Epimorphisms between knot groups and the images of meridians

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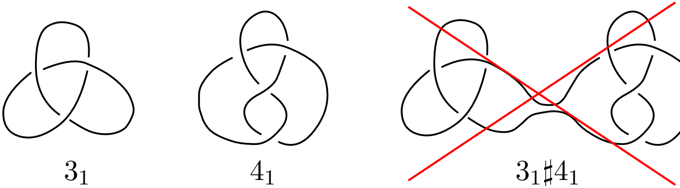
K : a *prime* knot in S^3



3_1 $\quad$ 4_1 $\quad$ $3_1 \sharp 4_1$

$* \in \partial(K \times D^2)$: a base point
$G(K)$: the knot group of K $\quad$ i.e. $G(K) = \pi_1(S^3 - K, *)$

$\mu_K \in G(K)$: meridian of K
i.e. $\mu_K = [\alpha_K]$ s.t. $\alpha_K \sim * \times \partial D^2$, $\quad lk(K, \alpha_K) = 1$

Definition.

K, K' : two prime knots

$$K \geq K' \iff {}^{\exists}\varphi : \ G(K) \twoheadrightarrow G(K')$$

$$K \geq_\mu K' \iff {}^{\exists}\varphi : \ \begin{array}{ccc} G(K) & \twoheadrightarrow & G(K') \\ \mu_K & \longmapsto & \mu_{K'} \end{array}$$

$$K \geq_\mu K' \Longrightarrow K \geq K'$$

Fact.

The relation "$\geq$" is a partial order on the set of prime knots.

- $K \geq K$
- $K \geq K'$, $K' \geq K \Longrightarrow K = K'$
- $K \geq K'$, $K' \geq K'' \Longrightarrow K \geq K''$

The relation "$\geq_\mu$" is also a partial order

Theorem. (Kitano-S. Horie-Kitano-Matsumoto-S.)

The partial order "$\geq_\mu$" on the set of prime knots with up to 11 crossings is given by

$$\left.\begin{array}{l}
8_5, 8_{10}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}, 9_1, 9_6, 9_{16}, 9_{23}, 9_{24}, 9_{28}, 9_{40}, \\
10_5, 10_9, 10_{32}, 10_{40}, 10_{61}, 10_{62}, 10_{63}, 10_{64}, 10_{65}, 10_{66}, 10_{76}, \\
10_{77}, 10_{78}, 10_{82}, 10_{84}, 10_{85}, 10_{87}, 10_{98}, 10_{99}, 10_{103}, 10_{106}, \\
10_{112}, 10_{114}, 10_{139}, 10_{140}, 10_{141}, 10_{142}, 10_{143}, 10_{144}, \\
10_{159}, 10_{164}, \\
11a_{43}, 11a_{44}, 11a_{46}, 11a_{47}, 11a_{57}, 11a_{58}, 11a_{71}, 11a_{72}, 11a_{73}, \\
11a_{100}, 11a_{106}, 11a_{107}, 11a_{108}, 11a_{109}, 11a_{117}, 11a_{134}, 11a_{139}, \\
11a_{157}, 11a_{165}, 11a_{171}, 11a_{175}, 11a_{176}, 11a_{194}, 11a_{196}, 11a_{203}, \\
11a_{212}, 11a_{216}, 11a_{223}, 11a_{231}, 11a_{232}, 11a_{236}, 11a_{244}, 11a_{245}, \\
11a_{261}, 11a_{263}, 11a_{264}, 11a_{286}, 11a_{305}, 11a_{306}, 11a_{318}, 11a_{332}, \\
11a_{338}, 11a_{340}, 11a_{351}, 11a_{352}, 11a_{355}, 11n_{71}, 11n_{72}, 11n_{73}, \\
11n_{74}, 11n_{75}, 11n_{76}, 11n_{77}, 11n_{78}, 11n_{81}, 11n_{85}, 11n_{86}, 11n_{87}, \\
11n_{94}, 11n_{104}, 11n_{105}, 11n_{106}, 11n_{107}, 11n_{136}, 11n_{164}, 11n_{183}, \\
11n_{184}, 11n_{185}
\end{array}\right\} \geq_\mu 3_1$$

