On the compatible contact structures of fibered Seifert links in homology 3-spheres

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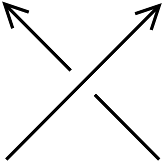

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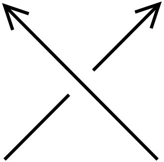

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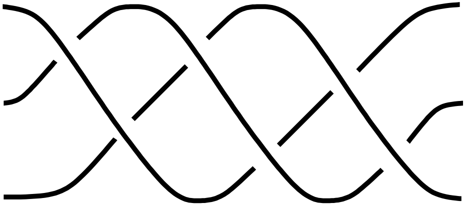
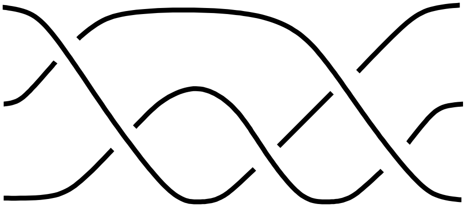
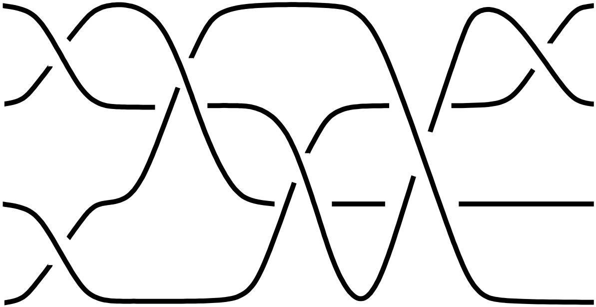
- **positive torus link**



- **positive braid**



- **strongly quasipositive braid**

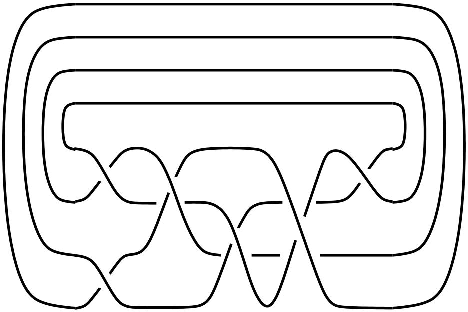


L : an oriented link in S^3.

Definition

L is **strongly quasipositive**

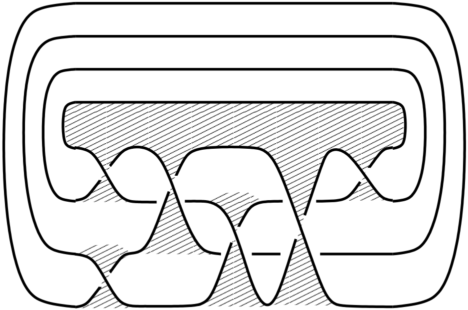
$\Leftrightarrow$ L is obtained as the closure of a strongly quasipositive braid



Definition

F **is a** **quasipositive surface** **in** S^3

$\Leftrightarrow$ ∂F **is a strongly quasipositive link.**

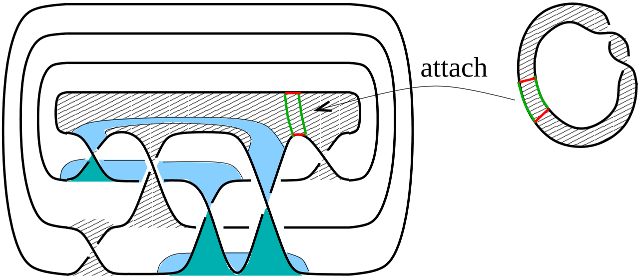


Theorem (Rudolph) — Rigidity of quasipositive surfaces

F : a quasipositive surface

H : a positive Hopf band

- **A surface obtained by plumbing F and H is again quasipositive.**
- **An incompressible subsurface of F is again quasipositive.**
- **A quasipositive surface is an incompressible subsurface of the fiber surface of a positive torus link in S^3.**

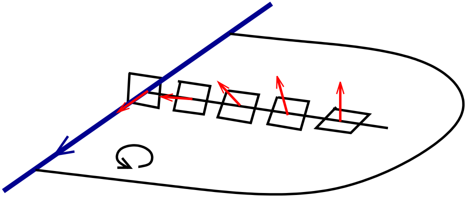


Theorem (Hedden, cf. Baader - I.)

