On the twisted Alexander polynomial for hyperbolic fibered links via twisted monodromy

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GCOE Workshop
Circle valued Morse theory and Alexander invariants

Notations

L : a fibered link in S^3, i.e.,

$$\begin{array}{ccc} F & \longrightarrow & E_L := S^3 \setminus N(L) \\ & & \downarrow \\ & & S^1 \end{array}$$

We can regard E_L as a mapping torus:

$$E_L = F \times [0,1]/(x,1) \sim (f(x),0)$$

where $f : F \to F$ is a diffeomorphism.

This diffeomorphism f is called a **monodromy**.

Classical result by Milnor

Theorem (J. Milnor)

For a fibered link exterior

$$E_L = F \times [0,1]/(x,1) \sim (f(x),0),$$

the characteristic polynomial of

$$f_* : H_1(F;\mathbb{Q}) \xrightarrow{\sim} H_*(F;\mathbb{Q})$$

is expressed as

$$\det(t\mathbf{1} - f_*) = \begin{cases} \Delta_K(t), & \text{if } L \text{ is a knot } K \\ (t-1)\Delta_L(t,\ldots,t), & \text{if } L \text{ has 2 or more compos.} \end{cases}$$

where $\Delta_K(t)$ and $\Delta_L(t_1,\ldots,t_n)$ are the Alexander polynomials.

The case of the twisted Alexander polynomial

Question

How about **the twisted Alexander polynomial** $\Delta_{L,\rho}(t)$ twisted by $\rho : \pi_1(E_L) \to \mathrm{SL}_N(\mathbb{C})$?

Theorem (J. Cha)

*For a fibered **knot** $K \subset S^3$,*
the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ for $\rho : \pi_1(E_K) \to \mathrm{SL}_N(\mathbb{C})$ is expressed as

$$\Delta_{K,\rho}(t) = \det(t\mathbf{1} - f_*)$$

where $f_* : H_1(F; \mathbb{C}^N_\rho) \xrightarrow{\sim} H_1(F; \mathbb{C}^N_\rho)$.

$$(E_K = F \times [0,1]/(x,1) \sim (f(x),0))$$

More general situations:

Theorem (J. Dubois and Y.)

For a fibered link in a closed 3-manifold M, the twisted Alexander polynomial $\Delta_{L,\rho}(t,\ldots,t)$ for $\rho : \pi_1(M_L) \to \mathrm{SL}_N(\mathbb{C})$ is expressed as

$$\Delta_{M,\rho}(t,\ldots,t) = \det(t\mathbf{1} - f_*)$$

where $f_* : H_1(F;\mathbb{C}^N_\rho) \xrightarrow{\sim} H_1(F;\mathbb{C}^N_\rho)$.

$(M_L = M \setminus N(L) = F \times [0,1]/(x,1) \sim (f(x),0),\ F$: *connected*$)$

Remark

We need some assumptions for ρ in the above theorem.

The contents of this talk

Precise statement and the proof of Main theorem

- The conditions on homomorphisms of $\pi_1(M_L)$.
- Sketch of the proof by using cut & paste method in Reidemeister torsion theory.

Explicit examples

In general, it is difficult to compute the twisted monodromy

$$f_* : H_1(F; \mathbb{C}^N_\rho) \xrightarrow{\sim} H_1(F; \mathbb{C}^N_\rho).$$

For special representations $(\pi_1(M_L) \to \mathrm{SL}_3(\mathbb{C}))$, we can relate

$H_1(F; \mathbb{C}^3_\rho)$ with an **affine variety** and

f_* with the **differential of a map on the variety**

We will discuss once-punctured torus bundles over S^1.

