An Estimate of the Rasmussen Invariant for Links

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Notations

L : an oriented link in the 3-sphere,

D_L : a diagram of L.

$x_+(D_L) = \#\{$positive crossings of $D_L\}$,

$x_-(D_L) = \#\{$negative crossings of $D_L\}$,

$w(D_L) = x_+(D_L) - x_-(D_L)$ writhe ,

$O(D_L) = \#\{$Seifert circles of $D_L\}$.

The Bennequin inequality

L : any oriented link.

$\chi(L)$ = the Euler characteristic of L

= the maximum Euler characteristic of the Seifert surface for L.

For any diagram D_L of L,

$$\chi(L) \leqq O(D_L) - w(D_L). \qquad \text{(BI)}$$

[Bennequin, (1983)]

The slice Bennequin inequality

L : any oriented link.

$\chi_s(L)$ = the slice Euler characteristic of L

= the maximum Euler chara. for an oriented compact 2-submanifold without closed components in B^4 with boundary L in ∂B^4.

For any diagram D_L of L,

$$\chi_s(L) \leqq O(D_L) - w(D_L). \qquad \text{(sBI)}$$

[Rudolph, (1993)]

Theorem 1

L : an oriented link, D_L : a diagram of L.

D_L^{0+} : the diagram obtained by smoothing all negative crossings of D_L.

$l_0(D_L) = \#\{\text{the split components of } D_L^{0+}\}$

If L is not splittable,

$$\chi_s(L) \leqq O(D_L) - w(D_L) - 2(l_0(D_L) - 1) \qquad \text{(sBI2)}$$

for any diagram D_L of L.

If L=K is a knot,

(sBI2) $\Leftrightarrow 2g^*(K) \geqq w(D_K) - O(D_K) + 2l_0(D_K) - 1.$

Rasmussen's knot invariant

$s(K)$: the Rasmussen invariant of a knot K (based on Lee's Khovanov homology).

Thm. [Rasmussen]

(R1) For any knot K, $2g^*(K) \geqq |s(K)|$.

(R2) s induces a homomorphism {knots}/conc $\rightarrow \mathbf{Z}$.

(R3) If a diagram D_K of a knot K is positive,

$$s(K) = w(D_K) - O(D_K) + 1.$$

Thm. [Plamenevskaya ('06), Shumakovitch ('07)]

(R4) For any knot K and any diagram D_K of K,

$$s(K) \geqq w(D_K) - O(D_K) + 1. \qquad \text{(RBI)}$$

The Rasmussen invariant for links

s : the oriented link invariant defined as an extension of the Rasmussen invariant [Beliakova-Wehrli (2008)].

Theorem. [Beliakova-Wehrli]

(BW1) s(a trivial r-comp. link) = $1 - r$.

(BW2) S : a cobordism between links L_0 and L_1

s.t. (each comp. of S)$\cap L_0$ is not empty

$\Rightarrow s(L_1) \geqq s(L_0) + \chi(S)$.

(BW3) $s(L_0 \sqcup L_1) = s(L_0) + s(L_1) - 1$.

Theorem 2

(i) For any oriented link L, $\chi_s(L) \leqq 1 - s(L)$.

(ii) If a diagram D_L of a link L is positive,

$$s(L) = w(D_L) - O(D_L) + 1.$$

(iii) If an oriented link L is not splittable,

$$s(L) \geqq w(D_L) - O(D_L) + 2l_0(D_L) - 1 \quad \text{(RBI2)}$$

for any diagram D_L of L.

Remark. Lobb gave an alternative proof of (RBI2) independently.

Ozsvath-Szabo's knot invariant

$\tau(K)$: the Ozsvath-Szabo's knot invariant of a knot K
(based on the knot Floer homology).

Thm. [Ozsvath-Szabo (2003)]

(OS1) For any knot K, $g^*(K) \geqq |\tau(K)|$.

(OS2) τ induces a homo. {knots}/conc $\to \mathbf{Z}$.

Thm. [Livingston (2004)]

(OS3) If a diagram D_K of a knot K is positive,

$$2\tau(K) = w(D_K) - O(D_K) + 1.$$

(OS4) For any knot K and any diagram D_K of K,

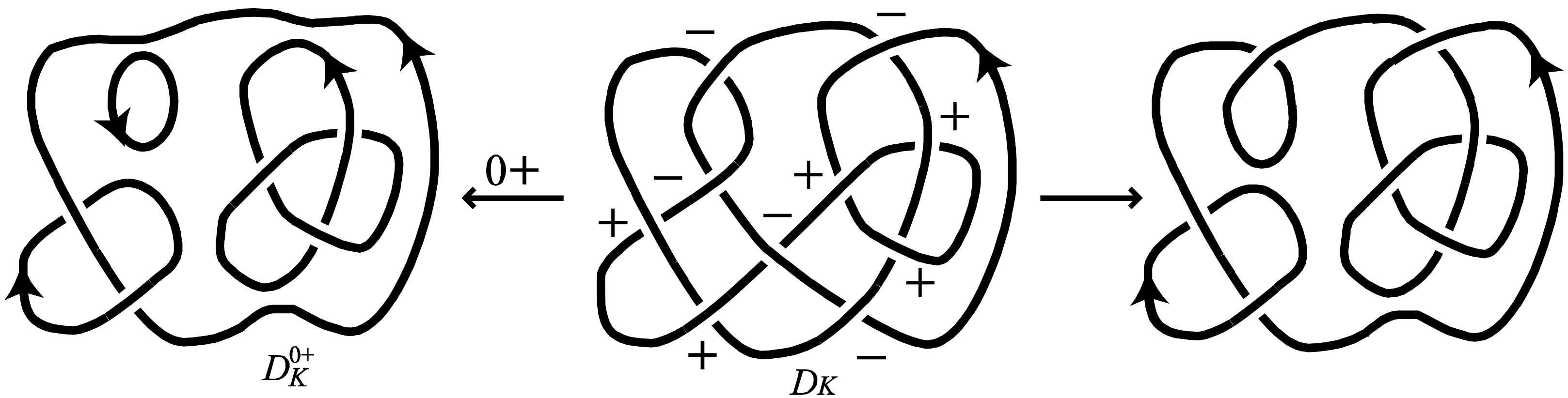
$$2\tau(K) \geqq w(D_K) - O(D_K) + 1. \qquad (\tau\text{BI})$$

Theorem 3

For any knot K and any diagram D_K of K,

$$2\tau(K) \geqq w(D_K) - O(D_K) + 2l_0(D_K) - 1. \qquad (\tau\mathrm{BI2})$$

A link cobordism for the proof of (sBI2) and (RBI2)



A knot cobordism for the proof of (τBI2)