Let F be a fiber surface in S^3. Then,

F is quasipositive

$\Leftrightarrow$ F is compatible with the tight contact structure on S^3.



↑ : Reeb vector field

□ : contact structure

§1. Compatible contact structures

M : an oriented, closed, smooth 3-manifold

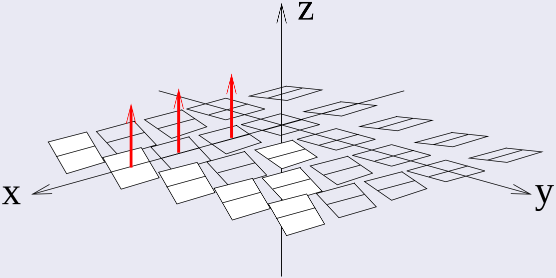
α : 1-form on M s.t. $\alpha \wedge d\alpha > 0$

$\xi = \ker \alpha$: (positive) contact structure

R_α : Reeb vector field of α $(\Leftrightarrow d\alpha(R_\alpha, \cdot) = 0,\ \alpha(R_\alpha) = 1)$

Example 1 (The standard contact structure on $\mathbb{R}^3$)

$\alpha = dz + xdy.$



Example 2

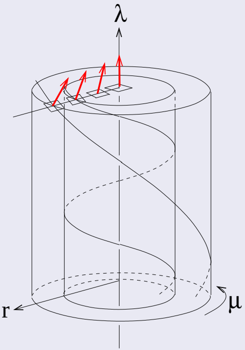
$\alpha = h_2(r)d\mu + h_1(r)d\lambda$.

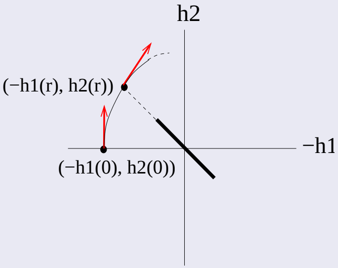
α is a positive contact form $\Leftrightarrow h_1 h_2' - h_2 h_1' > 0$

$\Leftrightarrow (-h_1)h_2' - h_2(-h_1') < 0$

$\Leftrightarrow (-h_1(r), h_2(r))$ moves in the clockwise orientation.

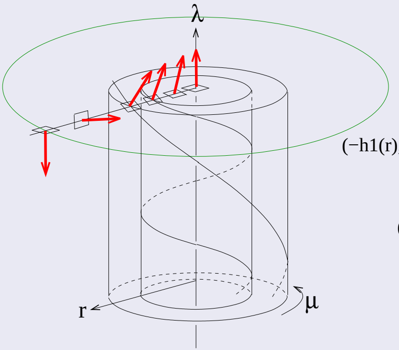
$R_\alpha = \frac{1}{h_1 h_2' - h_2 h_1'} \left(-h_1' \frac{\partial}{\partial \mu} + h_2' \frac{\partial}{\partial \lambda} \right)$.

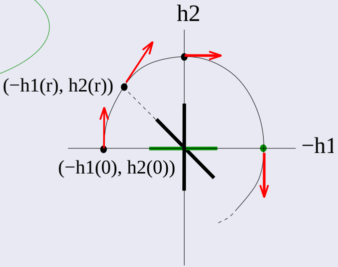




Example 3 (a half Lutz twist)

$$\alpha = h_2(r)d\mu + h_1(r)d\lambda.$$





Definition

A disk D in a contact manifold (M, ξ) is called overtwisted (OT for short) if ξ is tangent to D at each point in ∂D.

Definition

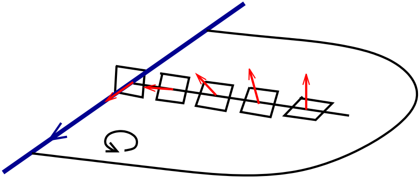
If (M, ξ) has an OT disk then ξ is called overtwisted.
Otherwise it is called tight.

Remark. S^3 **has a unique tight contact structure.**

Definition (cf. [Thurston-Winkelnkemper], [Etnyre])

$\xi = \ker \alpha$ is called **compatible** with a fibered link L in M if

- R_α is tangent to L in the same direction, and
- R_α is positively transverse to the interiors of the fiber surfaces of L.