$$\left.\begin{array}{l} 8_{18}, 9_{37}, 9_{40}, \\ 10_{58}, 10_{59}, 10_{60}, 10_{122}, 10_{136}, 10_{137}, 10_{138}, \\ 11a_{5}, 11a_{6}, 11a_{51}, 11a_{132}, 11a_{239}, 11a_{297}, 11a_{348}, 11a_{349}, \\ 11n_{100}, 11n_{148}, 11n_{157}, 11n_{165} \end{array}\right\} \geq_{\mu} 4_1$$

$$11n_{78}, 11n_{148} \geq_{\mu} 5_1$$

$$10_{74}, 10_{120}, 10_{122}, 11n_{71}, 11n_{185} \geq_{\mu} 5_2$$

$$11a_{352} \geq_{\mu} 6_1$$

$$11a_{351} \geq_{\mu} 6_2$$

$$11a_{47}, 11a_{239} \geq_{\mu} 6_3$$

To determine the partial order $\geq_\mu$ on the set of prime knots

For each pair of two prime knots K, K',
determine whether there exists an epimorphism

$$\begin{array}{cccc} \varphi : & G(K) & \twoheadrightarrow & G(K') \\ & \mu_K & \longmapsto & \mu_{K'} \end{array}$$

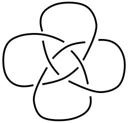
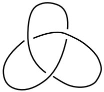
which preserves meridians.

The number of prime knots with up to 11 crossings is $\mathbf{801}$.
Then the number of cases to consider is ${}_{801}\mathrm{P}_2 = \mathbf{640,800}$.

the existence of an epimorphism which preserves meridians

We can construct such an epimorphism explicitly

Example. $8_{18} \geq_{\mu} 3_1$?

$8_{18} =$  , $3_1 =$ 

$$G(8_{18}) = \left\langle \begin{array}{l} x_1, x_2, x_3, \\ x_4, x_5, x_6, \\ x_7, x_8 \end{array} \middle| \begin{array}{l} x_4x_1\bar{x}_4\bar{x}_2, x_5x_3\bar{x}_5\bar{x}_2, x_6x_3\bar{x}_6\bar{x}_4, \\ x_7x_5\bar{x}_7\bar{x}_4, x_8x_5\bar{x}_8\bar{x}_6, x_1x_7\bar{x}_1\bar{x}_6, \\ x_2x_7\bar{x}_2\bar{x}_8 \end{array} \right\rangle$$

$$G(3_1) = \langle y_1, y_2, y_3 \,|\, y_3y_1\bar{y}_3\bar{y}_2,\ y_1y_2\bar{y}_1\bar{y}_3 \rangle$$

Example. $8_{18} \geq_\mu 3_1$?

$$G(8_{18}) = \left\langle \begin{array}{l|l} x_1, x_2, x_3, & x_4x_1\bar{x}_4\bar{x}_2, x_5x_3\bar{x}_5\bar{x}_2, x_6x_3\bar{x}_6\bar{x}_4, \\ x_4, x_5, x_6, & x_7x_5\bar{x}_7\bar{x}_4, x_8x_5\bar{x}_8\bar{x}_6, x_1x_7\bar{x}_1\bar{x}_6, \\ x_7, x_8 & x_2x_7\bar{x}_2\bar{x}_8 \end{array} \right\rangle$$

$$G(3_1) = \langle y_1, y_2, y_3 \mid y_3y_1\bar{y}_3\bar{y}_2,\ y_1y_2\bar{y}_1\bar{y}_3 \rangle$$

$$\varphi : G(8_{18}) \longrightarrow G(3_1)$$

$$\begin{array}{llll} \varphi(x_1) = y_1, & \varphi(x_2) = y_2, & \varphi(x_3) = y_1, & \varphi(x_4) = y_3, \\ \varphi(x_5) = y_3, & \varphi(x_6) = y_1y_3\bar{y}_1, & \varphi(x_7) = y_3, & \varphi(x_8) = y_1 \end{array}$$

$$\begin{array}{cccc} \varphi : & G(8_{18}) & \twoheadrightarrow & G(3_1) \\ & \varphi(\mu_{8_{11}}) & \longmapsto & \mu_{3_1} \end{array}$$

$$8_{18} \geq_\mu 3_1$$

the non-existence of an epimorphism which preserves meridians

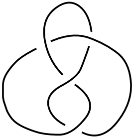
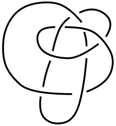
(1) By the (classical) Alexander polynomial

K : a knot
Δ_K : the Alexander polynomial of K

Fact.

