# Polynomial splittings of Ozsváth and Szabó's correction terms

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# Outline

## Section 1

# Motivation and History

$K_1, K_2$: two knots in $S^3$. $K_1$ is said to be smoothly concordant to $K_2$ ($K_1 \sim K_2$) if $\exists$ a smooth embedding $S^1 \times [0,1] \hookrightarrow S^3 \times [0,1]$ s.t.

$$\partial(S^1 \times [0,1]) = S^1 \times 0 \cup S^1 \times 1 = K_1 \cup (-K_2)$$

$C := \{\text{knot in } S^3\}/\sim$
$\sharp$: connected sum of knots.
Then $(C, \sharp)$ is called the smooth concordance group of knots in $S^3$.

Question:

1. Given a knot $K$, determine the order of $K$ in $C$.
2. (Independence Problem) Given $K_1$ and $K_2$, determine whether they are independent or not in $C$.

# Partial Answers to Q1

$K$: a knot in $S^3$

- Seifert surface of $K$: oriented surface in $S^3$ with $\partial F = K$
- Seifert form

$$\begin{aligned}\theta : H_1(F) \times H_1(F) &\to \mathbb{Z} \\ (\alpha, \beta) &\mapsto lk(\alpha, \beta^+)\end{aligned}$$

- A Seifert form $\theta$ is said to be null-concordant if $\exists$ a direct sum $Z$ of $H_1(F)$ such that $\mathrm{rank}(Z) = \frac{1}{2}\mathrm{rank}(H_1(F))$ and $\theta(Z, Z) = 0$, and then $Z$ is called a metabolizer of $\theta$.

# Partial Answers to Q1

Two Seifert forms $\theta_1$ and $\theta_2$ are algebraically concordant if $\theta_1 \oplus -\theta_2$ is null-concordant.

Two knots $K_1$ and $K_2$ are algebraically concordant ($K_1 \sim_{alg} K_2$) if their Seifert forms are algebraically concordant.

Let $C_{alg} = \{knot\}/\sim_{alg}$. Then $(C_{alg}, \sharp)$ is called algebraic concordance group.

**Fact:** If a knot $K$ has order one in $C$ (which is called smoothly slice knot), then $K$ has order one in $C_{alg}$ (which is called algebraically slice knot).

# Partial Answers to Q1

There are other invariants which examine the order of a knot. For instance:

- Signature
- twisted Alexander polynomial
- Casson-Gordon Invariants
- Von Neumann $\rho$-invariant
- Rasmussen invariant
- Ozsváth-Szabó $\tau$-invariant
- Ozsváth and Szabó's correction terms
- $\cdots$

# Partial Answer to Q2

**Q2:** Given two knots $K_1$ and $K_2$, and any two non-trivial integers $n_1$ and $n_2$, check whether $K = (n_1 K_1)\sharp(n_2 K_2)$ is slice or not.

1. Suppose the signatures of $K_i$ are $\sigma(K_i) = a_i$,
   the Rasmussen inv. are $s(K_i) = b_i$,
   and the Ozsváth-Szabó $\tau$-inv. are $\tau(K_i) = c_i$.
   Then $\sigma(K) = n_1 a_1 + n_2 a_2$, $s(K) = n_1 b_1 + n_2 b_2$ and $\tau(K) = n_1 c_1 + n_2 c_2$.
   If there is no non-trivial $(n_1, n_2)$ s.t $\sigma(K) = s(K) = \tau(K) = 0$, then $K_1$ and $K_2$ are independent in $C$.

# Partial Answer to Q2

2. (Polynomial Splitting) Suppose the Alexander polynomials of $K_1$ and $K_2$ are relatively prime in $\mathbb{Q}[t^{-1}, t]$.
   - ♣ (Levine) If $K$ has vanishing Levine obstruction, so do $n_1 K_1$ and $n_2 K_2$.
   - ♣ (Se-Goo Kim) If $K$ has vanishing Casson-Gordon-Gilmer obstruction, so do $n_1 K_1$ and $n_2 K_2$.
   - ♣ (Se-Goo Kim and Taehee Kim) If $K$ has vanishing von Neumann $\rho$-invariants associated with certain metabelian representations, so do $n_1 K_1$ and $n_2 K_2$.
   - ♣ In today's talk, we introduce similar property for Ozsváth and Szabó's correction terms.