↑ : Reeb vector field

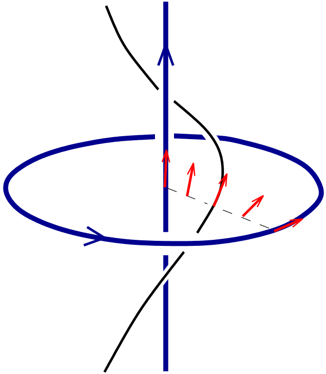
□ : contact structure

and theorems due to Giroux.

§2. Two examples

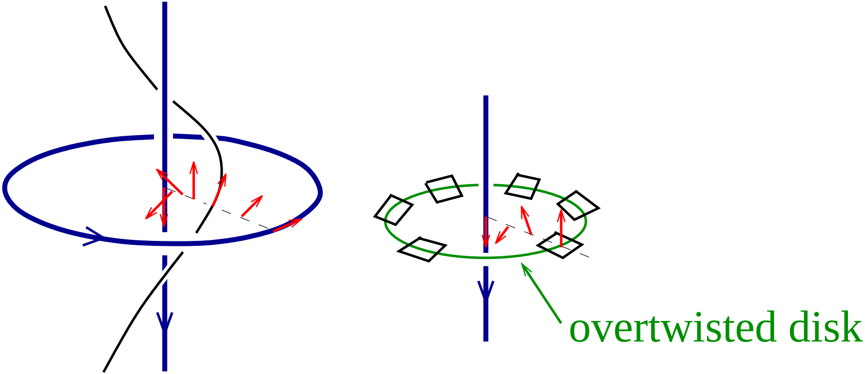
Positive Hopf link

We can set R_α s.t. it is tangent to the fibers of the Seifert fibration.



Negative Hopf link

We can set R_α s.t. it is tangent to the fibers of the Seifert fibration outside a small neighborhood of a link component.

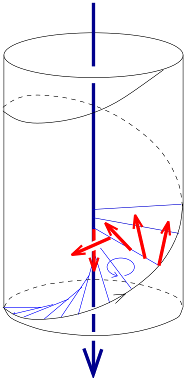


Thus, we can detect an overtwisted disk.

continue⟶

Negative Hopf link (continued)

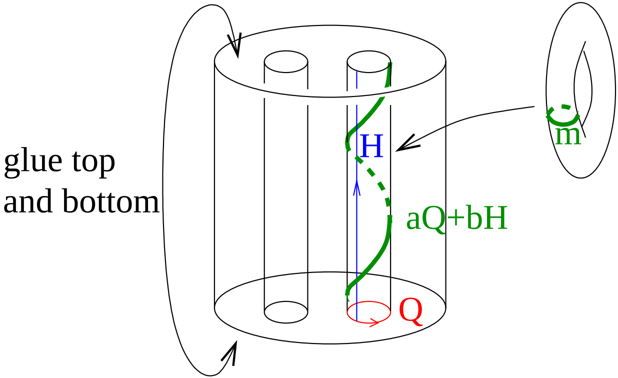
The Reeb vector field is positively transverse to the fiber surface of the negative Hopf link (after a small perturbation).



Thus, the compatible contact structure is OT.

§3. Main Result

$\mathcal{S} = S^2 \setminus \sqcup_{i=1}^{k} \text{int}\, D_i^2$
$\Sigma = (\mathcal{S} \times S^1) \cup \bigcup_{i=1}^{k} (D^2 \times S^1)_i$ with $\mathfrak{m}_i \mapsto a_i Q_i + b_i H$.



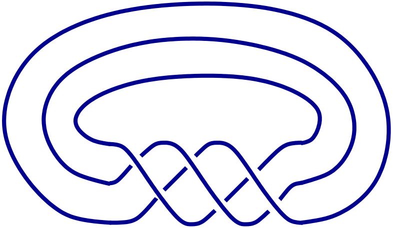
We assume that $\sum_{i=1}^{k} b_i a_1 \cdots a_{i-1} a_{i+1} \cdots a_k = 1$
s.t. Σ is a homology 3-sphere.

