Lens space surgery and Alexander polynomial

Circle valued Morse Theory and Alexander invariants

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Dehn surgery

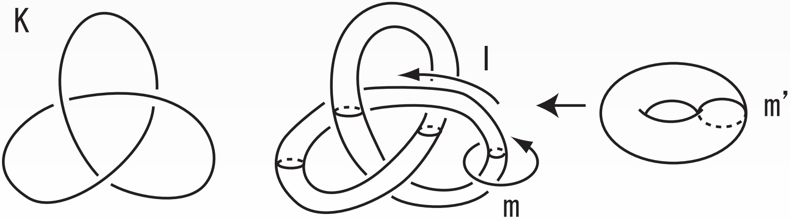
Definition

M:3-manifolds
$K \subset M$: knot

$$M(K,\varphi) = [M - \nu(K)] \cup_{\varphi} D^2 \times S^1,$$

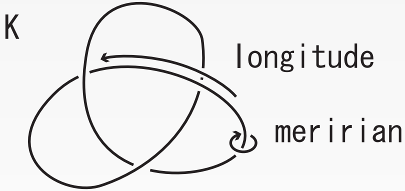
where

$$\varphi : \partial D^2 \times S^1 \to \partial \nu(K)$$



$$\varphi(\partial D^2) = p[\text{meridian}] + q[\text{longitude}]$$

This line $p[\text{meridian}] + q[\text{longitude}]$ is called "slope" of Dehn surgery.



When M is homology sphere, "longitude" and "meridian" are standard.
Then let write the resulting manifold:

$$M(K, p/q).$$

Known result

The Dehn surgery map

$$M \xrightarrow{(K,\ell)} M(K, p/q).$$

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$^{\exists}$a sequence of Dehn surgery s.t.

$$S^3 \overset{(K_1,\ell_1)}{\longrightarrow} M_1 \overset{(K_2,\ell_2)}{\longrightarrow} \cdots \overset{(K_n,\ell_n)}{\longrightarrow} M.$$

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2. All orientable 3-manifold can be obtained from a Dehn surgery of a link.

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But knot cannot make all 3-manifolds

Which kinds of manifolds can knot make?

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If

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Suppose that M is a homology sphere.
Which does $(K, p/r)$ satisfy

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If $K \subset S^3$ is satellite, then $r = 1$ and K is a 2-cable knot.

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as long as lens space surgeries over S^3.

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Lens surgery parameter

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$$M \overset{(K,5)}{\not\longrightarrow} L(5,2),\ L(5,3)$$

for any knot K and any homology sphere M

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Homeo type (orientation-pres.).

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What are Other Restrictions??

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A. Alexander polynomial

Alexander polynomial

The Alexander polynomial

$$K \longrightarrow \Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$$

isotopy invariant of knot with

$$\Delta_K(t) = \Delta(t^{-1}) \text{ and } \Delta_K(1) = 1$$

$$\chi(HFK(K, M)) = \Delta_K(t)$$

Main philosophy

Alenxander polynomial is
closely related to classification of Dehn surgery.

Theorem (Second Restriction(Kadokami-Yamada))

Suppose that M is a homology sphere.
If $M(K, p) = L(p, q)$, then
there exist h, g such that $\gcd(h, g) = 1$, $hg = \pm 1$, and

$$\Delta_K(t) \doteq \frac{(t^{hg} - 1)(t - 1)}{(t^h - 1)(t^g - 1)} \quad (t^p - 1).$$

Moreover

$$(p, h)$$

is a lens space surgery parameter of $M(K, p) = L(p, q)$.

$(\because)$ Surgery Formula of Reidemeister torsion.

Theorem (Third Restriction(Ozsváth-Szabó))

Suppose that M is an L-space homology sphere.
If $M(K, p) = L(p, q)$, then

$$2g(K) - 1 \leq p$$

($\because$) Heegaard Floer homology.

Theorem (Fourth Restriction(Ozsváth-Szabó))

Suppose that M is an L-space homology sphere.
If $M(K,p) = L(p,q)$, then there exists a sequence

$$0 = n_0 < n_1 < n_2 < \cdots < n_m = d$$

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($\because$) Knot Floer homology.

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First, Second, Third, and Fourth Restrictions are a complete criterion for whether a lens space $L(p, q)$ with paramter (p, k) is a Dehn surgery over an L-space homology sphere or not.

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Examples:
$S^3, \Sigma(2, 3, 5)$ and conected sums of copies of them.
We do not know other L-space homology sphere examples.