Section 2

# Introduction

# Correction Term

- $Y$: a $\mathbb{Q}HS^3$
  $\mathrm{Spin}^c(Y)$: the set of Spin$^c$-structures over $Y$
  Ozsváth and Szabó defined invariants

$$\begin{aligned} d(Y, \quad) : \mathrm{Spin}^c(Y) &\rightarrow \mathbb{Q} \\ s &\mapsto d(Y, s), \end{aligned}$$

  which are called the correction terms or $d$-invariants of $Y$.

- $K$: a knot in $S^3$
  $\Sigma^q(K)$: the $q$-fold cyclic branched cover of $S^3$ along $K$
  When $q$ is a prime power (by which we mean $q = p^r$ for some prime number $p$), $\Sigma^q(K)$ is a $\mathbb{Q}HS^3$. So we can consider the $d$-invariants for it.

# Properties

Correction terms have two important properties:

- (additivity) Let $Y_1$ and $Y_2$ be two $\mathbb{Q}HS^3$, and $s_i \in \mathrm{Spin}^c(Y_i)$ for $i = 1, 2$. Then

$$d(Y_1 \sharp Y_2, s_1 \sharp s_2) = d(Y_1, s_1) + d(Y_2, s_2).$$

- (vanishing) If $(Y, s)$ bounds a rational homology smooth 4-ball $(W, t)$, then $d(Y, s) = 0$.

# Slice Obstruction

**Theorem A (Ozsváth and Szabó)**

*Let $q$ be a prime power. If $K$ is a smoothly slice knot, then*

- $\exists$ *a subgroup* $M < H_1(\Sigma^q(K))$ *satisfying* $|M|^2 = |H_1(\Sigma^q(K))|$
- $d(\Sigma^q(K), s) = 0$ *for any* $s = s_0 + m$ *where* $m \in M$.

*Here* $s_0$ *is the unique spin-structure over* $\Sigma^q(K)$ *under certain restriction.*

**Remark:** $H_1(\Sigma^q(K))$ acts on $\mathrm{Spin}^c(\Sigma^q(K))$ freely and transitively. Under this setting, we identify these two sets, by sending $x \in H_1(\Sigma^q(K))$ to $s_0 + x \in \mathrm{Spin}^c(\Sigma^q(K))$.

# What is $M$ ?

Let $\Delta$ be the slice disk of $K$ in the 4-ball $B^4$, and $W^q(K)$ be the $q$-fold cyclic branched cover of $B^4$ along $\Delta$. Then consider the inclusion map

$$j : H_1(\Sigma^q(K)) \hookrightarrow H_1(W^q(K)).$$

Then $M$ is $Ker(j)$.

A $\mathrm{spin}^c$-structure $s$ over $\Sigma^q(K)$ extends to $W^q(K)$ iff $s = s_0 + m$ for some $m \in M$.

# Theorem 1

## Theorem (B)

*$K = K_1 \sharp K_2$. Suppose the Alexander polynomials of $K_1$ and $K_2$ are relatively prime in $\mathbb{Q}[t, t^{-1}]$.*

1. *If $K$ is smoothly slice, then for all but finitely many primes $q$ (or any of its prime power), the following holds.*
   - ◊ $\exists M_i < H_1(\Sigma^q(K_i))$ *satisfying* $|M_i|^2 = |H_1(\Sigma^q(K_i))|$
   - ◊ *the $d$-invariants $d(\Sigma^q(K_i), s)$ are constant on $M_i$, for both $i = 1$ and 2.*
2. *If $K$ is ribbon, the conclusions above hold for any prime power $q$.*

# $\mathcal{D}_p^q(K)$ and $\mathcal{T}_p^q(K)$

Let $(Y, L)$ be the pair of a $\mathbb{Q}HS^3$ and a null-homologous knot in $Y$. For each Spin$^c$-structure $s$ over $Y$, Grigsby, Ruberman and Strle defined the $\tau$-invariant $\tau_s(Y, L)$ for $(Y, L, s)$.