Definition

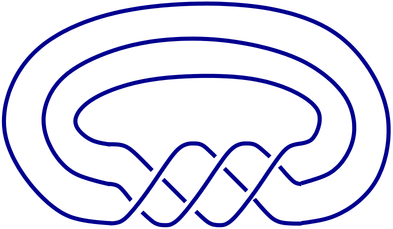
A link in Σ consisting of a union of orbits in Σ is called a **Seifert link** in Σ.

Definition

We say a Seifert link is **positively-twisted** (PT for short) if $a_1 a_2 \cdots a_n > 0$.



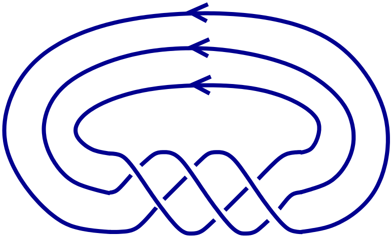
positively−twisted



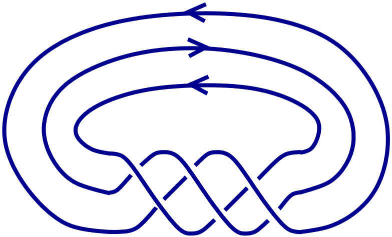
negatively−twisted

Definition

We say the orientation of a Seifert link L is canonical if the orientations of all components of L coincides with, or are opposite to, the orientation of the fibers of the Seifert fibration.



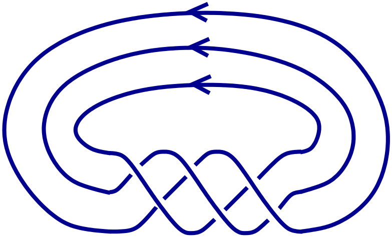
canonical



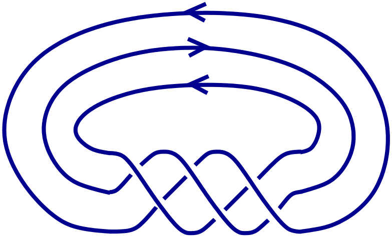
not canonical

Theorem 1

Let L be a fibered PT Seifert link in a homology S^3. Then, the contact structure compatible with L is tight
$\Leftrightarrow$ the orientation of L is canonical.

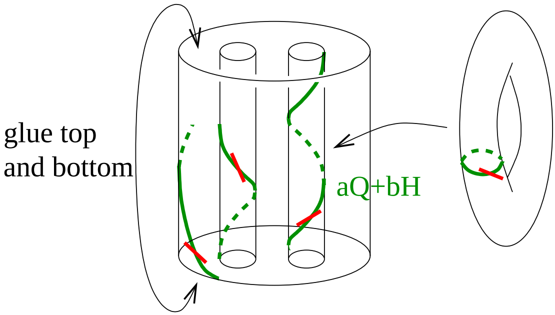


canonical



not canonical

Sketch of the proof.



- **Not canonical ⇒ OT**

  (1) $\exists$ **such a good** $\xi = \ker\alpha \Leftrightarrow \sum_{i=1}^{k} \frac{b_i}{a_i} = \frac{1}{a_1 \cdots a_k} > 0.$

  (2) Extend α **into** $(D^2 \times S^1)_i$ **as in the two examples.**

- **Canonical ⇒ tight**

  ξ **is symplectically fillable [McCarthy-Wolfson]**

  (cf. [Lisca-Matić])

§4. PT graph links in S^3

Σ_1, Σ_2 : homology S^3's.
L_i : a Seifert link in Σ_i.
S_i ; a link component of L_i.
$(\mathfrak{m}_i, \mathfrak{l}_i)$: a preferred meridian-longitude pair of $\Sigma_i \setminus \operatorname{int} N(S_i)$.

Definition

Glue $\Sigma_1 \setminus \operatorname{int} N(S_1)$ and $\Sigma_2 \setminus \operatorname{int} N(S_2)$ by the map

$$(\mathfrak{m}_1, \mathfrak{l}_1) \mapsto (\mathfrak{l}_2, \mathfrak{m}_2).$$

The obtained link $(L_1 \setminus S_1) \cup (L_2 \setminus S_2)$ is called a splice of L_1 and L_2 along S_1 and S_2.