Theorem (Main Theorem 2)

If lens space sugery parameter (p, k) *satisfies First, Second Third, and Fourth Restrrictions, then* (p, k) *is realized by a Dehn surgery* $S^3(K, p) = L(p, q)$ *with the lens surgery parameter* (p, k)*,*
or
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In the case of S^3

Greene obtained the same result in the case of S^3 ('10).

This means.

In other words some polynomial is a criterion for whether a lens space can be constructed by a Dehn surgery of a knot or not.

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$$t^{-\frac{(5-1)(7-1)}{2}} \frac{(t^{5\cdot 7}-1)(t-1)}{(t^5-1)(t^7-1)}$$

$$= \frac{1}{t^{12}} - \frac{1}{t^{11}} + \frac{1}{t^7} - \frac{1}{t^6} + \frac{1}{t^5} - \frac{1}{t^4} + \frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2 - t^4 + t^5 - t^6 + t^7 - t^{11} + t^{12}$$

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$$\Delta_{Pr(-2,3,5)}(t) = \frac{1}{t^5} - \frac{1}{t^4} + \frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2 - t^4 + t^5$$

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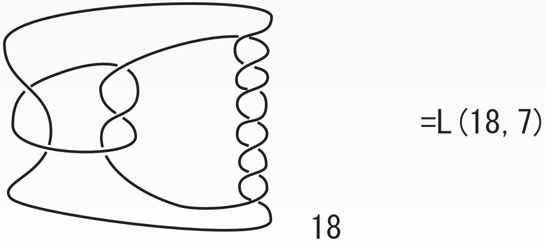

Figure: $L(18,7) = S^3(Pr(-2,3,7), 18)$

$p = 22, k = h = 5, -k^{-1} = g = 9$

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$$L(22,3) = \Sigma(2,3,5)(K,22)$$

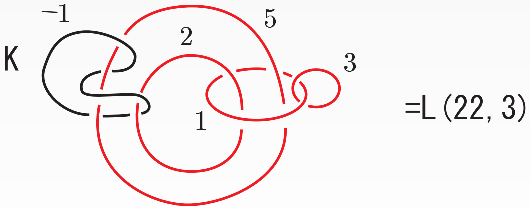

Figure: $\Sigma(2,3,5)(K,22) = L(22,3)$

$$p = 17, k = h = 3, -k^{-1} = g = 11$$

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$$L(17,2)=\Sigma(2,3,11)(K,17)$$

and $\Sigma(2,3,11)$ is a non-L-space.

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Key point

1. Expansion formula Kadokami-Yamada's polynomial.

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1. Take coprime integers (p, k).
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<u>Flow of proof</u>

1. Take coprime integers (p, k).
2. Compute $\Delta_{p,k}$ Kadokami-Yamada's polynomial.
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4. If so, find standard lens space surgery corresponding to (p, k).

Standard Lens space surgery

Definition

Let M be a homology sphere

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Let M be a homology sphere
$M = U_0 \cup_{\Sigma_2} U_1$*: genus 2 Heegaard splitting.*
K *is doubly primitive.*
$\overset{def}{\Leftrightarrow}$

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1. $K \subset \Sigma_2$
2. $[K] \in \pi_1(U_0)$ *and* $\pi_1(U_1)$ *are both primitive elements.*

Standard Lens space surgery

Definition

Let M be a homology sphere
$M = U_0 \cup_{\Sigma_2} U_1$*: genus 2 Heegaard splitting.*
K is doubly primitive.
$\overset{def}{\Leftrightarrow}$

1. $K \subset \Sigma_2$
2. $[K] \in \pi_1(U_0)$ *and* $\pi_1(U_1)$ *are both primitive elements.*

Theorem (Berge)

For any doubly primitive (M, K) there exists (p, k) s.t.

$$M(K, p) = L(p, q).$$

Berge's examples $\subset$ {doubly primitive knots in S^3} (Berge)

type	k	p
(I,II)	$(k, k') = 1$	$k_1 k' = \pm p + 1$
$(\mathrm{III}_{+,\pm})$	$d \mid k+1, \frac{k+1}{d}$: odd	$p = \pm(2k-1)d\ (k^2)$
$(\mathrm{III}_{-,\pm})$	$d \mid k-1, \frac{k-1}{d}$: odd	$p = \pm(2k+1)d\ (k^2)$
$(\mathrm{IV}_{+,\pm})$	$d \mid 2k+1$	$p = \pm(k-1)d\ (k^2)$
$(\mathrm{IV}_{-,\pm})$	$d \mid 2k-1$	$p = \pm(k+1)d\ (k^2)$
$(\mathrm{V}_{+,\pm})$	$d \mid k+1, d$: odd	$p = \pm(k+1)d\ (k^2)$
$(\mathrm{V}_{-,\pm})$	$d \mid k-1, d$: odd	$p = \pm(k-1)d\ (k^2)$
$(\mathrm{VII}_{\pm})$	$k^2 \pm k + 1\ (p)$	
$(\mathrm{VIII}_{\pm})$	$k^2 \pm k - 1\ (p)$	
(IX)	$11j + 2$	$22j^2 + 9j + 1$
(X)	$11j + 3$	$22j^2 + 13j + 2$

