# Non-commutative Reidemeister torsion, higher-order Alexander polynomials and circle valued Morse theory

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# 0. Introduction

(Milnor) For a fibered knot $K$,

$$\frac{\Delta_K(t)}{t-1} = \text{Lefschetz } \zeta := \exp\left(\sum_{k=1}^{\infty} \frac{\sharp\mathrm{Fix}(\varphi^k)}{k} t^k\right), \quad \varphi : \text{monodromy.}$$

(Fried) For a hyperbolic knot $K$,

$$\left(\frac{\Delta_K(t)}{t-1}\right)^2 = \text{Ruelle } \zeta(0), \quad \zeta(s) := \prod_{\gamma \in \mathrm{Conj}(\pi_1)} (1 - te^{-s\cdot \mathrm{length}(\gamma)})^{-1}.$$

Here

$$\frac{\Delta_K(t)}{t-1} = \text{(abelian) Reidemeister torsion} = \tau_{ab}.$$

# Background

$X$ : closed oriented Riemannian manifold

[Hutchings-Lee '99, Pajitnov '99] $f : X \to S^1$ : "generic" smooth map

$$\tau_{ab}(X) = \zeta_f \cdot \tau_{ab}^{Nov}(f).$$

(closed gradient flows $\rightsquigarrow$ $\zeta_f$, $\mathrm{Crit}(f) \rightsquigarrow \tau_{ab}^{Nov}(f)$.)

[Cochran '04, Harvey '05, Turaev (unpublished)] $\psi \in H^1(X)$
Higher-order Alexander polynomials $\Delta_\psi^{(n)}(t)$ introduced
(non-commutative coefficients)

[Friedl '07] $\dim X = 3$

$$\Delta_\psi^{(n)}(t) \text{ “=” } \tau_{\rho_n}(X), \quad \rho_n : \pi_1 X \to \pi_1 X / (\pi_1 X)_r^{(n+1)}.$$

# Results and related topics

[K. '10] a generalization to non-commutative coefficients

$$\tau_{\rho_n}(X) \;=\; \zeta_{f,\rho_n} \cdot \tau^{Nov}_{\rho_n}(f), \quad \underline{\rho_n \colon \pi_1 X \to \pi_1 X / (\pi_1 X)^{(n+1)}_r}.$$

[Goda-Pajitnov '09]

$$\tau_{\rho}(X) \;=\; \zeta_{f,\rho} \cdot \tau^{Nov}_{\rho}(f), \quad \underline{\rho \colon \pi_1 X \to GL(n, \mathbb{Z})}.$$

(Does $\tau_{\rho_n}(X)$ detect fiberedness of $3$-manifolds?
How about $L^2$-torsion?)

cf.
[Mazur-Wiles '84] (Iwasawa main conjecture (in number theory))

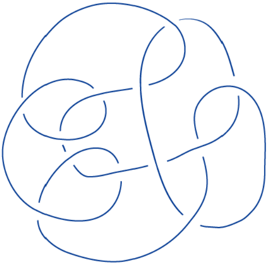
Iwasawa polynomial $I_p \sim p$-adic zeta function $\zeta_p$.

# Outline

1. Non-commutative torsion invariants
2. Circle-valued Morse theoretical presentation
3. Problems

# 1. Non-commutative torsion invariants

Is the knot $K = 12_{n57}$ $(g(K) = 2)$ fibered or not?



The answer (also for any prime knot with $12$ crossings) was given first by using twisted Alexander polynomials [Friedl-Kim ’06] and later in the study of homology cylinders [Goda-Sakasai ’11].

$$\pi_1 = \langle x, a, b \mid r_1, r_2\rangle,$$
$$r_1 =$$
$$b^{-1}axa^{-1}bx^2ba^{-1}xab^{-1}x^{-2}b^{-1}ax^{-1}a^{-1}bx^2bx^{-1}a^{-1}x^{-1}bx^2ba^{-1}x^{-1}ab^{-1}x^{-2},$$
$$r_2 = b^{-1}axa^{-1}xab^{-1}x^{-1}bx^{-1}.$$

$$\rho_0\colon \pi_1 \twoheadrightarrow \pi_1/\pi_1' = \langle t\rangle \ \rightsquigarrow\ \rho_0\colon \mathbb{Z}[\pi_1] \to \mathbb{Z}[t^{\pm 1}]$$

$$\rho_0(\tfrac{\partial r_1}{\partial a}) = t^4 - 2t^3 + t + 1$$
$$\rho_0(\tfrac{\partial r_1}{\partial b}) = -t^4 + 3t^3 - 2t^2 + 3t - 1,$$
$$\rho_0(\tfrac{\partial r_2}{\partial a}) = t^2 - t + 1,$$
$$\rho_0(\tfrac{\partial r_2}{\partial b}) = -t^2 + t - 1.$$

