Homotoy stability of Theorem of J. Mostovoy – Topology of spaces of holomorphic maps between complex projective spaces

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Let $\mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ be the space consisting of all continuous maps $f : \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ of degree d, and $\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \subset \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ the subspace of all based continuous maps $f : \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ of degree d. We denote by $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \subset \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ (resp. $\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \subset \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$) the corresponding subspace consisting of all holomorphic maps (resp. based holomorphic maps).

The motivation of this poster derives from the work of G. Segal [6], in which he describes that the Atiyah-Jones type result holds for the inclusion map $\mathrm{Hol}(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n)$ as follows.

Theorem 0.1 (G. Segal, [6]). *The inclusion maps*

$$\begin{cases} i_d : \mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) = \Omega^2_d \mathbb{C}\mathrm{P}^n \simeq \Omega^2 S^{2n+1} \\ j_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are homotopy equivalences up to dimension $(2n-1)d$. □

Remark. A map $f : X \to Y$ is called *a homotopy equivalence* (resp. *a homology equivalence*) *up to dimension* D if the induced homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is bijective when $k < D$ and surjective when $k = D$. Similarly, a map $f : X \to Y$ is called *a homotopy equivalence* (resp. *a homology equivalence*) *through dimension* D if the induced homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is bijective whenever $k \leq D$.

Segal also expected that a similar Atiyah-Jones type result would hold for the inclusion $\mathrm{Hol}(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ even if $2 \leq m \leq n$, and we would like to investigate this problem. For this purpose, we study the restriction fibration sequence

$$F_d(m,n) \to \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) \xrightarrow{r} \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^{m-1},\mathbb{C}\mathrm{P}^n),$$

where the map r is defined by the restriction $r(f) = f|\mathbb{C}\mathrm{P}^{m-1}$ and $F_d(m,n)$ denotes the fiber defined by

$$F_d(m,n) = r^{-1}(\psi_d^{m-1,n}) = \big\{ f \in \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) : f|\mathbb{C}\mathrm{P}^{m-1} = \psi_d^{m-1,n} \big\}.$$

Here, $\psi_d^{m,n} \in \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n)$ is the holomorphic map defined by $\psi_d^{m,n}([x_0 : \cdots : x_m]) = [(x_0)^d : \cdots : (x_m)^d : 0 : \cdots : 0]$ and we choose it as the base-point of $\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n)$. We remark that there is a homotopy equivalence $F_d(m,n) \simeq \Omega^{2m}\mathbb{C}\mathrm{P}^n$.

Let $H_d(m,n) \subset \mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n)$ be the subspace defined by $H_d(m,n) = F_d(m,n) \cap \mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n)$. We investigate the homotopy types of the subspaces $H_d(m,n)$, $\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n)$ and $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n)$ with the corresponding inclusion maps

$$\begin{cases} i'_d : H_d(m,n) \to F_d(m,n), \quad i_d : \mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) \to \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) \\ j_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n). \end{cases}$$

Recently, J. Mostovoy proved the following very remarkable important result.

Theorem 0.2 (J. Mostovoy, [5]). *If $2 \leq m \leq n$, the inclusion maps*

$$\begin{cases} i'_d : H_d(m,n) \to F_d(m,n), \quad i_d : \mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) \to \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) \\ j_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) \end{cases}$$

are homotopy equivalences through dimension $D(d;m,n)$ when $m < n$, and homology equivalences through dimension $D(d;m,n)$ when $m = n$.

Here, $\lfloor x \rfloor$ denotes the integer part of a number x and $D(d;m,n)$ is defined by $D(d;m,n) = (2n-2m+1)\Big(\lfloor \frac{d+1}{2} \rfloor + 1\Big) - 1.$ □

Since $\lim_{d\to\infty} D(d;m,n) = \infty$, we may regard $H_d(m,n)$ and $\mathrm{Hol}(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n)$ as finite dimensional homotopy (or homology) models for the infinite dimensional spaces $\Omega^{2m}\mathbb{C}\mathrm{P}^n$ and $\mathrm{Map}(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n)$, respectively. We know that the Atiyah-Jones type Theorem holds for several other cases, and the homotopy stability is usually satisfied for these cases. So one may expect that the homotopy stability may hold even if $m = n$. This is just my starting point and we shall announce that this holds when $m = n$, too. We remark that $H_d(m,n)$, $\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n)$ and $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n)$ are simply connected if $m < n$. So the homotopy stability follows from the homology stability if $m < n$. However, if $m = n$ they are not simply connected, and we need investigate their fundamental groups and homotopy types of universal coverings. Then our main result is as follows.

Theorem 0.3. *If $n \geq 2$, the inclusion maps*

$$\begin{cases} i'_d : H_d(n,n) \to F_d(n,n), \quad i_d : \mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \\ j_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are homotopy equivalences through dimension $D(d,n) = \lfloor \frac{d+1}{2} \rfloor$.

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