Groups, Homotopy and Configuration Spaces
Fred Cohen's 60th Birthday

Orbifold String Topology

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1 String topology

Theorem (Chas-Sullivan ’99): There is a product

$$H_*(\mathcal{L}M) \otimes H_*(\mathcal{L}M) \xrightarrow{\quad \circ \quad} H_*(\mathcal{L}M)$$

(*the string topology product*) and additional structure which gives $H_*(\mathcal{L}M)[n]$ a natural structure of BV-algebra. Moreover, $\circ$ is compatible with:

1. the intersection product in $H_*(M)$
2. the Pontryagin product in $H_*(\Omega M)$.

R.Cohen – J.Jones: Realized string product at homotopy theoretic level.

Definition: A *BV-algebra* is a graded vector space V with:

- A product $a \circ b$, $(V =$ graded comm. assoc. algebra$)$
- $\Delta : V \to V$, $\Delta^2 = 0$, $\deg \Delta = 1$
- $[a, b]$ graded Lie algebra with $\deg[\,,\,] = 1$.

such that:

(i) $[a, -]$ is a graded derivation of $\circ$:

$$[a, bc] \;=\; [a, b]c \;+\; (-1)^{|b|(|a|+1)} b[a, c]$$

(ii) Δ is a graded derivation of $[\,,\,]$

(iii) $\Delta(a \circ b) \;-\; (\Delta a) \circ b \;-\; (-1)^{|a|} a \circ \Delta b \;=\; (-1)^{|a|}[a, b]$

Question: When does $H*(\mathcal{L}X)$ have such a structure?

"Definition": An orbifold is a topological space X with an open cover $\{U_\alpha\}_{\alpha \in J}$ such that:

$$U_\alpha \approx M_\alpha / G_\alpha$$

where:

- $M_\alpha =$ smooth manifold
- $G_\alpha =$ finite group of diffeos

2 Groupoids

Definition:

1. A *groupoid* $\mathcal{G}$ is a small category in which every arrow is invertible.

$$\mathcal{G}_0 = \text{objects}, \qquad\qquad \mathcal{G}_1 = \text{arrows}$$

   Structure maps:

$$\begin{tikzcd} \mathcal{G}_1 \ {}_t\!\times_s \mathcal{G}_1 \arrow[r, "m"] & \mathcal{G}_1 \arrow[loop below, "i"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathcal{G}_0 \arrow[r, "e"] & \mathcal{G}_1 \end{tikzcd}$$

2. A groupoid is *smooth* if
   - $\mathcal{G}_0$ and $\mathcal{G}_1$ are smooth manifolds
   - the structure maps s, t, e, i, m are smooth
   - $s, t : \mathcal{G}_1 \to \mathcal{G}_0$ are submersions (so that $\mathcal{G}_1 \ {}_t\!\times_s \mathcal{G}_1$ is a mfd.)

3. A smooth groupoid is *proper* if $(s,t): \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is proper.
4. A smooth groupoid is *étale* if s,t are local diffeomorphisms.

- Morphism of groupoids: $\psi : \mathcal{G}' \to \mathcal{G}$

- Equivalence of gropoids: $\phi : \mathcal{G}' \xrightarrow{\simeq} \mathcal{G}$

- Morita equivalence: $\mathcal{G}' \xleftarrow{\simeq} \mathcal{H} \xrightarrow{\simeq} \mathcal{G}$

Theorem (Moerdijk–Pronk): The category of orbifolds is equivalent to the category of proper, étale groupoids, after inverting Morita equivalences.