Let $K \subset S^3$ be a knot. Let $q$ be a prime power and consider $(\Sigma^q(K), \tilde{K})$ where $\tilde{K}$ is the pre-image of $K$ in $\Sigma^q(K)$. GRS proved:

**Theorem**

*If $K$ is slice, then $\tau_s(\Sigma^q(K), \tilde{K}) = 0$ for any $s = s_0 + m$ where $m \in M$.*

# $\mathcal{D}_p^q(K)$ and $\mathcal{T}_p^q(K)$

Suppose $f : A \to \mathbb{Q}$ is a function on a finite abelian group and $H < A$ is a subgroup. GRS defined

$$S_H(f) = \sum_{h \in H} f(h)$$

In this talk, $A$ is $H_1(\Sigma^q(K))$ and $f$ is either $d$-invariant or $\tau$-invariant.

Given a prime $p$, let $\mathcal{G}_p$ be the set of all order $p$ subgroups of $A$. GRS discussed the following invariants for the case $q = 2$, but their methods work equally for any prime power.

# $\mathcal{D}_p^q(K)$ and $\mathcal{T}_p^q(K)$

Let

$$\mathcal{T}_p^q(K) = \begin{cases} \min\left\{ \left| \sum_{H\in\mathcal{G}_p} n_H S_H(\tau(\Sigma^q(K),\tilde{K}))\right| \;\middle|\; \begin{array}{l} n_H \in \mathbb{Z}_{\geqslant 0} \text{ \& at least} \\ \text{one is non-zero} \end{array} \right\} & \\ & \text{;if } p \text{ divides } |H_1(\Sigma^q(K))| \\ 0 & \text{;otherwise} \end{cases}$$

and

$$\mathcal{D}_p^q(K) = \begin{cases} \min\left\{ \left| \sum_{H\in\mathcal{G}_p} n_H S_H(d(\Sigma^q(K)))\right| \;\middle|\; \begin{array}{l} n_H \in \mathbb{Z}_{\geqslant 0} \text{ \& at least} \\ \text{one is non-zero} \end{array} \right\} & \\ & \text{;if } p \text{ divides } |H_1(\Sigma^q(K))| \\ 0 & \text{;otherwise} \end{cases}$$

# $\mathcal{D}_p^q(K)$ and $\mathcal{T}_p^q(K)$

GRS proved:

**Theorem**

*Let $p$ be a positive prime or 1. If $K$ has finite order in $C$, then $\mathcal{T}_p^q(K) = \mathcal{D}_p^q(K) = 0$.*

Given a function $f : A \to \mathbb{Q}$, we define

$$\begin{aligned} \bar{f} : A &\to \mathbb{Q} \\ \alpha &\to f(\alpha) - f(0) \end{aligned}$$

Then we define $\bar{\mathcal{D}}_p^q(K)$ and $\bar{\mathcal{T}}_p^q(K)$ by taking $\bar{d}(\Sigma^q(K))$ and $\bar{\tau}(\Sigma^q(K), \tilde{K})$.

# Theorem 2

We prove:

## Theorem (B)

*Let $p$ be a positive prime or 1. Suppose the Alexander polynomials of $K_1$ and $K_2$ are relatively prime in $\mathbb{Q}[t, t^{-1}]$.*

1. *If $n_1K_1\sharp n_2K_2$ is slice for some non-zero $n_1$ and $n_2$, then for all but finitely many primes $q$ (or any of its prime power), the following holds: $\bar{\mathcal{T}}_p^q(K_i) = \bar{\mathcal{D}}_p^q(K_i) = 0$ for $i = 1, 2$.*
2. *If $n_1K_1\sharp n_2K_2$ is ribbon for some non-zero $n_1$ and $n_2$, the conclusions above hold for any prime power $q$.*

# Application

## Proposition

*Let $T_k$ be the k-twist knot. Excluding the unknot, $T_1$ (which is the figure-8 knot) and $T_2$ (which is Stevedore's knot), no non-trivial linear combinations of twist knots are ribbon.*

**Remark:** This property was also proved by Se-Goo Kim, using Casson-Gordon invariant.

Section 3

# Proof of Theorem 1

### Theorem B (Kervaire, Levine, Kim)

*Given two knots $K_1$ and $K_2$, let $F_i$ be a Seifert surface of $K_i$, and $\theta_i$ be the Seifert form on $H_1(F_i)$ for $i = 1, 2$. Suppose the Alexander polynomials of $K_1$ and $K_2$ are relatively prime in $\mathbb{Q}[t^{-1}, t]$.*

*Then if $\theta_1 \oplus \theta_2$ is null-concordant with a metabolizer $Z$, then $\theta_i$ is null-concordant with metabolizer $Z_i = Z \cap H_1(F_i)$ for both $i = 1$ and $2$.*

