]

$$\tau_{ab}(X) = \zeta_f \cdot \tau_{ab}^{Nov}(f).$$

Today. A generalization to non-commutative coefficients

$$\tau_\rho(X) = \zeta_{f,\rho} \cdot \tau_\rho^{Nov}(f).$$

Related topics
[Goda-Pajitnov '09]
Similar presentation of Reidemeister torsion for $\rho\colon \pi_1 X \to GL_n(\mathbb{Z})$

$$\tau(C_{\rho,*}^{Nov}(f) \to C_*(\widetilde{X}) \otimes_{f_* \otimes \rho} \mathbb{Z}((t))^n) = \zeta_{f,\rho}.$$

[Mazur-Wiles '84] (Iwasawa main conjecture (in number theory))
Iwasawa polynomial $I_p \sim p$-adic zeta function ζ_p.

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Outline

1. (Non-commutative) Reidemeister torsion
2. Circle-valued Morse theory
3. Main theorem

1. (Non-commutative) Reidemeister torsion

$\mathbb{F}$: skew field

$\det : GL(n, \mathbb{F}) \to \mathbb{F}^\times_{ab}$ $(:= \mathbb{F}^\times/[\mathbb{F}^\times, \mathbb{F}^\times])$: Dieudonné determinant

(C_*, ∂_*) : acyclic finite dimensional chain complex / $\mathbb{F}$

$\boldsymbol{c}$: basis

$\rightsquigarrow$ $\tau(C_*, \boldsymbol{c}) \in \mathbb{F}^\times_{ab}$: algebraic torsion

Lemma. (Turaev)

$C_i = C'_i \oplus C''_i$ s. t.

(i) C'_i, C''_i are spanned by subbases of $\boldsymbol{c}$, and

(ii) $pr_{C''_{i-1}} \circ \partial_i : C'_i \to C''_{i-1}$ is an isomorphism.

$$\Rightarrow \tau(C_*, \boldsymbol{c}) = \pm \prod_i (\det pr_{C''_{i-1}} \circ \partial_i)^{(-1)^i}.$$

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Non-commutative topological torsion

$\rho\colon \mathbb{Z}[\pi_1 X] \to \mathbb{F}$: homomorphism s. t.
$H_*^\rho(X;\mathbb{F})\ (:= H_*(C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} \mathbb{F})) = 0$
$\rightsquigarrow\ \tau_\rho(X) := \tau(C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} \mathbb{F}, \langle \tilde{e} \otimes 1 \rangle) \in \mathbb{F}^\times_{ab} / \pm\rho(\pi_1 X)$
: Reidemeister torsion

$\tau_\rho(X)$ is a simple homotopy invariant ,in particular, a topological invariant.

Take homomorphisms $\rho\colon \pi_1 X \to \Gamma$ and $\alpha\colon \Gamma \twoheadrightarrow \langle t \rangle$ s. t.
$\mathbb{Z}[\Gamma]$ has the quotient (skew) field $\mathbb{Q}(\Gamma)$.

$\Gamma = \operatorname{Ker}\alpha \rtimes \langle t \rangle \ \rightsquigarrow\ \mathbb{Q}(\Gamma) = \mathcal{K}(t)$

If $H_*^\rho(X;\mathcal{K}(t)) = 0$, then $\tau_\rho(X) \in \mathcal{K}(t)^\times_{ab} / \pm\rho(\pi_1 X)$ is defined.

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Rational derived series

G : group

$\rightsquigarrow$ $G_r^{(0)} := G$,

$G_r^{(n)} := \{g \in G_r^{(n-1)} \ ; \ \exists k \in \mathbb{Z} \setminus 0$ s. t. $g^k \in [G_r^{(n-1)}, G_r^{(n-1)}]\}$.

: rational derived series

$G_r^{(n)}/G_r^{(n+1)}$: torsion free abelian

$\rightsquigarrow$ $G/G_r^{(n+1)}$: poly-torsion-free-abelian

$1 \triangleleft G_r^{(n)}/G_r^{(n+1)} \triangleleft \cdots \triangleleft G_r^{(1)}/G_r^{(n+1)} \triangleleft G/G_r^{(n+1)}$.

Theorem. (Passman)

Γ : poly-torsion-free-abelian

$\Rightarrow$ $\mathbb{Z}[\Gamma]$ has the quotient (skew) field $\mathbb{Q}(\Gamma)$.

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2. Circle-valued Morse theory

$f : X \to S^1$: Morse-Smale map, i. e.,
$\forall p, q \in \mathrm{Crit}\, f$, unstable manifold $\mathcal{D}(p) \pitchfork$ stable manifold $\mathcal{A}(q)$.

$\mathcal{O} := \{o : S^1 \to X \,;\, \frac{do}{ds} = -\nabla f\}/U(1)$: nondegenerate, i. e.,
for a return map ϕ around $x \in o(S^1)$,
$\det(id - (d\phi)_x : T_x X / T_x o(S^1) \to T_x X / T_x o(S^1)) \neq 0.$

$t \in \pi_1 S^1$: "downward" generator
Assume $f_* : \pi_1 X \to \langle t \rangle$ is surjective.

