Pointed harmonic volumes of hyperelliptic curves and the hyperelliptic mapping class group

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Dedicated to Professor Fred Cohen on his sixtieth birthday

1. Pointed harmonic volume

- X : compact Riemann surface of genus $g \geq 3$.
- $p \in X$: point.
- $H^1(X, \mathbb{Z}) = H_1(X, \mathbb{Z}) = H$. (Poincaré duality)
- $*$: Hodge $*$-operator.
  $H \cong$ { real harmonic 1-forms with $\mathbb{Z}$-periods }.
- $K :=$ kernel of the intersection pairing
  $H \otimes H \to \mathbb{Z}$.

Pointed harmonic volume

$I_p : K \otimes H \to \mathbb{R}/\mathbb{Z}$: homomorphism,

$$I_p(\omega_1 \otimes \omega_2 \otimes \omega_3) = \underbrace{\int_{\gamma_3} \omega_1 \omega_2}_{\text{Chen's iterated integral}} - \int_{\gamma_3} \eta.$$

- $\omega_1, \omega_2, \omega_3$: real harmonic 1-forms on X.
- γ_3: loop in X with the base point p and (Poincaré dual of $[\omega_3]$) $= [\gamma_3]$.
- η: 1-form on X such that

$$\begin{cases} d\eta = \omega_1 \wedge \omega_2, \\ \int_X \eta \wedge *\alpha = 0 \ (\forall \alpha : \text{closed 1-form on X}). \end{cases}$$

2 Facts

Motivation

Pointed Torelli theorem, (Pulte(1988))

$(X,p),(Y,q)$: pointed compact Riemann surfaces.
(With the possible exception of two points p)
J_p, J_q : augmentation ideals of the group rings $\mathbb{Z}\pi_1(X,p), \mathbb{Z}\pi_1(Y,q)$.

$\mathbb{Z}\pi_1(X,p)/J_p^3 \cong \mathbb{Z}\pi_1(Y,q)/J_q^3$
isomorphism of mixed Hodge structures
(Using the pointed harmonic volume)

$$\Rightarrow (X,p) \cong (Y,q).$$

Past result, (Tadokoro(2002))

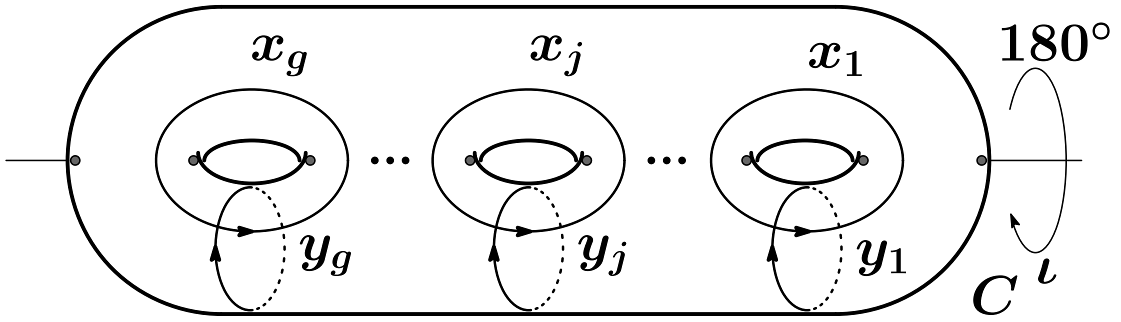
We determined the harmonic volumes I of all the hyperelliptic curves (I is a restriction map of I_p and does not depend on the choice of the base point).

Remark I and I_p depend only on the complex structure (and the base point).

Reference *The harmonic volumes of hyperelliptic curves*, to appear in Publications of RIMS.