Examples $\subset$ {doubly primitive knots in $\Sigma(2,3,5)$} (T.)

	p	k
A_1	$14J^2+7J+1$	$7J+2$
A_2	$20J^2+15J+3$	$5J+2$
B	$30J^2+9J+1$	$6J+1$
C_1	$42J^2+23J+3$	$7J+2$
C_2	$42J^2+47J+13$	$7J+4$
D_1	$52J^2+15J+1$	$13J+2$
D_2	$52J^2+63J+19$	$13J+8$
E_1	$54J^2+15J+1$	$27J+4$
E_2	$54J^2+39J+7$	$27J+10$

F_1	$69J^2+17J+1$	$23J+3$
F_2	$69J^2+29J+3$	$23J+5$
G_1	$85J^2+19J+1$	$17J+2$
G_2	$85J^2+49J+7$	$17J+5$
H_1	$99J^2+35J+3$	$11J+2$
H_2	$99J^2+53J+7$	$11J+3$
I_1	$120J^2+16J+1$	$12J+1$
I_2	$120J^2+20J+1$	$20J+2$
I_3	$120J^2+36J+3$	$12J+2$
J	$120J^2+104J+22$	$12J+5$
K	191	15

Absolutely least remainder

p:positive integer
residue system:

$$\lfloor \frac{-p}{2} \rfloor, \lfloor \frac{-p}{2} \rfloor + 1, \cdots, -1, 0, 1, \cdots, \lfloor \frac{p}{2} \rfloor - 1, \lfloor \frac{p}{2} \rfloor$$

$$[x]_p \text{ : the absolute least remainder of } x$$

ex)
$[23]_{20} = 3$
$[14]_{18} = -4$

Lemma (Expansion Formula(T.))

(p,k): Lens surgery parameter with $0 < k_1 < p$ and $k = k_1 \ (p)$.

$$q_2 = [-k_2^2]_p \qquad k_2 = [k_1^{-1}]_p$$

$$c = \frac{(k_1+1+p)(k_1-1)}{2} \qquad m = \frac{k_1 k_2 - 1}{p} \qquad e = sgn(k_2)$$

$$I_K = \begin{cases} \{1,2,\cdots,K\} & K > 0 \\ \{0,-1,-2,\cdots,K+1\} & K < 0 \end{cases}$$

$$t^{-\frac{(k_1-1)(k_2-1)}{2}} \frac{(t^{k_1 k_2}-1)(t-1)}{(t^{k_1}-1)(t^{k_2}-1)} = \sum_{|i| \leq p/2} a_i t^i$$

Then for any $i \in \mathbb{Z}$

$$a_i = -m + e\#\{j \in \{1,2,\cdots,k_1\} | [-q_2(j + k_1 i + c)]_p \in I_{k_2}\}$$

Continued Fraction

$$k_2^2 = \tau_0 p - q_2,\ p = \tau_1 q_2 - q_3,\ q_2 = \tau_2 q_3 - q_4, \cdots$$

$$, q_{n-1} = \tau_{n-1} q_n - q_{n+1}, q_n = \tau_n q_{n+1} \quad q_{n+1} = \pm 1$$

$$n_i = \det \begin{pmatrix} \tau_1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & \tau_2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \tau_{i-1} & 1 \\ 0 & \cdots & 0 & 0 & 1 & \tau_i \end{pmatrix}$$

$$q_i = q_{n+1} \det \begin{pmatrix} \tau_i & 1 & 0 & 0 & \cdots & 0 \\ 1 & \tau_2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \tau_{i-1} & 1 \\ 0 & \cdots & 0 & 0 & 1 & \tau_n \end{pmatrix}$$

Lemma

If the absolute least reduction $\Delta_{p,k}(t)$ *is flat, then there exist* u, v *s.t.* $2 \leq u < v \leq n$

$$|q_2| > \cdots > |q_{u-1}| > 2|k_2| > |k_2| > |q_{u-1}| > \cdots > |q_v| > \cdots > |q_{n+1}|$$

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5. $p = 191, k = 15$ $\Rightarrow$My (K)