$\Lambda := \{\sum_{\gamma \in \pi_1 X} a_\gamma \cdot \gamma \,;\, a_\gamma \in \mathbb{Z},\ (*)\}$: Novikov completion of $\mathbb{Z}[\pi_1 X]$
$(*)\ \forall k \in \mathbb{Z},\ \sharp\{\gamma \in \pi_1 X \,;\, a_\gamma \neq 0,\ \deg f_*(\gamma) \leq k\} < \infty.$
Λ_{ab} : Novikov completion of $\mathbb{Z}[H_1(X)]$

e. g., $X = S^1,\ f = id \ \Rightarrow\ \Lambda = \Lambda_{ab} = \mathbb{Z}((t))\ (:= \mathbb{Z}[[t]][t^{-1}])$.

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Novikov complex

Choose a lift $\tilde{p} \in \widetilde{X}$ for each $p \in \mathrm{Crit}\, f$.

$C_i^{Nov}(f) := \sum_{p \text{ of index } i} \tilde{p} \cdot \Lambda$: Novikov complex
$\partial_i^f : C_i^{Nov}(f) \to C_{i-1}^{Nov}(f) \colon \tilde{p} \cdot \gamma \mapsto \sum_{q \text{ of index } i-1, \gamma'} n(\tilde{p} \cdot \gamma, \tilde{q} \cdot \gamma') \tilde{q} \cdot \gamma'$,
$n(\tilde{p} \cdot \gamma, \tilde{q} \cdot \gamma') :=$ signed number of flows from $\tilde{p} \cdot \gamma$ to $\tilde{q} \cdot \gamma'$.

Theorem. (Pajitnov)

$C_*^{Nov}(f) \simeq C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} \Lambda$: chain homotopic.

$\rho \colon \Lambda \to \mathbb{F}$: homomorphism s. t. $H_*^{\rho}(X; \mathbb{F}) \; (= H_*(C_*^{Nov}(f) \otimes_{\Lambda} \mathbb{F})) = 0$
$\rightsquigarrow \; \tau_{\rho}^{Nov}(f) := \tau(C_*^{Nov}(f) \otimes_{\Lambda} \mathbb{F}, \langle \tilde{p} \otimes 1 \rangle) \in \mathbb{F}_{ab}^{\times} / \pm \rho(\pi_1 X)$
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Lefschetz-type zeta function

$[o] \in \mathcal{O}$, ϕ : return map around $x \in o(S^1)$

$\rightsquigarrow$ $p(o) := \deg(o\colon S^1 \to o(S^1))$

$\epsilon(o) := \operatorname{sgn}\det(id - (d\phi)_x)$

$$\zeta_f := \exp\left(\sum_{[o]\in\mathcal{O}} \frac{\epsilon(o)}{p(o)}[o]\right) \in \Lambda_{ab} \otimes \mathbb{Q}.$$

Lemma.

$$\zeta_f = \prod_{[o]\in\mathcal{O}, p(o)=1} (1 - (-1)^{i_-(o)}[o])^{(-1)^{i_+(o)+i_-(o)+1}} \in \Lambda_{ab},$$

where $i^{\pm}(o)$ are the number of real eigen values of $\det(id - (d\phi)_x)$ which are > 1 and < -1 respectively.

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3. Main theorem

Take (ρ, α) s. t.

(i) $\pi_1 X \xrightarrow{\rho} \Gamma$, (ii) $\mathbb{Z}[\Gamma]$ has the quotient (skew) field $\mathbb{Q}(\Gamma)$.

$$\begin{array}{ccc} \pi_1 X & \xrightarrow{\rho} & \Gamma \\ & \searrow^{f_*} & \downarrow \alpha \\ & & \langle t \rangle \end{array}$$

(e. g. , $\Gamma = \pi_1 X / (\pi_1 X)_r^{(n+1)}$.)

$\Gamma = \mathrm{Ker}\,\alpha \rtimes \langle t \rangle \rightsquigarrow \mathbb{Q}(\Gamma) = \mathcal{K}(t)$

$\rho \colon \mathbb{Z}[\pi_1 X] \to \mathcal{K}(t) \hookrightarrow \mathcal{K}((t))$ $(:= \mathcal{K}[[t]][t^{-1}])$

$\rightsquigarrow \tau_\rho(X) \in \mathcal{K}((t))^\times_{ab} / \pm \rho(\pi_1 X)$

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$N \lhd \mathcal{K}((t))^{\times}_{ab}$:
$\forall k \in \mathbb{Z}, \exists$ a representative in $\mathcal{K}((t))^{\times}$ which equals 1 up to degree k.
$\rightsquigarrow \mathcal{K}((t))^{\times}_{\overline{ab}} := \mathcal{K}((t))^{\times}_{ab}/N$

Remark. $\mathcal{K}^{\times}_{ab} \hookrightarrow \mathcal{K}((t))^{\times}_{\overline{ab}}$

For each $[o] \in \mathcal{O}$ with $p(o) = 1$, choose a path σ_o from $*$ to $o(S^1)$.

$$\zeta_{f,\rho} = \prod_{[o]\in\mathcal{O}, p(o)=1} (1 - (-1)^{i_-(o)}\rho([\sigma_o o \overline{\sigma}_o]))^{(-1)^{i_+(o)+i_-(o)+1}} \in \mathcal{K}((t))^{\times}_{\overline{ab}}.$$

Lemma

$\zeta_{f,\rho}$ does not depend on the choices of σ_o and the order of the product.

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Statement

Main Theorem

$H_*^\rho(X;\mathcal{K}((t))) = 0$

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$\rho\colon \pi_1 X \to H_1(X)$
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$X = M$: 3-manifold

Corollary.

If $\rho_n\colon \pi_1 M \to \pi_1 M/(\pi_1 M)_r^{(n+1)}$ is not cyclic and $\Delta_M^{(n)} \neq 0$, then

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