Review of Twisted Alexander polynomial

To define Twisted Alexander polynomial, we need

- A **surjective** homomorphism from $\pi_1(E_L)$ onto an free abelian group $\mathbb{Z}^n$ ($n \leqq b_1(E_L)$):

$$\varphi : \pi_1(E_L) \to \mathbb{Z}^n = \langle t_1, \ldots, t_n \,|\, t_i t_j = t_j t_i (\forall i, j) \rangle$$

- a representation of $\pi_1(E_L)$:

$$\rho : \pi_1(E_L) \to \mathrm{SL}_N(\mathbb{C})$$

“Representation” means that $\pi_1(E_L)$ acts on a vector space via the homomorphism ρ,

$$\gamma \cdot \boldsymbol{v} = \rho(\gamma)(\boldsymbol{v}) \quad \text{for} \quad \gamma \in \pi_1(E_L), \boldsymbol{v} \in \mathbb{C}^N$$

Definition of Twisted Alexander polynomial

Choose a presentation of $\pi_1(E_L)$:

$$\pi_1(E_L) = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_{k-1} \rangle.$$

Definition (M. Wada)

For $\varphi : \pi_1(E_L) \to \mathbb{Z}^n$ and $\rho : \pi_1(E_L) \to \mathrm{SL}_N(\mathbb{C})$,

the twisted Alexander polynomial $\Delta^{\varphi}_{L,\rho}(t_1, \ldots, t_n)$ of L is given by

$$\Delta^{\varphi}_{L,\rho}(t_1, \ldots, t_n) = \frac{\det\left(\varphi \otimes \rho\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{\substack{1 \leqq i,j \leqq k, \\ j \neq \ell}}}{\det\left(\varphi \otimes \rho(x_\ell - 1)\right)}$$

when $\det\left(\varphi \otimes \rho(x_\ell - 1)\right) \neq 0$.

Example of Twisted Alexander polynomial

Let K be the figure eight knot.

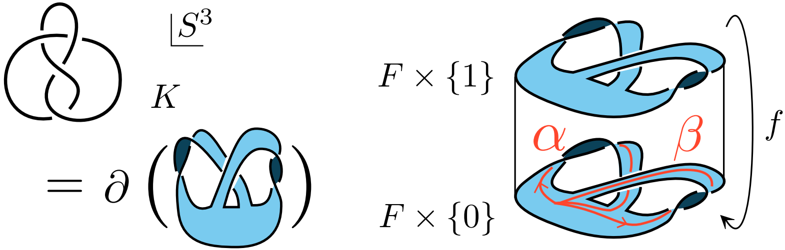


Figure: the fibered structure of E_K

The knot group $\pi_1(E_K)$ is expressed as

$$\pi_1(E_K) = \langle \mu, \alpha, \beta \mid \mu\alpha\mu^{-1} = \alpha\beta,\ \mu\beta\mu^{-1} = \beta\alpha\beta \rangle$$

where μ is a meridian (a lift of the base circle).

Put the relators as

$$r_1 = \mu\alpha\mu^{-1}\beta^{-1}\alpha^{-1}, \quad r_2 = \mu\beta\mu^{-1}\beta^{-1}\alpha^{-1}\beta^{-1}.$$

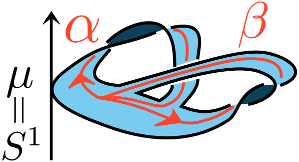
Then

$$\Delta^{\varphi}_{K,\rho}(t) = \frac{\det\begin{pmatrix} \varphi\otimes\rho\left(\frac{\partial}{\partial\alpha}r_1\right) & \varphi\otimes\rho\left(\frac{\partial}{\partial\beta}r_1\right) \\ \varphi\otimes\rho\left(\frac{\partial}{\partial\alpha}r_2\right) & \varphi\otimes\rho\left(\frac{\partial}{\partial\beta}r_2\right)\end{pmatrix}}{\det(\varphi\otimes\rho(\mu-1))}$$
$$= \frac{\det\begin{pmatrix} \varphi\otimes\rho(\mu-1) & \varphi\otimes\rho(-\alpha) \\ \varphi\otimes\rho(-\beta) & \varphi\otimes\rho(\mu-\beta\alpha-1)\end{pmatrix}}{\det(\varphi\otimes\rho(\mu-1))}$$

in which we simplify words with $r_1 = 1$ and $r_2 = 1$.