From now on:

- M smooth, compact, connected, oriented manifold

- G finite group of diffeos acting on M

Definition: A *"global quotient orbifold"* X is one represented by a groupoid of the form

$$M \times G \underset{t}{\overset{s}{\rightrightarrows}} M$$

Notation: $\mathsf{X} = [M/G]$

Fact: The classifying space of X is

$$B\mathsf{X} = M_G = M \times_G EG$$

3 The free loop space

- $\forall g \in G, \qquad \mathcal{P}_g(M) = \{\gamma : [0,1] \to M \mid \gamma(0) \cdot g = \gamma(1)\}$

$$\begin{aligned} \mathcal{P}_G(M) &= \coprod_{g \in G} \mathcal{P}_g(M) \\ &= \bigcup_{g \in G} \mathcal{P}_g(M) \times \{g\} \end{aligned}$$

- G-action on $\mathcal{P}_G(M)$:

$$\begin{aligned} \mathcal{P}_g(M) \times G &\longrightarrow \mathcal{P}_g(M) \\ ((\gamma, g), h) &\longmapsto (\gamma_h, h^{-1}gh) \end{aligned}$$

- Define the loop orbifold LX:

$$\mathsf{LX} := [\mathcal{P}_G(M)/G] \qquad \Longrightarrow \qquad B\mathsf{LX} = \mathcal{P}_G(M) \times_G EG$$

Lupercio–Uribe: Construct loop groupoids for any smooth groupoid.

In the case of $\mathsf{X} = [M/G]$ we have:

Lemma: There is weak equivalence: $\tau : \mathcal{L}B\mathsf{X} \to B(\mathsf{L}\mathsf{X})$.

Proof: There is a map of fibrations

$$\begin{tikzcd} \mathcal{L}B\mathsf{X} \arrow[rr, "\tau"] \arrow[dr, "\epsilon_0"'] & & B(\mathsf{L}X) \arrow[dl, "|ev_0|"] \\ & B\mathsf{X} & \end{tikzcd}$$

which is an equivalence on fibers. $\square$

Two extreme cases:

- $G = \{e\}, \quad \mathcal{L}(M \times_G EG) = \mathcal{L}M$

- $M = \{*\}, \quad \mathcal{L}(M \times_G EG) = \mathcal{L}BG \cong \coprod_{(g)} BC(g)$

In the last case:

$$\bigoplus_{(g)} H_*(C(g); k) \ \cong \ HH_*(kG)$$

4 Principal G-bundles

- G-bundles $/S^1$: $\quad G \to Q \to S^1 \quad$ are classified by holonomy

$$\mathrm{Bun}_G(S^1) = [S^1, BG]_* = \pi_1(G) = G$$

- $\mathrm{Bun}_G(S^1, M)$ = iso classes of G-equiv. maps from principal G-bundles to M.

- G-action on $\mathrm{Bun}_G(S^1, M)$: Given $\quad \beta : Q_g \to M \ , \quad k \in G$

$$(\beta, k) \quad \longmapsto \quad \beta_{k^{-1}gk} : Q_{k^{-1}gk} \to M$$

Also:

$$\mathrm{Bun}_G(S^1, M) \;=\; \coprod_{g \in G} \mathrm{Bun}_g(S^1, M)$$

Proposition: The loop orbifold $\mathsf{LX} = [\mathcal{P}_G(M)/G]$ is isom. to $[\mathrm{Bun}_G(S^1, M) \,/\, G]$.

Thus,

$$\mathrm{Bun}_G(S^1, M) \times_G EG \;=\; \begin{cases} B\mathsf{LX} = \mathcal{P}_G(M) \times_G EG \\ \\ \mathcal{L}B\mathsf{X} = \mathcal{L}(M \times_G EG) \end{cases}$$

5 Orbifold string product

Given $g, h \in G$, there is a composition product:

$$* \quad : \quad \mathcal{P}_g(M)\ {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_h(M) \quad \longrightarrow \quad \mathcal{P}_{gh}(M)$$

Equivalently, the following is a pull-back:

$$\begin{tikzcd}
\mathcal{P}_g(M)_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h(M) \arrow[r] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_g(M) \times \mathcal{P}_h(M) \arrow[d, "\epsilon_1 \times \epsilon_0"] \\
M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

$\nu_\Delta =$ tubular neighborhood of Δ

$\approx$ normal bundle of embedding $\Delta M \subset M \times M$

$\cong$ tangent bundle TM

$$\begin{aligned}\widetilde{\nu} &= (\epsilon_1 \times \epsilon_0)^{-1}(\nu_\Delta) \quad \subset \quad \mathcal{P}_g(M) \times \mathcal{P}_h(M) \\ &= ev_0^*(TM)\end{aligned}$$

