A partial order in the knot table

Teruaki Kitano and Masaaki Suzuki

K : a prime knot

$G(K)$: the knot group of K i.e. $G(K) = \pi_1(E(K)) = \pi_1(S^3 - K)$

Definition.

K, K' : two prime knots

If there exists a surjective homomorphism $G(K) \twoheadrightarrow G(K')$,

$$K \geq K'$$

This relation "$\geq$" is a partial order on the set of prime knots.

We determine this partial order "$\geq$" on the set of knots in Rolfsen's knot table, which lists all the prime knots of 10 crossings or less. The numbering of the knots follows that of Rolfsen's book.

Main Theorem.

The above partial order on the knots in Rolfsen's table is given as below:

$$\left.\begin{array}{l} 8_5, 8_{10}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}, 9_1, 9_6, 9_{16}, 9_{23}, 9_{24}, 9_{28}, 9_{40}, \\ 10_5, 10_9, 10_{32}, 10_{40}, 10_{61}, 10_{62}, 10_{63}, 10_{64}, 10_{65}, 10_{66}, 10_{76}, 10_{77}, \\ 10_{78}, 10_{82}, 10_{84}, 10_{85}, 10_{87}, 10_{98}, 10_{99}, 10_{103}, 10_{106}, 10_{112}, 10_{114}, \\ 10_{139}, 10_{140}, 10_{141}, 10_{142}, 10_{143}, 10_{144}, 10_{159}, 10_{164} \end{array}\right. \geq 3_1,$$

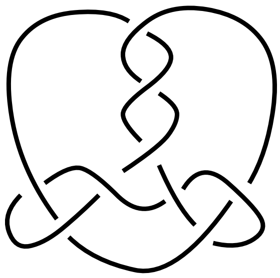
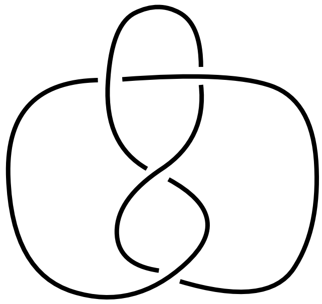
$$8_{18}, 9_{37}, 9_{40}, 10_{58}, 10_{59}, 10_{60}, 10_{122}, 10_{136}, 10_{137}, 10_{138} \geq 4_1,$$

$$10_{74}, 10_{120}, 10_{122} \geq 5_2.$$

Existence of surjective homomorphisms

we can construct explicitly a surjective homomorphism between the groups of each pair of knots which appears in the list of the main theorem.

For example, we show a surjective homomorphism from $G(9_{37})$ to $G(4_1)$.

$9_{37} =$  , $4_1 =$ 

We fix presentations of $G(9_{37})$ and $G(4_1)$ as follows:

$$G(9_{37}) = \left\langle \begin{array}{c} y_1, y_2, y_3, y_4, y_5, \\ y_6, y_7, y_8, y_9 \end{array} \,\middle|\, \begin{array}{c} y_8y_1y_8^{-1}y_2^{-1}, y_7y_2y_7^{-1}y_3^{-1}, y_9y_4y_9^{-1}y_3^{-1}, y_3y_4y_3^{-1}y_5^{-1}, \\ y_1y_6y_1^{-1}y_5^{-1}, y_5y_6y_5^{-1}y_7^{-1}, y_2y_7y_2^{-1}y_8^{-1}, y_4y_9y_4^{-1}y_8^{-1} \end{array} \right\rangle,$$

$$G(4_1) = \langle x_1, x_2, x_3, x_4 \mid x_4x_2x_4^{-1}x_1^{-1},\ x_1x_2x_1^{-1}x_3^{-1},\ x_2x_4x_2^{-1}x_3^{-1} \rangle.$$

The following mapping $\varphi : G(9_{37}) \to G(4_1)$ is a surjective homomorphism:

$$\varphi(y_1) = x_2, \varphi(y_2) = x_3, \varphi(y_3) = x_1x_4x_1^{-1}, \varphi(y_4) = x_3, \varphi(y_5) = x_1,$$
$$\varphi(y_6) = x_1^{-1}x_4x_1, \varphi(y_7) = x_4, \varphi(y_8) = x_1, \varphi(y_9) = x_4.$$

Then we get

$$9_{37} \geq 4_1.$$

Non-existence of surjective homomorphisms

we prove the non-existence of surjective homomorphisms by using the Alexander polynomial and the twisted Alexander invariants.