For example

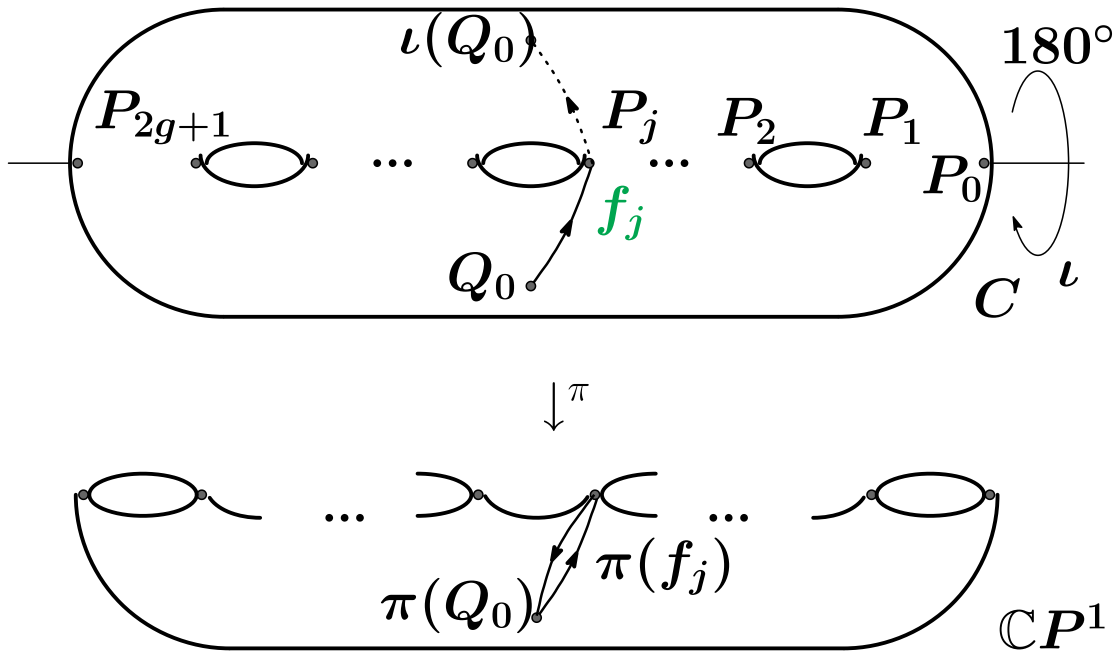
$$I\big((x_i \otimes y_i - x_3 \otimes y_3) \otimes y_2\big) = \begin{cases} 1/2 & (i=1) \\ 0 & (i \geq 2) \end{cases}$$



3. The pointed harmonic volumes of hyperelliptic curves with Weierstrass points

Hyperelliptic curve

$\pi : C \longrightarrow \mathbb{C}P^1$: 2-fold branch covering.



- $\iota \curvearrowright C$: hyperelliptic involution.
- P_ν: Weierstrass points (branch points of π).

<u>Fact</u> $I_\nu = I_{P_\nu} = 0$ or $1/2 \bmod Z$, using ι

Main Theorem (2005)

We determine the pointed harmonic volumes $I_\nu = I_{P_\nu}$ of all the hyperelliptic curves with Weierstrass points.

<u>For example</u>

$$I_\nu(x_i \otimes x_j \otimes y_i) = \begin{cases} 1/2 & (\nu = 2j-1 \text{ or } 2j), \\ 0 & (\text{otherwise}). \end{cases}$$

$$I_\nu((x_i \otimes y_i - x_1 \otimes y_1) \otimes y_i) = \begin{cases} 0 & (\nu > 2i-1), \\ 1/2 & (\nu \leq 2i-1). \end{cases}$$

4. Combinatorial formula and the hyperelliptic mapping class group Δ_g

- Σ_g: oriented closed surface of genus $g \geq 3$.
- $\Gamma_g = \mathrm{Diff}_+(\Sigma_g)/\mathrm{Diff}_0(\Sigma_g)$: mapping class group.
- $\Delta_g = \{\varphi \in \Gamma_g;\ \varphi\iota = \iota\varphi\}$: hyperelliptic M.C.G.
- $\Delta_{g,\nu} = \{\varphi \in \Delta_g;\ \varphi(P_\nu) = P_\nu\} \subset \Delta_g$.

Key fact

$$I_\nu = I_{P_\nu} \in H^0(\Delta_{g,\nu}; \mathrm{Hom}(K \otimes H, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

- $f_0, f_1, \ldots, f_{2g+1} \in H_1\big(C, \{Q_0, \iota(Q_0)\}; \mathbb{Z}/2\mathbb{Z}\big)$: paths in Figure in 3.

Identification

$$\left.\begin{aligned} x_i &= f_{2i-1} + f_{2i} \\ y_i &= f_0 + f_1 + \cdots + f_{2i-1} \end{aligned}\right\} \in H_1(C; \mathbb{Z}/2\mathbb{Z})$$

Counting function

$n\colon K \otimes H \ni A \mapsto n(A) \in \mathbb{Z}_{\geq 0}$: counting function

$$\Big(A = \textstyle\sum_{p,q,r} A_{p,q,r} f_p \otimes f_q \otimes f_r \in (K \otimes H) \otimes \mathbb{Z}/2\mathbb{Z}\Big)$$

$$n(A) \overset{\text{def.}}{=} \sharp\Big\{(p,q,r)\Big|\ A_{p,q,r} = 1 \text{ and } \diamondsuit\Big\}$$

Combinatorial formula for I_ν

$$I_\nu(A) = \begin{cases} 0 & (n(A)\colon \text{even}), \\ 1/2 & (n(A)\colon \text{odd}). \end{cases}$$

$\diamondsuit \cdots\ p, q, r \neq \nu$ and $\sharp\{p,q,r\} = 2$.