$$\Delta_K(t) = \det\begin{pmatrix} \rho_0(\overline{\tfrac{\partial r_1}{\partial a}}) & \rho_0(\overline{\tfrac{\partial r_2}{\partial a}}) \\ \rho_0(\tfrac{\partial r_1}{\partial b}) & \rho_0(\tfrac{\partial r_2}{\partial b}) \end{pmatrix} = t^4 - 2t^3 + 3t^2 - 2t + 1.$$

Neubirth's fibering obstruction ($\Delta_K$ : monic & $\deg \Delta_K = 2g(K)$) is satisfied.

Let us try metabelian computations!

$\rho_1 : \pi_1 \twoheadrightarrow \pi_1/\pi_1'' = \pi_1'/\pi_1'' \rtimes \langle t \rangle = \langle a_0, a_1, b_0, b_1 \rangle_{ab} \rtimes \langle t \rangle$

$\rightsquigarrow\ \rho_1 : \mathbb{Z}[\pi_1] \to \mathbb{Z}[a_0^{\pm 1}, a_1^{\pm 1}, b_0^{\pm 1}, b_1^{\pm 1}; t^{\pm 1}]$.

$$\rho_1(\tfrac{\partial r_1}{\partial a}) = a_0^2 a_1^{-1} b_0^{-2} b_1 t^4 - (a_0^2 a_1^{-1} b_0^{-2} b_1 + a_0 a_1^{-1} b_0^{-1} b_1) t^3 + (a_0^2 a_1^{-4} b_0^{-1} b_1^3 - a_1^{-1} + a_0 a_1^{-1} b_0^{-1} b_1) t + b_0^{-1}$$

$$\rho_1(\tfrac{\partial r_1}{\partial b}) = -a_0^2 a_1^{-2} b_0^{-2} b_1^2 t^4 + (a_0 a_1^{-1} b_0^{-1} b_1 + a_0 a_1^{-2} b_0^{-1} b_1^2 + a_1^{-1} b_1) t^3 - (a_0^2 a_1^{-4} b_0^{-1} b_1^3 + 1) t^2 + (a_0 a_1^{-1} b_0^{-1} + a_0 a_1^{-2} b_0^{-1} b_1 + a_1^{-1}) t - b_0^{-1},$$

$$\rho_1(\tfrac{\partial r_2}{\partial a}) = a_0 a_1^{-1} b_0^{-1} t^2 - a_0 a_1^{-1} b_0^{-1} t + b_0^{-1},$$

$$\rho_1(\tfrac{\partial r_2}{\partial b}) = -b_1^{-1} t^2 + b_1^{-1} t - b_0^{-1}.$$

$$\det \begin{pmatrix} \rho_1(\frac{\partial r_1}{\partial a}) & \rho_1(\frac{\partial r_2}{\partial a}) \\ \rho_1(\frac{\partial r_1}{\partial b}) & \rho_1(\frac{\partial r_2}{\partial b}) \end{pmatrix} = \rho_1(\tfrac{\partial r_1}{\partial a})\rho_1(\tfrac{\partial r_2}{\partial b}) - \underline{\rho_1(\tfrac{\partial r_1}{\partial a})\rho_1(\tfrac{\partial r_2}{\partial a})\rho_1(\tfrac{\partial r_1}{\partial a})^{-1}\rho_1(\tfrac{\partial r_1}{\partial b})}$$

$$= (a_0 b_1 + a_1 b_0 - b_0 b_1) t^4 + (\dots) t^3 + (\dots) t^2 + (\dots) t + \dots.$$

A non-monic polynomial appears! $\rightsquigarrow$ $12_{n57}$ : non-fibered!

# Higher-order Alexander polynomials

$\psi \in H^1(X)$ $(= \mathrm{Hom}(\pi, \langle t \rangle))$ : surjective, $\pi := \pi_1 X$

$\pi_r^{(0)} := \pi$
$\pi_r^{(n+1)} := \mathrm{Ker}(\pi_r^{(n)} \to H_1(\pi_r^{(n)})/$ torsion) : rational derived series

$\Gamma_n := \pi / \pi_r^{(n+1)}$ : torsion-free solvable group

**Fact.**

$\mathbb{Z}[\Gamma_n]$ has the quotient skew field $\mathcal{K}_n(t) := \mathbb{Z}[\Gamma_n](\mathbb{Z}[\Gamma_n] \setminus 0)^{-1}$