K, K' : two knots
If Δ_K can not be divided by $\Delta_{K'}$,
$\Longrightarrow$ there exists no epimorphism $G(K) \longrightarrow\!\!\!\!\!\rightarrow G(K')$.

Example. $4_1 \geq_\mu 8_{21}$?

$4_1 =$  , $8_{21} =$ 

$\Delta_{4_1} = t^2 - 3t + 1, \qquad \Delta_{8_{21}} = t^4 - 4t^3 + 5t^2 - 4t + 1$

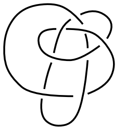
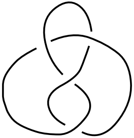
$$\frac{\Delta_{4_1}}{\Delta_{8_{21}}} = \frac{t^2 - 3t + 1}{t^4 - 4t^3 + 5t^2 - 4t + 1}$$

$\Delta_{8_{21}}$ can not divide Δ_{4_1}

$$4_1 \not\geq 8_{21}$$

$$4_1 \not\geq_\mu 8_{21}$$

Example. $8_{21} \geq_\mu 4_1$?

$8_{21} =$  , $4_1 =$ 

$$\Delta_{8_{21}} = t^4 - 4t^3 + 5t^2 - 4t + 1, \qquad \Delta_{4_1} = t^2 - 3t + 1$$

$$\frac{\Delta_{8_{21}}}{\Delta_{4_1}} = \frac{t^4 - 4t^3 + 5t^2 - 4t + 1}{t^2 - 3t + 1} = t^2 - t + 1$$

Δ_{4_1} can divide $\Delta_{8_{21}}$!

We cannot determine the existence of an epimorphism from $G(8_{21})$ onto $G(4_1)$ by the Alexander polynomial.

the non-existence of an epimorphism which preserves meridians

(1) By the (classical) Alexander polynomial
(2) By the twisted Alexander polynomial

Fix Wirtinger presentation of $G(K)$
$\Delta_{K,\rho}$: the twisted Alexander polynomial of K
$\Delta^N_{K,\rho}, \Delta^D_{K,\rho}$: the numerator and denominator of $\Delta_{K,\rho}$

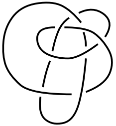
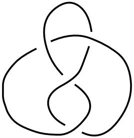
Theorem. (Kitano-S.-Wada)

If there exists a representation $\rho' : G(K') \to SL(2; \mathbb{Z}/p\mathbb{Z})$ such that

$$\Delta^N_{K,\rho} \text{ is not divisible by } \Delta^N_{K',\rho'} \text{ or } \Delta^D_{K,\rho} \neq \Delta^D_{K',\rho'}$$

for any representation $\rho : G(K) \to SL(2; \mathbb{Z}/p\mathbb{Z})$,
then there exists no epimorphism $G(K) \longrightarrow\!\!\!\!\!\rightarrow G(K')$ which preseves meridians.

Example. $8_{21} \geq_\mu 4_1$?

$8_{21} =$  , $4_1 =$ 

$\Delta_{8_{21}} = t^4 - 4t^3 + 5t^2 - 4t + 1$, $\Delta_{4_1} = t^2 - 3t + 1$

$$\frac{\Delta_{8_{21}}}{\Delta_{4_1}} = \frac{t^4 - 4t^3 + 5t^2 - 4t + 1}{t^2 - 3t + 1} = t^2 - t + 1$$

Δ_{4_1} divides $\Delta_{8_{21}}$!

We cannot determine the existence of an epimorphism from $G(8_{21})$ onto $G(4_1)$ by the Alexander polynomial.

