Hyperplane Arrangements and Lefschetz's Hyperplane Section Theorem

Masahiko Yoshinaga*

Groups, Homotopy and Configuration Spaces
5th-11th July, 2005, Tokyo

The purposes

Recently Dimca-Papadima [1] and Randell [2] proved that the complement $\mathsf{M}(\mathcal{A})$ of a complex hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ is homotopy equivalent to a minimal CW-complex, namely, a CW-complex whose number of k-cells is equal to its k-th betti number for each $k \geq 0$.

The goal here is to provide

- a presentation for the fundamental group $\pi_1(\mathsf{M}(\mathcal{A}))$,
- a minimal chain complex computing local system homology

obtained from the study of attaching maps for a real arrangement $\mathcal{A}$.

We concentrate on 2-dimensional case. See [3] for details.

*RIMS, Kyoto Univ., yosinaga@kurims.kyoto-u.ac.jp

Counting chambers

Let

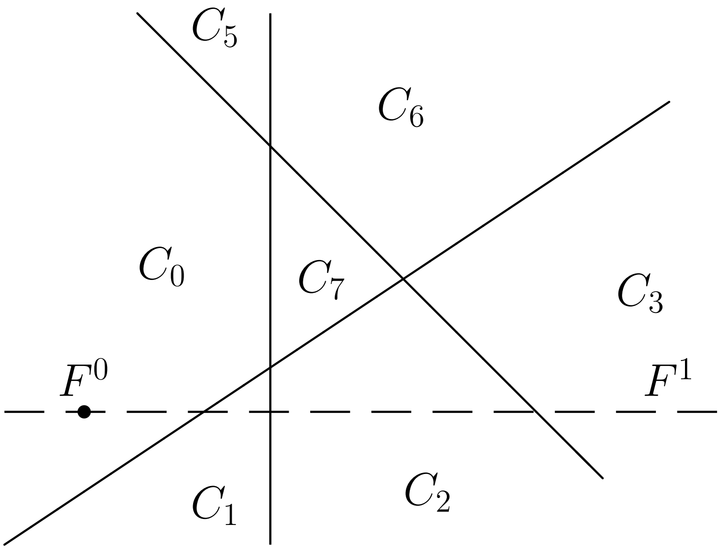
$$F^0 \subset F^1 \subset F^2 = V_{\mathbb{R}} = \mathbb{R}^2$$

be a generic flag. And define

$$\mathsf{ch}_k(\mathcal{A}) = \left\{ C \;\middle|\; \begin{array}{l} C \cap F^{k-1} = \emptyset \\ C \cap F^k \neq \emptyset \end{array} \right\},$$

where C is a chamber.

Fact: $\sharp\mathsf{ch}_k(\mathcal{A}) = b_k(\mathsf{M}(\mathcal{A}))$.

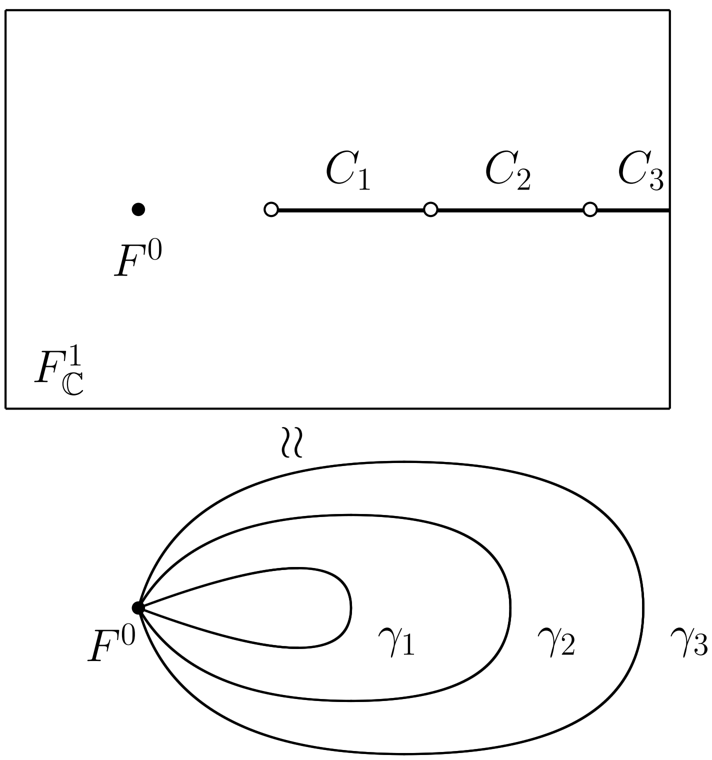


$\mathsf{ch}_2 = \{C_5, C_6, C_7\}$
$\mathsf{ch}_1 = \{C_1, C_2, C_3\}$
$\mathsf{ch}_0 = \{C_0\}$
$\pi(\mathcal{A}, t) = 1 + 3t + 3t^2$

The minimal CW-decomposition

$\mathsf{ch}_k(\mathcal{A})$ indexes k-cells.