$(\Gamma_n = \mathrm{Ker}\,\psi \rtimes \langle t \rangle)$

$H_1(X; \mathcal{K}_n[t^{\pm 1}]) \cong \oplus_i \mathcal{K}_n[t^{\pm 1}]/p_i(t)\mathcal{K}_n[t^{\pm 1}]$ ($\mathcal{K}_n[t^{\pm 1}]$ : PID)
$\rightsquigarrow$ $\Delta_\psi^{(n)}(t) := \prod_i p_i(t) \in (\mathcal{K}_n(t)^\times_{ab} / \langle a t^l \rangle^{a \in \mathcal{K}_n^\times}_{l \in \mathbb{Z}}) \cup 0.$ ($\Delta_\psi^{(0)}(t)$ : classical).
$\rightsquigarrow$ $\delta_n(\psi) := \deg \Delta_\psi^{(n)} \in \mathbb{Z}$ : Cochran-Harvey invariant

# Non-commutative Reidemeister torsion

$\rho_n \colon \mathbb{Z}[\pi] \to \mathbb{Z}[\Gamma_n] \to \mathcal{K}_n(t)$
($\det \colon GL(n, \mathcal{K}_n(t)) \to K_1(\mathcal{K}_n(t)) = \mathcal{K}_n(t)^\times_{ab}$ : Dieudonné determinant)

If $H_*(X; \mathcal{K}_n(t)) = 0$, then we define the Reidemeister torsion associated to $\rho_n$: $\tau_{\rho_n}(X) \in \mathcal{K}_n(t)^\times_{ab} / \pm \Gamma_n$. ($\tau_{\rho_0}(X)$ "=" $\tau_{ab}(X)$)
Otherwise we set $\tau_{\rho_n}(X) = 0$.

## Lemma.

$C_i(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} \mathcal{K}_n(t) = C'_i \oplus C''_i$ s. t. $pr_{C''_{i-1}} \circ \partial_i \colon C'_i \to C''_{i-1}$ : isomorphism

$$\Rightarrow \; \tau_{\rho_n}(X) = \prod_i (\det pr_{C''_{i-1}} \circ \partial_i)^{(-1)^i}.$$

[Friedl '07]
If $\dim X = 3$ and $\rho_n(\pi)$ is not cyclic, then $\Delta^{(n)}_\psi(t) = \tau_{\rho_n}(X)$.

# The leading coefficient

$$
\begin{aligned}
c_n : \mathcal{K}_n(t)^{\times}_{ab}/\pm\Gamma_n &\to (\mathcal{K}_n)^{\times}_{ab}/\pm \operatorname{Ker}\psi \cdot \langle t_*(p)p^{-1}\rangle_{p\in\mathbb{Z}[\operatorname{Ker}\psi]\setminus 0} \\
&: (a_l t^l + a_{l-1}t^{l-1} + \dots)(b_m t^m + b_{m-1}t^{m-1} + \dots)^{-1} \mapsto a_l b_m^{-1}
\end{aligned}
$$

**Lemma.**

The map $c_n$ is a well-defined homomorphism.

If $H_*^{\rho_n}(X;\mathcal{K}_n(t)) = 0$, then we set

$$
c_n(\psi) := c(\tau_{\rho_n}(X)) \ \in (\mathcal{K}_n)^{\times}_{ab}/\pm \operatorname{Ker}\psi \cdot \langle t_*(p)p^{-1}\rangle_{p\in\mathbb{Z}[\operatorname{Ker}\psi]\setminus 0}.
$$

Later we will see monicness of $\tau_{\rho_n}(X)$ for a fibered manifold over a circle as a corollary of the main theorem.

# 2. Circle-valued Morse theoretical presentation

$f : X \to S^1$ : Morse-Smale map, i. e.,
$\forall p, q \in \mathrm{Crit}\, f$, unstable manifold $\mathcal{D}(p)$ ⋔ stable manifold $\mathcal{A}(q)$.

$\mathcal{O} := \{o : S^1 \to X\ ;\ \frac{do}{ds} = -\nabla f\}/U(1)$ : nondegenerate, i. e.,
$\det(id - (d\phi)_x : T_x X/T_x o(S^1) \to T_x X/T_x o(S^1)) \neq 0,\ \phi$ : return map.

$t \in \pi_1 S^1$ : "downward" generator
Assume $f_* : \pi_1 X \to \langle t \rangle$ is surjective.