For a certain representation $\rho' : G(4_1) \longrightarrow SL(2;\mathbb{Z}/3\mathbb{Z})$,

$$\Delta^N_{4_1,\rho'} = t^4 + t^2 + 1, \qquad \Delta^D_{4_1,\rho'} = t^2 + t + 1$$

Table of the twisted Alexander polynomials of $G(8_{21})$ for all representations $\rho : G(8_{21}) \longrightarrow SL(2;\mathbb{Z}/3\mathbb{Z})$

	$\Delta^N_{8_{21},\rho_i}$	$\Delta^D_{8_{21},\rho_i}$
ρ_1	$t^8 + t^4 + 1$	$t^2 + 1$
ρ_2	$t^8 + t^7 + 2t^6 + 2t^4 + 2t^2 + t + 1$	$t^2 + t + 1$
ρ_3	$t^8 + t^7 + 2t^6 + 2t^4 + 2t^2 + t + 1$	$t^2 + 2t + 1$
ρ_4	$t^8 + 2t^7 + 2t^6 + 2t^4 + 2t^2 + 2t + 1$	$t^2 + t + 1$
ρ_5	$t^8 + 2t^7 + 2t^6 + 2t^4 + 2t^2 + 2t + 1$	$t^2 + 2t + 1$

$$8_{21} \not\geq_\mu 4_1$$

To determine the partial order $\geq_\mu$ on the set of prime knots

For each pair of two prime knots K, K',
determine whether there exists an epimorphism

$$\begin{array}{rccc} \varphi: & G(K) & \longrightarrow\!\!\!\!\to & G(K') \\ & \mu_K & \longmapsto & \mu_{K'} \end{array}$$

which preserves meridians.

The number of prime knots with up to 11 crossings is $\mathbf{801}$.

Then the number of cases to consider is ${}_{801}\mathrm{P}_2 = \mathbf{640,800}$.

$\mathbf{146}$ cases: existence of an epimorphism
$\mathbf{637,501}$ cases : non-existence by the Alexander polynomial
$\mathbf{3,153}$ cases : non-existence by the twisted Alexander poly.

Definition.

K, K' : two prime knots

$$K \geq K' \iff {}^{\exists}\varphi : \ G(K) \twoheadrightarrow G(K')$$

$$K \geq_\mu K' \iff {}^{\exists}\varphi : \begin{array}{ccc} G(K) & \twoheadrightarrow & G(K') \\ \mu_K & \longmapsto & \mu_{K'} \end{array}$$

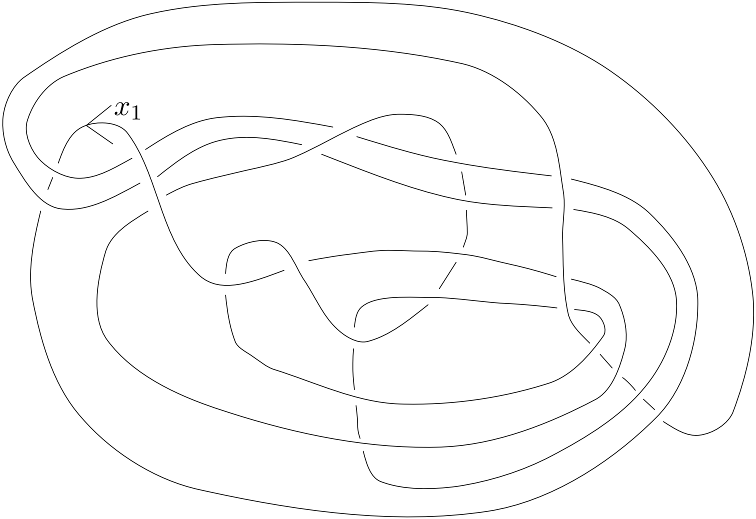
$$K \geq_\mu K' \Longrightarrow K \geq K'$$

Problem.

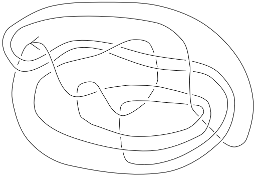
$$K \geq K' \overset{?}{\Longrightarrow} K \geq_\mu K'$$

Problem.