Choose $\varphi : \pi_1(E_K) \to \mathbb{Z}$ as the induced hom. from the fibration

$$\varphi : \pi_1(E_K) \to \pi_1(S^1) = \langle t \rangle$$
$$\mu \mapsto t$$
$$\alpha, \beta \mapsto 1$$



and $\rho : \pi_1(E_K) \to \mathrm{SL}_N(\mathbb{C})$ such that $\rho(r_1) = \mathbf{1}$ and $\rho(r_2) = \mathbf{1}$:

$$\rho : \mu \mapsto M, \quad \alpha \mapsto A, \quad \beta \mapsto B \in \mathrm{SL}_N(\mathbb{C})$$

The twisted Alexander polynomial $\Delta^{\varphi}_{K,\rho}(t)$ turns into

$$\Delta^{\varphi}_{K,\rho}(t) = \frac{\det \begin{pmatrix} tM - \mathbf{1} & -A \\ -B & tM - BA - \mathbf{1} \end{pmatrix}}{\det(tM - \mathbf{1})}.$$

Some remarks on the homomorphism φ

Remark

- For every fibered knot $K \subset S^3$, the induced homomorphism

$$\varphi : \pi_1(E_K) \to \pi_1(S^1) = \mathbb{Z}$$

agrees with the abelianization homomorphism

$$\pi_1(E_K) \to H_1(E_K;\mathbb{Z}) = \pi_1(E_K)/[\pi_1(E_K), \pi_1(E_K)].$$

- For every fibered link $L \subset S^3$, the induced homomorphism φ factors through the abelianization homomorphism.

$$\begin{array}{ccc} \pi_1(E_L) & \xrightarrow{\ \varphi\ } & \pi_1(S^1) = \langle t \rangle \\ & \searrow \quad \nearrow & t_1 = \cdots = t_m = t \\ & H_1(E_L;\mathbb{Z}) := \langle t_1, \ldots, t_m \mid t_i t_j = t_j t_i \ (\forall i,j) \rangle & \end{array}$$

Homomorphism onto an abelian group

We assume that

- L is a fibered link in a closed 3-manifold M and;
- p denotes the fibration of M_L over S^1.

$$\begin{array}{ccl} F & \longrightarrow & M_L := M \setminus N(L) \\ & p \downarrow & \\ & S^1 & \end{array}$$

Hereafter we set the surjective homomorphism

$$\varphi : \pi_1(M_L) \to \mathbb{Z}$$

as the induced homomorphism from the fibration, i.e.,

$$\varphi = p_* : \pi_1(M_L) \to \pi_1(S^1) = \langle t \rangle.$$

Note that $\mathrm{Ker}\,\varphi = \pi_1(F)$.

Representations of $\pi_1(M_L)$ into $\mathrm{SL}_N(\mathbb{C})$

We denote by ρ an $\mathrm{SL}_N(\mathbb{C})$-representation of $\pi_1(M_L)$:

$$\pi_1(M_L) \to \mathrm{SL}_N(\mathbb{C}) \curvearrowright \mathbb{C}^N.$$

We assume that the ρ satisfies that

- the homology of local system given by ρ and φ is trivial:

$$H_*(M_L; \mathbb{C}(t)^N_\rho) = \mathbf{0} \quad \text{and;}$$

- the restriction $\rho|_{\pi_1(F)}$ on $\pi_1(F)$ is irreducible.
  ($\Leftrightarrow$ $\rho(\pi_1(F))$ has no common eigenvector in $\mathbb{C}^N$.)

Under these assumptions, $\Delta_{L,\rho}$ is well-defined as a Laurent polynomial.