$\mathbb{Z}[\pi] = \mathbb{Z}[\mathrm{Ker}\, f_*][t^{\pm 1}] \subset \mathbb{Z}[\mathrm{Ker}\, f_*]((t))\ (:= \mathbb{Z}[\mathrm{Ker}\, f_*][[t]][t^{-1}]) =: \Lambda$
$\mathbb{Z}[H_1(X)] \subset \Lambda_{ab}$ : Novikov completion

e. g., $X = S^1,\ f = id \ \Rightarrow\ \Lambda = \Lambda_{ab} = \mathbb{Z}((t))$.

# Novikov torsion

Choose a lift $\tilde{p} \in \widetilde{X}$ for each $p \in \mathrm{Crit} f$.

$C_i^{Nov}(f) := \sum_{p \text{ of index } i} \tilde{p} \cdot \Lambda$ : Novikov complex
$\partial_i^f : C_i^{Nov}(f) \to C_{i-1}^{Nov}(f)$: $\tilde{p} \cdot \gamma \mapsto \sum_{q \text{ of index } i-1, \gamma'} n(\tilde{p} \cdot \gamma, \tilde{q} \cdot \gamma')\tilde{q} \cdot \gamma'$,
$n(\tilde{p} \cdot \gamma, \tilde{q} \cdot \gamma') :=$ signed number of flows from $\tilde{p} \cdot \gamma$ to $\tilde{q} \cdot \gamma'$.

**Theorem (Pajitnov).**

$C_*^{Nov}(f) \simeq C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} \Lambda$ : chain homotopic.

$\rho_n : \mathbb{Z}[\pi] \to \mathcal{K}_n(t)$ extends to $\rho_n : \Lambda \to \mathcal{K}_n((t))$
$\rightsquigarrow \tau_{\rho_n}^{Nov}(f) :=$ torsion of $C_*^{Nov}(f) \otimes_\Lambda \mathcal{K}_n((t)) \in \mathcal{K}_n(t)_{ab}^\times / \pm \Gamma_n$.
$(\tau_{\rho_0}^{Nov}(f)$ "=" $\tau_{ab}^{Nov}(f)$.)

# (Abelian) Lefschetz-type zeta function

$[o] \in \mathcal{O}$, $\phi$ : return map around $x \in o(S^1)$
$\rightsquigarrow$ $p(o) := \deg(o\colon S^1 \to o(S^1))$
$\epsilon(o) := \operatorname{sgn}\det(id - (d\phi)_x)$

$$\zeta_f := \exp\left(\sum_{[o]\in\mathcal{O}} \frac{\epsilon(o)}{p(o)}[o]\right) \in \Lambda_{ab} \otimes \mathbb{Q}.$$

## Lemma.

$$\zeta_f = \prod_{[o]\in\mathcal{O}, p(o)=1} (1 - (-1)^{i_-(o)}[o])^{(-1)^{i_0(o)+1}} \in \Lambda_{ab},$$

where $i_-(o)$, $i_0(o)$ are the number of real eigenvalues of $\det(id - (d\phi)_x)$ which are in $(-\infty, -1)$ and $(-1, 1)$ respectively.

# Non-commutative Lefschetz-type zeta function

$N \lhd \mathcal{K}_n((t))^\times_{ab}$ : consisting of elements s. t.
$\forall k \in \mathbb{Z}, \exists$ representative in $\mathcal{K}_n((t))^\times$ which equals 1 up to degree $k$.
$\rightsquigarrow \mathcal{K}_n((t))^\times_{\overline{ab}} := \mathcal{K}_n((t))^\times_{ab}/N \ (\sim \mathcal{K}_n((t))^\times/\overline{[\mathcal{K}_n((t))^\times, \mathcal{K}_n((t))^\times]})$.

**Remark.** $(\mathcal{K}_n)^\times_{ab}/\langle p^{-1}t_*(p)\rangle \hookrightarrow \mathcal{K}_n((t))^\times_{\overline{ab}}$

For each $[o] \in \mathcal{O}$ with $p(o) = 1$, choose a path $\sigma_o$ from $*$ to $o(S^1)$.

$$\zeta_{f,\rho_n} := \prod_{[o]\in\mathcal{O}, p(o)=1} (1 - (-1)^{i_-(o)}\rho([\sigma_o o \bar{\sigma}_o]))^{(-1)^{i_0(o)+1}} \in \mathcal{K}_n((t))^\times_{\overline{ab}}.$$

## Lemma

$\zeta_{f,\rho_n}$ does not depend on the choices of $\sigma_o$ and the order of the product.