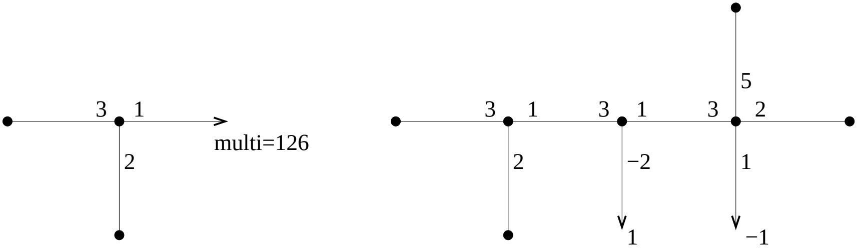
Remark. The ambient space of the splice is again a homology S^3.

Definition

A link obtained by iterating splicing of Seifert links is called a graph link

Definition

A link obtained by iterating splicing of PT Seifert links is called a PT graph link.

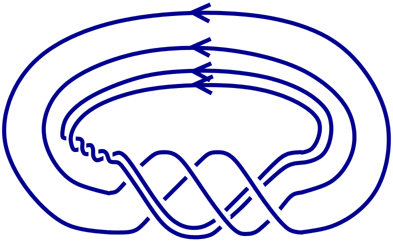


Remark. We use "multilink"

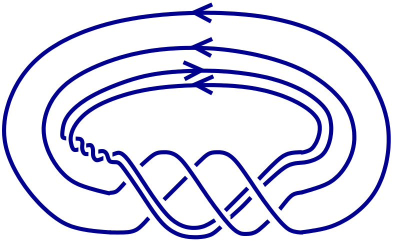
(= "rational open book" in [Baker-Etnyre-HornMorris])

Theorem 2

Let L be a fibered PT graph link in S^3. Then, the contact structure compatible with L is tight $\Leftrightarrow$ the orientation of L is canonical.



canonical



not canonical

Sketch of the proof.

We prepare two contact forms compatible with L.

α_1 : obtained by Thurston-Winkelnkemper's construction.

α_2 : obtained by splicing Seifert multilinks.

- Canonical $\Rightarrow$ tight

  Canonical $\Rightarrow$ each splice is like a positive cabling $\Rightarrow$ tight.

- Not canonical $\Rightarrow$ OT

  (1) Choose α_1 and let $N(S_i)$ be a solid torus for splicing. Then $(N(S_i), \ker \alpha_1)$ does not contain a half Lutz twist.

  (2) Choose α_2 then $\exists$ a half Lutz tube N' with OT disk D.

  (3) $\exists$ contactom. $\phi : (S^3, \ker \alpha_2) \to (S^3, \ker \alpha_1)$ s.t. $N(S_i) \subset \phi(N')$ and $\phi(\partial D) \subset \phi(N') \setminus N(S_i)$.

  (4) $\phi(D)$ remains in $(S^3, \ker \alpha_1)$ as an OT disk after the splicings. □

§5. Corollary

$f : \mathbb{C}^2 \to \mathbb{C}$: a polynomial map s.t.

- $f(0) = 0$.
- f has an isolated singularity at $0 \in \mathbb{C}^2$.

Theorem (Milnor)

$f/|f| : S^3_\varepsilon \setminus \{f = 0\} \to S^1$ is a locally trivial fibration.

Remark. The monodromy is a product of positive Dehn twists, and hence its compatible contact structure is tight.

Theorem (Pichon-Seade)

$f\bar{g}/|f\bar{g}| : S^3_\varepsilon \setminus \{f\bar{g} = 0\} \to S^1$ is a locally trivial fibration in most cases.

Corollary (answer to a question of Pichon)

Suppose that $f\bar{g}/|f\bar{g}| : S^3_\varepsilon \setminus \{f\bar{g} = 0\} \to S^1$ is a locally trivial fibration. Then, its compatible contact structure is overtwisted.

Remark. The link $S^3_\varepsilon \cap \{g = 0\}$ and $S^3_\varepsilon \cap \{\bar{g} = 0\}$ are ambient isotopic as oriented links.

Question. How about "strongly quasipositive links with reversed orientations"?

Question

Let L be an oriented link in S^3 obtained from a non-splittable strongly quasipositive link with several components by reversing the orientations of some, but not all, link components. Is L not strongly quasipositive?

True for Seifert links and fibered PT graph links in S^3.

Question (contact version)

Let L be as above. Suppose further that L is fibered. Is the contact structure compatible with L overtwisted?