Statement of Main Theorem

Theorem

Let L be a fibered link in a closed 3-manifold M:

$$\begin{array}{ccc} F \to M_L := M \setminus N(L) & & \\ p \downarrow & = & F \times [0,1]/(x,1) \sim (f(x),0). \\ S^1 & & \end{array}$$

We assume that

- $\varphi = p_* : \pi_1(M_L) \to \pi_1(S^1) = \langle t \rangle$ *and;*
- $\rho : \pi_1(M_L) \to \mathrm{SL}_N(\mathbb{C})$ *satisfies that* $H_*(M_L; \mathbb{C}(t)^N_\rho) = 0$ *and* $\rho|_{\pi_1 F}$ *is irreducible.*

Then

$$\Delta_{L,\rho}(t) = \det(t\mathbf{1} - f_*)$$

where $f_* : H_1(F; \mathbb{C}^N_\rho) \xrightarrow{\sim} H_1(F; \mathbb{C}^N_\rho)$ *(twisted monodromy).*

Idea of Proof

Main Tool

Mayer-Vietoris argument in Reidemeister torsion:

$$\Delta_{L,\rho}(t) = \text{Reidemeister torsion of } (M_L, \varphi \otimes \rho).$$

The Right Hand Side is an invariant of

$$C_*(M_L; \mathbb{C}(t)^N_\rho) := C_*(\widetilde{M_L}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M_L)]} (\mathbb{C}(t) \otimes \mathbb{C}^N)$$

where

- $\widetilde{M_L}$ is the universal cover of M_L and
- $\mathbb{C}(t) \otimes \mathbb{C}^N = \mathbb{C}(t)^N$ is a left $\mathbb{Z}[\pi_1(M_L)]$-module via $\varphi \otimes \rho$.

Details in Mayer-Vietoris argument

Decompose $M_L = F \times [0,1]/\sim$ as

$$M_L = \overline{N(F \times \{1\})} \cup F \times [\epsilon, 1-\epsilon]$$

where $N(F)$ is an open tubular neighbourhood of a fiber F.
Note that

- $\overline{N(F)} \simeq F \times [0,1]$;
- $\partial \overline{N(F)} = F_+ \cup F_-, \quad F_\pm = F$;
- the gluing map is given by

$$F_+ \cup F_- \xrightarrow{f \cup id} F \times \{\epsilon\} \cup F \times \{1-\epsilon\}$$

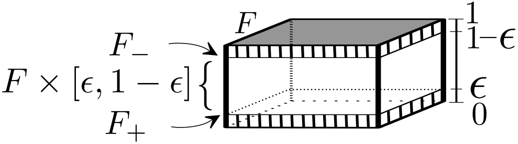


Figure: the decomposition of M_L

Computation

From the decomposition:

$$F_+ \cup F_- \to \overline{N(F)} \cup F \times [\epsilon, 1-\epsilon] \xrightarrow{\text{identified along } F_\pm} M_L$$

it follows that

$$\begin{aligned}\Delta_{L,\rho}(t) &= \frac{\mathrm{Tor}(\overline{N(F)}) \cdot \mathrm{Tor}(F \times [\epsilon, 1-\epsilon])}{\mathrm{Tor}(F_+ \cup F_-) \cdot \mathrm{Tor}(\mathcal{H})} \\ &= \frac{\mathrm{Tor}(F) \cdot \mathrm{Tor}(F)}{\mathrm{Tor}(F) \cdot \mathrm{Tor}(F) \cdot \mathrm{Tor}(\mathcal{H})} \\ &= \mathrm{Tor}(\mathcal{H})^{-1}\end{aligned}$$

where $\mathcal{H}$ is the chain complex given by Mayer-Vietoris exact sequence with the coefficient $\mathbb{C}(t)^N_\rho$.

$$\cdots \to H_i(F_+ \cup F_-) \to H_i(\overline{N(F)}) \oplus H_i(F \times [\epsilon, 1-\epsilon]) \to H_i(M_L) \to \cdots.$$