$\underline{k=1}$.



Cahmber: $C_i \longleftrightarrow 1$-cell: γ_i.

$\underline{k=2}$. Let Q be a defining equation of $\mathcal{A}$. Let f be a defining equation of the line F^1. Then

$$\varphi = \left|\frac{f^\lambda}{Q}\right| : \mathsf{M}(\mathcal{A}) \to \mathbb{R}_{\geq 0}$$

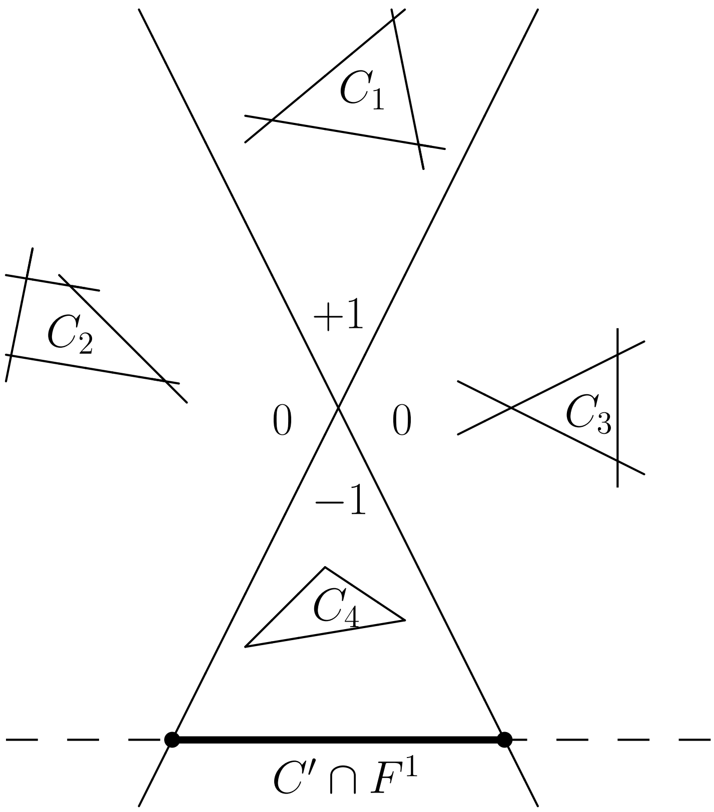
is a Morse function. $C \in \mathsf{ch}_2(\mathcal{A})$ is a stable manifold. The corresponding unstable manifold σ_C is the 2-cell attaching to $\mathsf{M}(\mathcal{A}) \cap F^1_{\mathbb{C}}$.

The degree map

The degree map is a map

$$\deg : \mathsf{ch}_2(\mathcal{A}) \times \mathsf{ch}_1(\mathcal{A}) \to \{-1, 0, 1\}$$

defined below.
Let $C_1, \ldots, C_4 \in \mathsf{ch}_2(\mathcal{A})$ and $C' \in \mathsf{ch}_1(\mathcal{A})$.



$$\deg(C_1, C') = 1$$
$$\deg(C_2, C') = \deg(C_3, C') = 0$$
$$\deg(C_4, C') = -1.$$

The degree map will be used to describe the fundamental group and the boundary map of twisted minimal chain complex.

Fundamental group

$\mathrm{ch}_1(\mathcal{A}) = \{C_1, \ldots, C_n\}$.
Let $\gamma_1, \ldots, \gamma_n$ be the corresponding 1-cells.

Given $C \in \mathrm{ch}_2(\mathcal{A})$, define a word $R(C)$ by

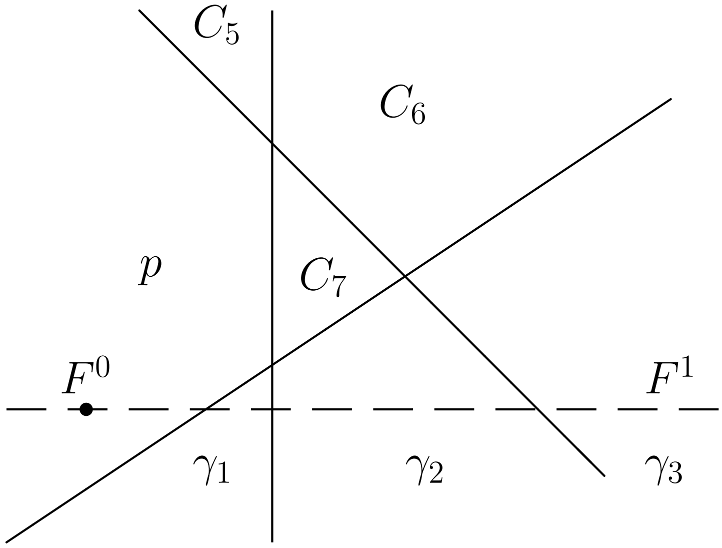
$$R(C) := \gamma_1^{e_1}\gamma_2^{e_2}\cdots\gamma_n^{e_n}\gamma_1^{-e_1}\cdots\gamma_n^{-e_n},$$