Does there exist an epimorphism between knot groups which does not preserve meridians?

$K =$ 

Wirtinger presentation of $G(K)$



generators : $x_1, x_2, \ldots, x_{24}$
relators :

$$
\begin{array}{llll}
x_6x_2\bar{x}_6\bar{x}_1, & x_{10}x_2\bar{x}_{10}\bar{x}_3, & x_6x_3\bar{x}_6\bar{x}_4, & x_{22}x_4\bar{x}_{22}\bar{x}_5, \\
x_1x_6\bar{x}_1\bar{x}_5, & x_{17}x_7\bar{x}_{17}\bar{x}_6, & x_{23}x_7\bar{x}_{23}\bar{x}_8, & x_{13}x_9\bar{x}_{13}\bar{x}_8, \\
x_3x_9\bar{x}_3\bar{x}_{10}, & x_1x_{10}\bar{x}_1\bar{x}_{11}, & x_{22}x_{12}\bar{x}_{22}\bar{x}_{11}, & x_6x_{13}\bar{x}_6\bar{x}_{12}, \\
x_{23}x_{14}\bar{x}_{23}\bar{x}_{13}, & x_{17}x_{14}\bar{x}_{17}\bar{x}_{15}, & x_{18}x_{16}\bar{x}_{18}\bar{x}_{15}, & x_6x_{17}\bar{x}_6\bar{x}_{16}, \\
x_1x_{17}\bar{x}_1\bar{x}_{18}, & x_{16}x_{19}\bar{x}_{16}\bar{x}_{18}, & x_{24}x_{19}\bar{x}_{24}\bar{x}_{20}, & x_{12}x_{21}\bar{x}_{12}\bar{x}_{20}, \\
x_4x_{21}\bar{x}_4\bar{x}_{22}, & x_1x_{23}\bar{x}_1\bar{x}_{22}, & x_6x_{23}\bar{x}_6\bar{x}_{24}, & x_{18}x_{24}\bar{x}_{18}\bar{x}_1
\end{array}
$$

where $\bar{x}_i = {x_i}^{-1}$.

Wirtinger presentation of $G(3_1)$:

$$G(3_1) = \langle y_1, y_2 \,|\, y_1 y_2 y_1 = y_2 y_1 y_2 \rangle = \langle 1, 2 \,|\, 121 = 212 \rangle$$

We define a map $f : G(K) \to G(3_1)$ as follows:

$$f(x_1) = 12\bar{1}2\bar{1},$$
$$f(x_2) = 1\bar{2}1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}2\bar{1},$$
$$f(x_3) = 12\bar{1}2\bar{1},$$
$$f(x_4) = 1\bar{2}12\bar{1}2\bar{1}21\bar{2}\bar{1}2\bar{1},$$
$$f(x_5) = 212\bar{1}2\bar{1}21\bar{2}\bar{1}\bar{2},$$
$$f(x_6) = 1\bar{2}12\bar{1}2\bar{1}2\bar{1},$$
$$f(x_7) = 1\bar{2}1\bar{2}\bar{2}\bar{2}122\bar{1}\bar{1}222\bar{1}2\bar{1},$$
$$f(x_8) = 1\bar{2}12\bar{1}2\bar{1}2\bar{1},$$
$$f(x_9) = 1\bar{2}1\bar{2}\bar{2}12\bar{1}2\bar{1}22\bar{1}2\bar{1},$$
$$f(x_{10}) = 1\bar{2}12\bar{1}2\bar{1}2\bar{1},$$
$$f(x_{11}) = 122\bar{1}\bar{1},$$
$$f(x_{12}) = 1\bar{2}\bar{2}122\bar{1}\bar{1}22\bar{1},$$
$$f(x_{13}) = 12\bar{1}2\bar{1},$$
$$f(x_{14}) = 1\bar{2}1\bar{2}\bar{2}\bar{2}12\bar{1}2\bar{1}222\bar{1}2\bar{1},$$
$$f(x_{15}) = 12\bar{1}2\bar{1},$$
$$f(x_{16}) = 1\bar{2}\bar{2}12\bar{1}2\bar{1}22\bar{1},$$
$$f(x_{17}) = 1\bar{2}1\bar{2}1\bar{2}\bar{1}\bar{2}12\bar{1}2\bar{1}212\bar{1}2\bar{1}2\bar{1},$$
$$f(x_{18}) = 22\bar{1},$$
$$f(x_{19}) = 1\bar{2}\bar{2}1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}22\bar{1},$$
$$f(x_{20}) = 22\bar{1},$$
$$f(x_{21}) = 1\bar{2}12\bar{1}\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1}212\bar{1}2\bar{1}21\bar{2}\bar{1}2\bar{1},$$
$$f(x_{22}) = 22\bar{1},$$
$$f(x_{23}) = 1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1},$$
$$f(x_{24}) = 1\bar{2}12\bar{1}\bar{1}221\bar{2}\bar{1}2\bar{1}.$$