# Proof of Theorem 1

- If $K$ is smoothly slice, let $\Delta$ be the slice disk of $K$ in the 4-ball $B^4$.
  $F \cup \Delta$ bounds a 3-manifold $R$ in $B^4$
  Consider
  $$\iota : H_1(F) \to H_1(R)/\mathrm{Tor}$$
  Then $Z := Ker(\iota) \subset H_1(F)$ is a metabolizer of the Seifert form on $H_1(F)$
- Let $W^q(K)$ be the $q$-fold cyclic branched cover of $B^4$ along $\Delta$. Then $\exists$ the following CD (horizontal sequences are exact):

$$\begin{array}{ccccccccc}
\longrightarrow & \bigoplus_{1\leqslant i\leqslant q} H_1(F) & \xrightarrow{f} & \bigoplus_{1\leqslant i\leqslant q} H_1(F) & \xrightarrow{g} & H_1(\Sigma^q(K)) & \longrightarrow & 0 \\
 & \Big\downarrow & & \Big\downarrow \bigoplus_{1\leqslant i\leqslant q} \bar{\iota} & & \Big\downarrow j & & \\
\longrightarrow & \bigoplus_{1\leqslant i\leqslant q} H_1(R) & \longrightarrow & \bigoplus_{1\leqslant i\leqslant q} H_1(R) & \xrightarrow{h} & H_1(W^q(K)) & \longrightarrow & 0
\end{array}$$

★ Fix a basis for $H_1(F)$. Then

$$f = \begin{pmatrix} G & I-G & 0 & 0 & \cdots & 0 \\ 0 & G & I-G & 0 & \cdots & 0 \\ 0 & 0 & G & I-G & \cdots & 0 \\ & \vdots & & \vdots & & \\ I-G & 0 & 0 & 0 & \cdots & G \end{pmatrix},$$

where $G = (A - A^t)^{-1}A$ while $A$ is the Seifert matrix. It is known that $f$ is a presentation matrix of $H_1(\Sigma^q(K))$.

Defien $f_1$ and $f_2$ for $K_1$ and $K_2$. Then $f = f_1 \oplus f_2$.

★ $\bar{\iota}$ and $j$ are induced by the inclusion maps.

- Let $M := Ker(j)$. Then by the commutativity of the diagram we have the following fact:

  Let $\mathrm{Tor}$ denote the torsion part of $H_1(R)$. If $|\mathrm{Tor}|$ and $|H_1(\Sigma^q(K))|$ are relatively prime, then $g(\bigoplus_{1 \leqslant i \leqslant q} Z) = M$.

**Lemma**

1. *If $K$ is ribbon, $R$ can be chosen to be a handlebody. Then $H_1(R)$ is torsion free.*
2. *Given a knot $K$ and a prime number $p$, $\exists$ only finitely many prime numbers $q$ for which $p$ divides $H_1(\Sigma^{q^r}(K))$ for some $r \in \mathbb{N}$.*

Let $S$ be the set of primes $q$ for which $|\mathrm{Tor}|$ and $|H_1(\Sigma^{q^r}(K))|$ are NOT relatively prime for some $r \in \mathbb{N}$. Then by the lemma above, it is a finite set. In particular, if $K$ is ribbon, $S$ is empty.

# Theorem 1

**Theorem**

*$K = K_1 \sharp K_2$. Suppose the Alexander polynomials of $K_1$ and $K_2$ are relatively prime in $\mathbb{Q}[t, t^{-1}]$.*

1. *If $K$ is smoothly slice, then for all but finitely many primes $q$ (or any of its prime power), the following holds.*
   - $\Diamond$ $\exists M_i < H_1(\Sigma^q(K_i))$ *satisfying* $|M_i|^2 = |H_1(\Sigma^q(K_i))|$
   - $\Diamond$ *the $d$-invariants $d(\Sigma^q(K_i), s)$ are constant on $M_i$, for both $i = 1$ and 2.*
2. *If $K$ is ribbon, the conclusions above hold for any prime power $q$.*

# Proof of Theorem 1

1. Suppose $K$ is smoothly slice. Then for all the prime powers except for those with prime numbers in $S$, $g(\bigoplus_{1 \leqslant i \leqslant q} Z) = M$. By Theorem B, $Z$ decomposes as $Z = Z_1 \oplus Z_2$. So $M = M_1 \oplus M_2$, where $M_i = g(\bigoplus_{1 \leqslant i \leqslant q} Z_i)$.