Torsion of Mayer-Vietoris exact sequence

By the irreducibolity of $\rho|_{\pi_1(F)}$ and $H_*(M_L; \mathbb{C}(t)^N_\rho) = \mathbf{0}$, the Mayer-Vietoris sequence $\mathcal{H}$ turns into

$$0 \to H_1(F_+) \oplus H_1(F_-) \to H_1(\overline{N(F)}) \oplus H_1(F \times [\epsilon, 1-\epsilon]) \to 0.$$

Moreover this isomorphism is expressed as, (by Friedl-Kim's Proposition),

$$0 \to H_1(F) \xrightarrow{t \cdot f_* - id} H_1(F) \to 0$$

where the coefficiet is $\mathbb{C}(t)^N_\rho$.
We can deduce that

$$\Delta_{L,\rho}(t) = \mathrm{Tor}(\mathcal{H})^{-1} = \det(t \cdot f_* - \mathbf{1}).$$

Note that

$$\Delta_{L,\rho}(t) \doteq \Delta_{L,\rho}(t^{-1}) \doteq \det(t\mathbf{1} - f_*) \quad \text{up to a factor } t^k \ (k \in \mathbb{Z}).$$

Review of once–punctured torus bundles

Let

- $\Sigma_{1,1}$ be the once-punctured torus and;
- $f : \Sigma_{1,1} \to \Sigma_{1,1}$ an ori. pres. diffeomorphism such that

$$f|_{\partial \Sigma_{1,1}} = id \quad \text{on } \partial \Sigma_{1,1}.$$

- T_f the mapping torus of f, i.e.,
  $T_f = \Sigma_{1,1} \times [0,1]/(x,1) \sim (f(x),0)$.

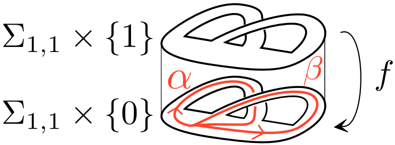


Figure: Once–punctured torus bundle

Review of once–punctured torus bundles

Once–punctured torus bundles are classified by the induced isomorphism:

$$A_f : H_1(\Sigma_{1,1}; \mathbb{Z}) \simeq \mathbb{Z}^2 \to H_1(\Sigma_{1,1}; \mathbb{Z}) \simeq \mathbb{Z}^2.$$

Since f is ori. pres., $A_f \in \mathrm{SL}_2(\mathbb{Z})$, generated by

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For example, the figure knot exterior E_K corresponds to

$$LR = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

The Alexander polynomial for once–punctured torus bundles

We can define the Alexander polynomial of T_f corresponding to

$$\varphi : \pi_1(T_f) \to \pi_1(S^1) = \langle t \rangle \quad (\leftrightarrow \widehat{T}_f\text{: infinite cyclic cover of } T_f)$$

induced from the fibration.
The Alexander polynomial of T_f is expressed as

$$\Delta_{T_f}(t) = t^2 - (\mathrm{tr}\, A_f)t + 1$$

where $A_f : H_1(\Sigma_{1,1};\mathbb{Z}) \to H_1(\Sigma_{1,1};\mathbb{Z})$ induced from the monodromy f.

Relation to the character variety

We consider the composition $Ad \circ \rho$:

$$\pi_1(T_f) \xrightarrow{\rho} \mathrm{SL}_2(\mathbb{C}) \curvearrowright \mathfrak{sl}_2(\mathbb{C}) \quad \text{(adjoint action)}$$
$$\gamma \mapsto \rho(\gamma) \qquad \rho(\gamma)\cdot \boldsymbol{v} = \rho(\gamma)\boldsymbol{v}\rho(\gamma)^{-1}$$

where $\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Note that

- $Ad \circ \rho : \pi_1(T_f) \to \mathrm{SL}_3(\mathbb{C}) \curvearrowright \mathbb{C}^3 \simeq \mathfrak{sl}_2(\mathbb{C})$.
- If ρ is irreducible on $\pi_1(\Sigma_{1,1})$, then

$$H_1(\Sigma_{1,1}; \mathfrak{sl}_2(\mathbb{C})_\rho) \simeq T^*_{[\rho]}X(\Sigma_{1,1})$$