Theorem.

The above mapping $f : G(K) \to G(3_1)$ is an epimorphism which does not map a meridian of K to a meridian of 3_1. Moreover, there does not exist an epimorphism from $G(K)$ onto $G(3_1)$ which preserves meridians.

To prove the above theorem

(1) f is a group homomorphism.
(2) f is surjective.
(3) f does not map a meridian of K to a meridian of 3_1.
(4) There does not exist an epimorphism which preserves meridians.

Corollary.

$$K \geq K' \not\Rightarrow K \geq_\mu K'$$

(1) f is a group homomorphism.

The relators of $G(K)$ vanish under the mapping f.

$$
\begin{aligned}
&f(x_6x_2\bar{x}_6\bar{x}_1)\\
&\quad= 1\bar{2}12\bar{1}2\bar{1}2\bar{1}\cdot 1\bar{2}1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}2\bar{1}\cdot 1\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1}\cdot 1\bar{2}1\bar{2}\bar{1}\\
&\quad= e\\
&f(x_{10}x_2\bar{x}_{10}\bar{x}_3)\\
&\quad= 1\bar{2}12\bar{1}2\bar{1}2\bar{1}\cdot 1\bar{2}1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}2\bar{1}\cdot 1\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1}\cdot 1\bar{2}1\bar{2}\bar{1}\\
&\quad= e\\
&f(x_6x_3\bar{x}_6\bar{x}_4)\\
&\quad= 1\bar{2}12\bar{1}2\bar{1}2\bar{1}\cdot 12\bar{1}2\bar{1}\cdot 1\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1}\cdot 1\bar{2}12\bar{1}\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1}\\
&\quad= e\\
&\qquad\cdots
\end{aligned}
$$

(2) f is surjective.

We find elements of $G(K)$ which are mapped to 1 and 2.

$$
\begin{aligned}
& f(x_{18}x_6\bar{x}_1\bar{x}_1x_{18}x_6\bar{x}_1) \\
&\quad = f(x_{18})f(x_6)\overline{f(x_1)}\,\overline{f(x_1)}f(x_{18})f(x_6)\overline{f(x_1)} \\
&\quad = 22\bar{1}\cdot 1\bar{2}12\bar{1}2\bar{1}2\bar{1}\cdot 1\bar{2}1\bar{2}\bar{1}\cdot 1\bar{2}1\bar{2}\bar{1}\cdot 22\bar{1}\cdot 1\bar{2}12\bar{1}2\bar{1}2\bar{1}\cdot 1\bar{2}1\bar{2}\bar{1} \\
&\quad = 212\bar{1}\bar{2}1\bar{2}\bar{1}2122\bar{1}\bar{1} = 121\bar{1}\bar{2}1\bar{2}\bar{1}121\bar{1}\bar{1} = 1, \\
& f(x_1\bar{x}_6\bar{x}_{18}x_1x_{18}x_6\bar{x}_1\bar{x}_1x_{18}x_6\bar{x}_1) \\
&\quad = \cdots \\
&\quad = 2
\end{aligned}
$$

(3) f does not map a meridian of K to a meridian of 3_1.