where $e_i = \deg(C, C_i)$.

Theorem

The fundamental group $\pi_1(\mathsf{M}(\mathcal{A}))$ has the following presentation:

$$\langle \gamma_1, \ldots, \gamma_n \,|R(C); C \in \mathrm{ch}_2(\mathcal{A})\rangle .$$



$$R(C_5) = \gamma_2\gamma_3^{-1}\gamma_2^{-1}\gamma_3$$
$$R(C_6) = \gamma_1\gamma_3^{-1}\gamma_1^{-1}\gamma_3$$
$$R(C_7) = \gamma_1\gamma_2^{-1}\gamma_1^{-1}\gamma_2$$

$$\pi_1(\mathsf{M}(\mathcal{A})) = \mathbb{Z}^3$$

Local system

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$. Choose a nonzero complex number $q_i \in \mathbb{C}^*$ for each $i = 1, \ldots, n$.

We consider the rank one local system $\mathcal{L}$ on $\mathsf{M}(\mathcal{A})$ such that the local monodromy around H_i is q_i^2.

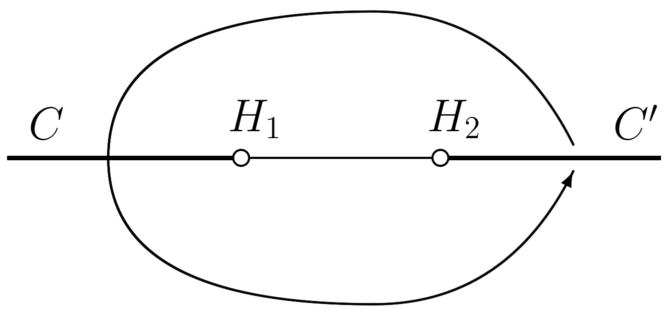
Definition

For two chambers C and C',

$$\mathrm{Sep}(C, C') := \{i | H_i \text{ separates } C,\ C'\}.$$

And define

$$\Delta(C, C') := \prod_{i \in \mathrm{Sep}(C,C')} q_i - \prod_{i \in \mathrm{Sep}} q_i^{-1}$$



$$\Delta(C, C') = q_1 q_2 - \frac{1}{q_1 q_2}.$$

Twisted minimal chain complex

$\mathcal{A} = \{H_1, \dots, H_n\}$,
$\mathsf{ch}_0(\mathcal{A}) = \{p\}$,
$\mathsf{ch}_1(\mathcal{A}) = \{\gamma_1, \dots, \gamma_n\}$
as above. Put

$$\mathcal{C}_0 := \mathbb{C}[p]$$
$$\mathcal{C}_1 := \mathbb{C}[\gamma_1] \oplus \cdots \oplus \mathbb{C}[\gamma_n]$$
$$\mathcal{C}_2 := \bigoplus_{C \in \mathsf{ch}_2} \mathbb{C}[\sigma_C]$$

and define $\partial : \mathcal{C}_k \to \mathcal{C}_{k-1}$ as

$$\partial([\gamma_i]) = \Delta(\gamma_i, p)[p]$$
$$\partial([\sigma_C]) = \sum_{i=1}^{n} \deg(C, \gamma_i)\Delta(C, \gamma_i)[\gamma_i].$$

Theorem
Then $(\mathcal{C}_\bullet, \partial)$ is a chain complex, namely, $\partial^2 = 0$ and

$$H_k(\mathcal{C}_\bullet, \partial) \cong H_k(\mathsf{M}(\mathcal{A}), \mathcal{L}).$$

Reference

[1] A. Dimca, S. Papadima, Hypersurface complements, Milor fibers and higher homotopy groups of arrangements. Ann. of Math. (2) **158** (2003), 473–507

[2] R. Randell, Morse theory, Milnor fibers and minimality of hyperplane arrangements. Proc. Amer. Math. Soc. **130** (2002), 2737–2743

[3] M. Yoshinaga, Hyperplane arrangements and Lefschetz's hyperplane section theorem. (Preprint)