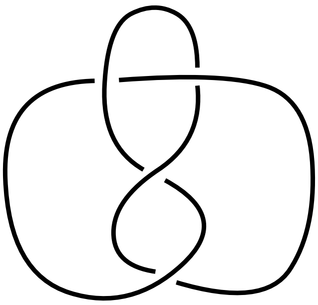
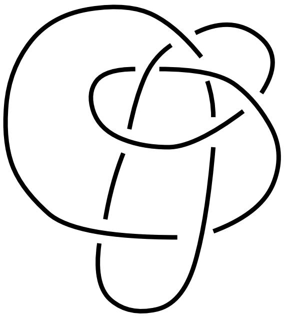
(1) By the (classical) Alexander polynomial

Δ_K : the Alexander polynomial of a knot K

Fact.

If Δ_K can not be divided by $\Delta_{K'}$,
then there exists no surjective homomorphism $\varphi : G(K) \twoheadrightarrow G(K')$.

For example, we consider whether there exists a surjective homomorphism between $G(4_1)$ and $G(8_{21})$.

$4_1 =$  , $8_{21} =$ 

$$\Delta_{4_1} = t^2 - 3t + 1, \qquad \Delta_{8_{21}} = t^4 - 4t^3 + 5t^2 - 4t + 1$$

Thus Δ_{4_1} can not be divided by $\Delta_{8_{21}}$. We get $4_1 \not\geq 8_{21}$. However, $\Delta_{8_{21}}$ can be divided by Δ_{4_1}. In fact,

$$\frac{\Delta_{8_{21}}}{\Delta_{4_1}} = t^2 - t + 1.$$

Then we can not determine whether there exists a surjective homomorphism from $G(8_{21})$ to $G(4_1)$ by the Alexander polynomial.

(2) By the twisted Alexander invariant

$\Delta_{K,\rho}$: the twisted Alexander invariant of a knot K
associated to a representation $\rho : G(K) \to SL(2;\mathbb{Z}/p\mathbb{Z})$

$\Delta^N_{K,\rho}, \Delta^D_{K,\rho}$: the numerator and the denominator of $\Delta_{K,\rho}$ respectively

Theorem.(Kitano-Suzuki-Wada)

If there exists a representation $\rho' : G(K') \to SL(2;\mathbb{Z}/p\mathbb{Z})$ such that for any representation $\rho : G(K) \to SL(2;\mathbb{Z}/p\mathbb{Z})$,

$$\Delta^N_{K,\rho} \text{ is not divisible by } \Delta^N_{K',\rho'} \text{ or } \Delta^D_{K,\rho} \neq \Delta^D_{K',\rho'},$$

then there exists no surjective homomorphism $\varphi : G(K) \twoheadrightarrow G(K')$.

For example, we prove the non-existence of a surjective homomorphism from $G(8_{21})$ to $G(4_1)$. We have a certain representation $\rho' : G(4_1) \longrightarrow SL(2;\mathbb{Z}/3\mathbb{Z})$ such that the twisted Alexander invariant is $\Delta^N_{4_1,\rho'} = t^4+t^2+1, \Delta^D_{4_1,\rho'} = t^2+t+1$. On the other hand, for any $\rho : G(8_{21}) \longrightarrow SL(2;\mathbb{Z}/3\mathbb{Z})$, there exists no $\Delta^D_{8_{21},\rho}, \Delta^N_{8_{21},\rho}$ such that $\Delta^D_{8_{21},\rho} = \Delta^D_{4_1,\rho'}$ and $\Delta^N_{8_{21},\rho}$ can be divided by $\Delta^N_{4_1,\rho'}$. Then we get

$$8_{21} \not\geq 4_1.$$

In fact, all the twisted Alexander invariants $\Delta^D_{8_{21},\rho}, \Delta^N_{8_{21},\rho}$ are as follows:

	$\Delta^N_{8_{21},\rho_i}$	$\Delta^D_{8_{21},\rho_i}$
ρ_1	t^8+t^4+1	t^2+1
ρ_2	$t^8+t^7+2t^6+2t^4+2t^2+t+1$	t^2+t+1
ρ_3	$t^8+t^7+2t^6+2t^4+2t^2+t+1$	t^2+2t+1
ρ_4	$t^8+2t^7+2t^6+2t^4+2t^2+2t+1$	t^2+t+1
ρ_5	$t^8+2t^7+2t^6+2t^4+2t^2+2t+1$	t^2+2t+1