Check the image of a meridian of K.

x_1 : meridian of K
1 : meridian of 3_1 $\Longrightarrow$ $f(x_1) \not\sim 1$?

$$\begin{array}{rccc} \rho: & G(3_1) & \longrightarrow & SL(2;\mathbb{Z}) \\ & 1 & \longmapsto & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ & 2 & \longmapsto & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \end{array}$$

$$\rho(f(x_1)) = \rho(12\bar{1}2\bar{1}) = \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}$$

$$\mathrm{tr}\ \rho(f(x_1)) \neq \mathrm{tr}\ \rho(1) \quad \Longrightarrow \quad f(x_1) \not\sim 1$$

(4) There does not exist an epimorphism which preserves meridians.

By making use of the twisted Alexander polynomial.

Fix Wirtinger representations of the knot groups.
$\Delta_{K,\rho}$: the twisted Alexander polynomial of K
$\Delta^N_{K,\rho}, \Delta^D_{K,\rho}$: the numerator and denominator of $\Delta_{K,\rho}$

Theorem. (Kitano-S.-Wada)

If there exists a representation $\rho' : G(K') \to SL(2;\mathbb{Z}/p\mathbb{Z})$ such that

$$\Delta^N_{K,\rho} \text{ is not divisible by } \Delta^N_{K',\rho'} \text{ or } \Delta^D_{K,\rho} \neq \Delta^D_{K',\rho'}$$

for any representation $\rho : G(K) \to SL(2;\mathbb{Z}/p\mathbb{Z})$,
then there exists no epimorphism $G(K) \twoheadrightarrow G(K')$ which preseves meridians.

For a certain representation $\rho' : G(3_1) \longrightarrow SL(2;\mathbb{Z}/5\mathbb{Z})$,

$$\Delta^N_{3_1,\rho'} = t^4 + 2t^3 + 2t^2 + 2t + 1, \qquad \Delta^D_{3_1,\rho'} = t^2 + 2t + 1$$

Table of the twisted Alexander polynomials of $G(K)$
for all representations $\rho : G(K) \longrightarrow SL(2;\mathbb{Z}/5\mathbb{Z})$

	Δ^N_{K,ρ_i}	Δ^D_{K,ρ_i}
ρ_1	$t^8 + 3t^6 + 3t^4 + 3t^2 + 1$	$t^2 + 1$
ρ_2	$t^8 + 4t^6 + t^4 + 4t^2 + 1$	$t^2 + 1$
ρ_3	$t^8 + t^7 + 4t^5 + 4t^4 + 4t^3 + t + 1$	$t^2 + 3t + 1$
ρ_4	$t^8 + 2t^7 + t^5 + 3t^4 + t^3 + 2t + 1$	$t^2 + t + 1$
ρ_5	$t^8 + 2t^7 + t^6 + 2t^5 + 4t^4 + 2t^3 + t^2 + 2t + 1$	$t^2 + t + 1$
ρ_6	$t^8 + 3t^7 + 4t^5 + 3t^4 + 4t^3 + 3t + 1$	$t^2 + 4t + 1$
ρ_7	$t^8 + 3t^7 + t^6 + 3t^5 + 4t^4 + 3t^3 + t^2 + 3t + 1$	$t^2 + 4t + 1$
ρ_8	$t^8 + 4t^7 + t^5 + 4t^4 + t^3 + 4t + 1$	$t^2 + 2t + 1$

$$K \not\geq_\mu 3_1$$

Theorem. (Kitano-S.-Wada)

If there exists a representation $\rho' : G(K') \to SL(2; \mathbb{Z}/p\mathbb{Z})$ such that

$$\Delta_{K,\rho} \text{ is not divisible by } \Delta_{K',\rho'}$$

for any representation $\rho : G(K) \to SL(2; \mathbb{Z}/p\mathbb{Z})$,
then there exists no epimorphism $G(K) \twoheadrightarrow G(K')$.

Problem.

Is the converse true?
If there exists no epimorphism $G(K) \twoheadrightarrow G(K')$, then does there exist a prime number p and $\rho' : G(K') \to SL(2; \mathbb{Z}/p\mathbb{Z})$ such that

$$\Delta_{K,\rho} \text{ is not divisible by } \Delta_{K',\rho'}$$

for any $\rho : G(K) \to SL(2; \mathbb{Z}/p\mathbb{Z})$?