($X(\Sigma_{1,1})$ the character variety of $\pi_1(\Sigma_{1,1})$)

The character variety $X(\Sigma_{1,1})$ of $\pi_1(\Sigma_{1,1})$ is

$$Hom(\pi_1(\Sigma_{1,1}), \mathrm{SL}_2(\mathbb{C}))//\text{conjugation}.$$

This space $X(\Sigma_{1,1})$ is identified with

$$\begin{aligned} X(\Sigma_{1,1}) &\simeq \mathbb{C}^3 \\ [\rho] &\mapsto (\mathrm{tr}\,\rho(\alpha), \mathrm{tr}\,\rho(\beta), \mathrm{tr}\,\rho(\alpha\beta)) \end{aligned}$$

where $\pi_1(\Sigma_{1,1}) = \langle \alpha, \beta \rangle$.

A diffeomorphism $f : \Sigma_{1,1} \to \Sigma_{1,1}$ induces

$$\begin{aligned} f^* : X(\Sigma_{1,1}) &\to X(\Sigma_{1,1}) \\ [\rho] &\mapsto [\rho \circ f_*] \quad \text{and;} \end{aligned}$$

$${}^t(df^*) : T^*_{[\rho \circ f_*]} X(\Sigma_{1,1}) \to T^*_{[\rho]} X(\Sigma_{1,1}) \simeq H_1(\Sigma_{1,1}; \mathfrak{sl}_2(\mathbb{C})_\rho)$$

The character varieties of once–punctured torus bundles

If T_f is an once–punctured torus bundle over S^1:

$$T_f = \Sigma_{1,1} \times [0,1]/(x,1) \sim (f(x),0),$$

then the fundamental group $\pi_1(T_f)$ has the presentation:

$$\langle \mu, \alpha, \beta \,|\, \mu\alpha\mu^{-1} = f_*(\alpha), \mu\beta\mu^{-1} = f_*(\beta)\rangle.$$

Hence every $\rho : \pi_1(T_f) \to \mathrm{SL}_2(\mathbb{C})$ satisfies that

$$\mathrm{tr}\,\rho(\alpha) = \mathrm{tr}\,\rho(f_*(\alpha)),\ \mathrm{tr}\,\rho(\beta) = \mathrm{tr}\,\rho(f_*(\beta)),\ \mathrm{tr}\,\rho(\alpha\beta) = \mathrm{tr}\,\rho(f_*(\alpha\beta)).$$

This means that

$$X(T_f)(:= Hom(\pi_1(T_f), \mathrm{SL}_2(\mathbb{C}))//\text{conj.}) \subset \mathbf{Fix(f^*)}$$

in $X(\Sigma_{1,1}) \simeq \mathbb{C}^3$ ($f^* : \mathbb{C}^3 \to \mathbb{C}^3$).

If the $f \in Diff_+(\Sigma_{1,1}, \partial\Sigma_{1,1})$ is pseudo-Anosov, i.e., $|\operatorname{tr} A_f| > 2$, then

- T_f is a hyperbolic 3-manifold and;
- $\exists X_0 \subset X(T_f)$ such that

$$\dim X_0 = 1 \quad \& \quad X_0 \ni [\rho_0]$$

where $\rho_0 : \pi_1(T_f) \to \mathrm{SL}_2(\mathbb{C})$ corresponding to the complete hyperbolic structure.

Hence the differential df^* at $[\rho] \in X_0 \subset Fix(f^*) \subset X(\Sigma_{1,1}) \simeq \mathbb{C}^3$:

$$df^* : T_{[\rho]}X(\Sigma_{1,1}) \to T_{[\rho]}X(\Sigma_{1,1})$$

has the eigenvalues **1**, $\lambda_{\mathbf{1}}$ and $\lambda_{\mathbf{2}}$.