Have a Thom–collapse map:

$$\mathcal{P}_g \times \mathcal{P}_h \longrightarrow \mathcal{P}_g \times \mathcal{P}_h \,/\, (\mathcal{P}_g \times \mathcal{P}_h - \widetilde{\nu}) \approx (\mathcal{P}_g \ {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_h)^{\epsilon_\infty^*(TM)}$$

$$\begin{tikzcd}
\mathcal{P}_g \times \mathcal{P}_h \arrow[r, "\widetilde{\tau}"] \arrow[d, "\epsilon_1 \times \epsilon_0"'] & (\mathcal{P}_g \ {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_h)^{TM} \arrow[r, "\widetilde{*}"] \arrow[d, "\epsilon_\infty"'] & (\mathcal{P}_{gh})^{TM} \arrow[d, "\epsilon_{1/2}"] \\
M \times M \arrow[r, "\tau"] & M^{TM} \arrow[r, equal] & M^{TM}
\end{tikzcd}$$

Consider the composite:

$$H_p(\mathcal{P}_g) \otimes H_q(\mathcal{P}_h) \xrightarrow{\times} H_{p+q}(\mathcal{P}_g \times \mathcal{P}_h) \xrightarrow{\widetilde{*} \circ \tau} H_{p+q}(\mathcal{P}_{gh}^{TM}) \xrightarrow{u_*} H_{p+q-d}(\mathcal{P}_{gh})$$

Taking direct sum over all $g \in G$:

$$\star \quad : \quad H_p(\mathcal{P}_G) \otimes H_q(\mathcal{P}_G) \longrightarrow H_{p+q-d}(\mathcal{P}_G)$$

To obtain an honest orbifold invariant, need also:

1. A good morphism: $\qquad \theta_* : H_*(\mathcal{L}B\mathsf{X}) \to H_*(\mathcal{P}_G)$

$$\begin{tikzcd}
H_*(\mathcal{P}_G \times EG) \arrow[rr, "\pi_*"] \arrow[d] & & H_*(\mathcal{P}_G) \\
H_*(\mathcal{P}_G \times_G EG) \arrow[u, bend right] \arrow[urr, "\theta_*"'] & &
\end{tikzcd}$$

2. The map induced by:

$$\sigma \ : \ \mathcal{P}_G \longrightarrow \mathcal{P}_G \times_G EG$$

Definition: The orbifold string product

$$\circ \quad : \quad H_p(\mathcal{L}B\mathsf{X}) \otimes H_q(\mathcal{L}B\mathsf{X}) \longrightarrow H_{p+q-d}(\mathcal{L}B\mathsf{X})$$

is defined by the following composite:

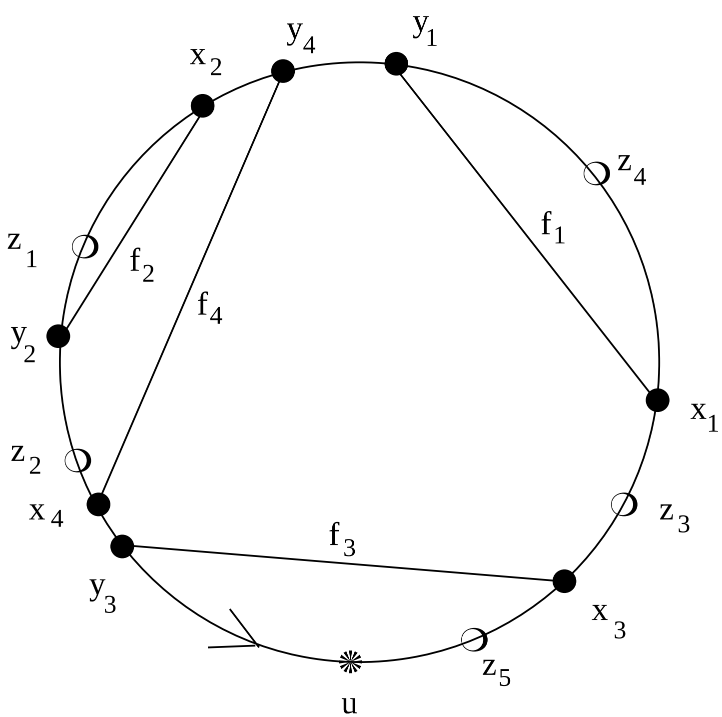
$$H_*(\mathcal{P}_G \times_G EG)^{\otimes 2} \xrightarrow{\theta_* \otimes \theta_*} H_*(\mathcal{P}_G)^{\otimes 2} \xrightarrow{\star} H_{*-d}(\mathcal{P}_G) \xrightarrow{\sigma_*} H_{*-d}(\mathcal{P}_G \times_G EG$$