# Statement

$\rho\colon \pi \to \Gamma$ : homomorphism to a torsion-free solvable group

## Theorem (K.).

$$\tau_\rho(X) = \zeta_{f,\rho} \cdot \tau_\rho^{Nov}(f) \ \in \mathcal{K}((t))^\times_{ab}/\pm\Gamma.$$

$\rho\colon \pi \to H_1(X)$
$\rightsquigarrow$ Hutchings-Lee's, (acyclic version of) Pajitnov's

$\dim X = 3, \Gamma = \Gamma_n$
$\rightsquigarrow$ dynamical and Morse theoretical presentation of $\Delta^{(n)}_{f_*}(t^{-1})$

## Corollary (K.).

If $X$ is fibered over a circle and $\psi \in H^1(X)$ is induced by the fibration, then $c_n(\psi) = 1$ for all $n$.

# 3. Problems

$\dim X = 3$

[Friedl-Vidussi '11]
Twisted Alexander polynomials associated to finite representations detect fiberedness of 3-manifolds.

$Q1$. For what class of 3-manifolds do $\Delta_{\psi}^{(n)}(t)$ or $\tau_{\rho_n}(X)$ detect fiberedness?

**Proposition (K.).**

For any $n$, there are infinitely many knots s. t. $c_i(\psi) = 1$ for $i \leq n$ and $c_{n+1}(\psi) \neq 1$.

We can also construct infinitely many non-fibered knots where $c_n(\psi) = 1$ and $\delta_n(\psi) = ||\psi||_T$ for all $n$.

# Detection of fiberedness

In [Friedl-Vidussi '11] it is essential to prove the theorem for $3$-manifolds with residually finite solvable fundamental groups.

$$\pi : \text{residually } (*) \iff \pi \hookrightarrow \prod_{N \triangleleft \pi,\ \pi/N\,:\,(*)} \pi/N$$

[Ascenbrenner-Friedl (preprint)]
For almost all primes $p$ there exists a finite covering $\widehat{X}$ s. t. $\pi_1 \widehat{X}$ is residually $p$ and, in particular, residually finite solvable.

**Conjecture.**

If $\pi_1 X$ is residually (finite) solvable and $c_n(\psi) = 1$ and $\delta_n(\psi) = ||\psi||_T$ for all $n$, then $\psi$ is indeced by a fibration over a circle.

# $L^2$-torsion

$X$ : hyperbolic 3-manifold

$\mathbb{C}[\pi] \subset l^2(\pi) := \{ f : \pi \to \mathbb{C} \,;\ \sum_{\gamma\in\pi} |f(\gamma)|^2 < \infty \}$
$\rightsquigarrow\ C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} l^2(\pi)$

$\det_{\mathcal{N}(\pi)} : M_{m,n}(l^2(\pi)) \to \mathbb{R}$ : Fuglede-Kadison determinant
$\rightsquigarrow\ \tau^{(2)}(X) =$ torsion of $C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} l^2(\pi)\ \in \mathbb{R}$.

[Hess-Schick '98, Lott '92, Lück-Schick '99, Mattai '92]

$$\tau^{(2)}(X) = -\frac{1}{6\pi} \operatorname{Vol}(X).$$

$Q2$. How about a circle-valued Morse theoretical presentation of $L^2$-torsion?

# $L^2$-Lefschetz zeta function

$(\prod_{[o]\in\mathcal{O},p(o)=1} \det_{\mathcal{N}(\pi)}(1-(-1)^{i_-(o)}\rho([\sigma_o o\overline{\sigma}_o]))^{(-1)^{i_0(o)+1}} = 1.)$

[Goda-Pajitnov '09] Morse-Novikov theory for 'half-transversal flows'

· no critical points of index 0 and 3

· $\mathcal{D}(p) \pitchfork \mathcal{A}(q)$ $(\mathrm{ind}(p)=2, \mathrm{ind}(q)=1)$

· $\exists$ regular level surgface $\Sigma$ s. t. the diffeomorphism
$\Sigma \setminus \sqcup_p \mathcal{A}_{X\setminus\Sigma}(p) \to \Sigma \setminus \sqcup_p \mathcal{D}_{X\setminus\Sigma}(p)$ extends to $\Sigma$ entirely.

$$\zeta_f^{(2)} := \prod_i \det_{\mathcal{N}(\pi)}(Id - t\varphi_* : C_i(\widetilde{\Sigma}) \otimes_{\mathbb{Z}[\pi]} l^2(\pi) \to C_i(\widetilde{\Sigma}) \otimes_{\mathbb{Z}[\pi]} l^2(\pi)).$$

**Problem.**

How to describe $\tau^{(2)}(X)/\zeta_f^{(2)}$ in terms of $C_*^{Nov}(f)$?