   It is not hard to see $|M_i|^2 = |H_1(\Sigma^q(K_i))|$ for $i = 1, 2$.

   Moreover,

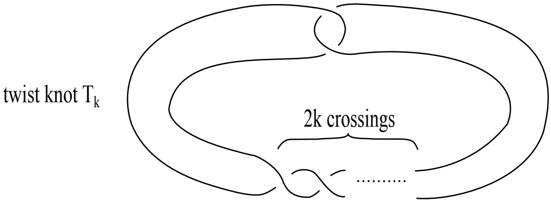
$$
\begin{aligned}
& \text{For any } m_1 \in M_1,\ (m_1, 0) \in M, \\
\Rightarrow \quad & d(\Sigma^q(K), s_0 + (m_1, 0)) = 0 \text{ by Theorem A} \\
\Rightarrow \quad & d(\Sigma^q(K_1), s_0 + m_1) + d(\Sigma^q(K_2), s_0) = 0 \\
\Rightarrow \quad & d\text{-invariant of } \Sigma^q(K_1) \text{ is constant on } M_1. \\
& \text{The same fact holds for } K_2.
\end{aligned}
$$

2. If $K$ is ribbon, the set $S$ is empty.

## Section 4

## Application

# Twist Knots




Twist knot $T_k$ is

- of infinite order in the algebraic concordance group $C_{alg}$ if $k < 0$
- algebraically slice if $k \geq 0$ and $4k + 1$ is a square
- of finite order in $C_{alg}$ otherwise

# Proof of Application

The Alexander polynomial of $k$-twist knot is

$$\Delta_{T_k}(t) = -kt^2 + (2k+1)t - k.$$

It is easy to check that for any two twist knots, their Alexander polynomials are relatively prime in $\mathbb{Q}[t^{-1}, t]$.

Excluding the unknot, 1-twist knot and 2-twist knot, suppose there are$\{k_i\}_{i=1}^l$ and $\{n_i\}_{i=1}^l$ for which $K = \sharp_{i=1}^l (n_i T_{k_i})$ is a ribbon knot. Then

- by Levine's splitting theorem, each $n_i T_{k_i}$ is algebraically slice.
- by our Theorem 2, each $T_{k_i}$ has vanishing $\bar{\mathcal{D}}_p^q$ and $\bar{\mathcal{T}}_p^q$ for any prime $p$ and prime power $q$.

We consider the case when $q = 2$, namely the double branched covers of twist knots.

# Proof of Application

It is known that $T_k$ has infinite order in the algebraic concordance group $C_{alg}$ if $k < 0$. So each $k_i$ for $1 \leq i \leq l$ is a positive integer.

$\Sigma^2(T_k) = L(4k+1, 2) =: L_k$. Assume that $k \geq 0$.
Let $p$ be a prime dividing $4k+1$. Then

$$\bar{\mathcal{D}}_p^2(T_k) = \left| \sum_{j=0}^{p-1} \bar{d}(L_k, s_0 + j) \right| = \left| \sum_{j=0}^{p-1} (d(L_k, s_0 + j) - d(L_k, s_0)) \right|$$

There is formula of correction terms for lens spaces, which gives:

$$d(L_k, s_0 + j) = \frac{1}{4} - \frac{j^2}{8k+2} + \begin{cases} \frac{1}{4} & \text{if } j \text{ is odd} \\ \frac{-1}{4} & \text{if } j \text{ is even} \end{cases}$$

for $0 \leq j \leq 2k$.

# Proof of Application

By calculation $d(L_k, s_0) = 0$. So

$$\bar{\mathcal{D}}_p^2(T_k) = \left| \sum_{j=0}^{p-1} d(L_k, s_0 + j) \right| = \mathcal{D}_p^2(T_k)$$

In GRS' paper, the authors discussed $\mathcal{D}_p^2(T_k)$ for $k > 0$ and showed:

$$\mathcal{D}_p^2(T_k) > 0$$

except for the case $k = 0, 1, 2$.

Therefore those $T_{k_i}$ which make $\bar{\mathcal{D}}_p^2$ vanishes are restricted to $T_0$, $T_1$ and $T_2$.