Theorem (L-U-X): For $\mathsf{X} = [M/G]$, the homology $H_*(\mathcal{L}B\mathsf{X})$ has a BV-algebra structure and realizes the Chas-Sullivan construction when $G = \{e\}$.

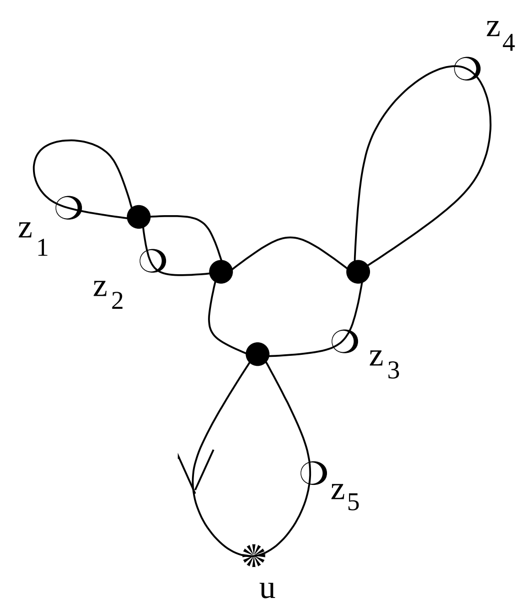
6 Operadic structure

- **Cohen, Godin, Voronov:** BV-structure on $H_*(\mathcal{L}M)$ comes from an action of the *flat chord diagram operad* $\mathcal{MC}_*$ on $\mathcal{L}M$

Marked chord diagram



"Cactus" associated



- **G-marked chord diagrams:** Principal G-bundles / $\mathbf{c} \in \mathcal{MC}_*$

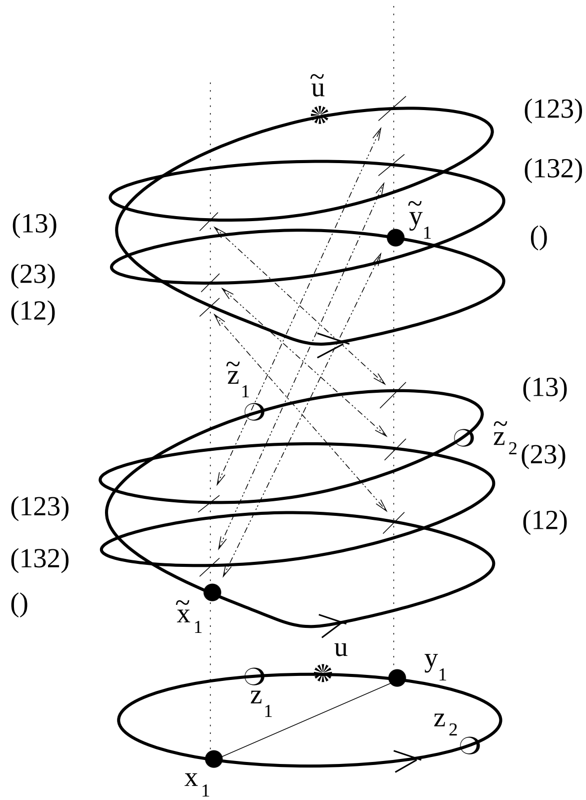


Fig. 2: Σ_3-chord diagram over marked chord diagram.

- $G\mathcal{MC}_*$ acts on $\mathrm{Bun}_G(S^1, M)$ and induces BV-structure on $H_*(\mathcal{L}M_G)$.