Fact

$$\lambda_1\lambda_2 = 1.$$

The twisted Alexander polynomial via Trace of df^*

Summarized above, we have

$$\begin{aligned}
\Delta_{T_f, Ad\circ\rho}(t) &= \det(t\mathbf{1} - f_*) \\
&\qquad (f_* : H_1(\Sigma_{1,1}; \mathfrak{sl}_2(\mathbb{C})_\rho) \to H_1(\Sigma_{1,1}; \mathfrak{sl}_2(\mathbb{C})_\rho)) \\
&= \det(t\mathbf{1} - {}^t(df^*)) \\
&\qquad ({}^t(df^*) : T^*_{[\rho]}X(\Sigma_{1,1}) \to T^*_{[\rho]}X(\Sigma_{1,1})) \\
&= \det(t\mathbf{1} - df^*) \\
&\qquad (df^* : T_{[\rho]}X(\Sigma_{1,1}) \to T_{[\rho]}X(\Sigma_{1,1})) \\
&= \mathbf{t^3 - (\mathrm{tr}\, df^*)t^2 + (\mathrm{tr}\, df^*)t - 1}. \\
&(= (t-1)(t-\lambda_1)(t-\lambda_2))
\end{aligned}$$

Recursive formula for f^*

For every $f \in Diff_+(\Sigma_{1,1}, \partial\Sigma_{1,1})$, we can compute

$$f^* : X(\Sigma_{1,1}) \to X(\Sigma_{1,1})$$

from the presentation

$$A_f : H_1(\Sigma_{1,1}; \mathbb{Z}) \to H_1(\Sigma_{1,1}; \mathbb{Z})$$

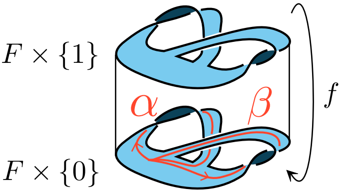
as a word in $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.
The relation is given by

$$A_f = \pm id \Rightarrow f^* : (x, y, z) \mapsto (x, y, z),$$
$$A_f = R \Rightarrow f^* : (x, y, z) \mapsto (x, z, xz - y),$$
$$A_f = L \Rightarrow f^* : (x, y, z) \mapsto (z, y, yz - x).$$

Example of the figure eight knot exterior

If we consider the figure eight knot exterior E_K, then

$$A_f = LR.$$



Hence the map f^* on $X(\Sigma_{1,1}) = \mathbb{C}^3$ is expressed as

$$f^*(x,y,z) = \begin{pmatrix} f_1^*(x,y,z) \\ f_2^*(x,y,z) \\ f_3^*(x,y,z) \end{pmatrix} = \begin{pmatrix} z \\ yz - x \\ yz^2 - xz - y \end{pmatrix}$$

since $(x,y,z) \xrightarrow{L} (z,y,yz-x) \xrightarrow{R} (z, yz-x, z(yz-x)-y)$.

Twisted Alexander polynomial for 4_1 knot and $Ad \circ \rho$

From the Jacobi matrix, we can see that

$$\mathrm{tr}(df^*) = \mathrm{tr}\begin{pmatrix} 0 & 0 & 1 \\ -1 & z & y \\ -z & z^2-1 & 2yz-x \end{pmatrix} = z + 2yz - x.$$

Since

$$[\rho] \in X(E_K) \subset \mathrm{Fix}(f^*) = \{(x,y,z) \in C^3 \,|\, x = z, x + y = xy\},$$

we have

$$\mathrm{tr}(df^*) = 2(x+y) = 2(\mathrm{tr}\,\rho(\alpha) + \mathrm{tr}\,\rho(\beta)).$$

Hence for the figure eight knot K and $Ad \circ \rho : \pi_1(E_K) \to \mathrm{SL}_3(\mathbb{C})$,

$\Delta_{K,Ad\circ\rho}(t) = t^3 - 2(\mathrm{tr}\,\rho(\alpha) + \mathrm{tr}\,\rho(\beta))t^2 + 2(\mathrm{tr}\,\rho(\alpha) + \mathrm{tr}\,\rho(\beta))t + 1.$