A Reduction Theorem for Fusion Systems of Blocks

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1. Introduction

Let p be a prime number.

Fusion systems:

- Puig, 1990: *full Frobenius systems*
- Broto, Levi, Oliver, 2000: *saturated fusion systems*
  - provide an axiomatic framework for studying p-fusion in finite groups
  - useful in determining many properties of finite groups and of the p-completion of their classifying spaces
  - underlie the theory of p-local finite groups

2. Fusion Systems

Definition: A *category* $\mathcal{F}$ *on a finite p-group* P

objects: the subgroups of P

morphisms: between Q and R, is the set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ of injective group homomorphisms from Q to R, with the following properties:

(a) if $Q \leq R$ then the inclusion of Q in R is a morphism in $\mathrm{Hom}_{\mathcal{F}}(Q, R)$.

(b) for any $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ the induced isomorphism $Q \simeq \phi(Q)$ and its inverse are morphisms in $\mathcal{F}$.

(c) composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

Definition. A *fusion system $\mathcal{F}$ on P* is a category on P satisfying:

(1) $\mathrm{Hom}_P(Q,R) \subset \mathrm{Hom}_{\mathcal{F}}(Q,R)$ for all $Q,R \leq P$.

(2) $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

(3) Every $\phi \in \mathrm{Iso}_{\mathcal{F}}(Q,R)$ such that $N_P(R)$ is maximal in the $\mathcal{F}$-isomorphism class of R extends to $\mathrm{Hom}_{\mathcal{F}}(N_P(Q), N_P(R))$ (up to modifying ϕ by an element of $\mathrm{Aut}_{\mathcal{F}}(R)$)

Example. Let G be a finite group and $P \in \mathrm{Syl}_p(G)$. Then $\mathcal{F} := \mathcal{F}_P(G)$ is a fusion system on P (objects: subgroups of P; morphisms: the conjugations by elements of G).

3. Brauer Category

Let k be an algebraically closed field of characteristic p.

The group algebra kG decomposes into *block algebras*

$$kG = kGb_1 \bigoplus kGb_2 \bigoplus \cdots \bigoplus kGb_n$$

b_i is called a *block* of kG; it is a central primitive idempotent (i.e. $b_i \in Z(kG)$ and $b_i b_i = b_i$).

To every block b of kG we associate up to isomorphism a *defect group* P which is a p-subgroup of G maximal under the assumption that $\mathrm{Br}_P(b) \neq 0$
($\mathrm{Br}_P : (kG)^P \to kC_G(P)$ is the Brauer morphism).

Definition. [Brauer, Broué]
A *b-Brauer pair* (Q, e_Q) consists of:
- a p-subgroup Q of G, $\mathrm{Br}_Q(b) \neq 0$;
- a block e_Q of $kC_G(Q)$, $\mathrm{Br}_Q(b)e_Q \neq 0$.

There exists a partial order $\leq$ on the set of b-Brauer pairs. Here are some of its important properties:

(a) there exists e_P such that (P, e_P) is maximal with respect to this partial order iff P is a b-defect.

(b) if $(Q, e_Q) \leq (R, e_R)$ then $Q \leq R$.

(c) for any b-Brauer pair (R, e_R) and $Q \leq R$ there exists an unique b-Brauer pair $(Q, e_Q) \leq (R, e_R)$.

${}^{g}(P, e_P) = ({}^{g}P, {}^{g}e_P)$ is also a b-Brauer pair, so G acts by conjugation on the set of b-Brauer pairs. The maximal b-Brauer pairs are in the same G-conjugacy class.

Proposition. The category $\mathcal{F}_{(P,e_P)}(G, b)$ of b-Brauer pairs contained in a maximal b-Brauer pair (P, e_P), with the action of G by conjugation is a fusion system on P.

Definition. We say that b is a $\mathcal{F}$*-block* if $\mathcal{F}_{(P,e_P)}(G, b)$ is isomorphic to $\mathcal{F}$.

4. Main Result

Let $\mathcal{F}$ be a fusion system on a finite p-group P and $P' \leq P$.

Definition. We say that P' is *strongly $\mathcal{F}$-closed* if for any subgroup R of P' and any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ we have $\phi(R) \leq P'$.

Definition. Let $\mathcal{F}'$ a fusion subsystem of $\mathcal{F}$ on P'. We say that $\mathcal{F}'$ is *normal* in $\mathcal{F}$ if P' is strongly $\mathcal{F}$-closed and if for every $\phi \in \mathrm{Iso}_{\mathcal{F}}(Q, Q')$ and any two subgroups R, R' of $Q \cap P'$ we have

$$\phi \circ \mathrm{Hom}_{\mathcal{F}'}(R, R') \circ \phi^{-1} \subseteq \mathrm{Hom}_{\mathcal{F}'}(\phi(R), \phi(R'))\,.$$

Theorem 1 [Kessar, S. 05]. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two fusion systems on P, $\mathcal{F}_1$ containing $\mathcal{F}_2$. Suppose that:

a) P has no proper non-trivial strongly $\mathcal{F}_2$-closed subgroup.

b) if $\mathcal{F}$ is a fusion system on P containing $\mathcal{F}_2$, then $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$.

c) if $\mathcal{F}$ is a non-trivial fusion system normal in $\mathcal{F}_1$ or $\mathcal{F}_2$ then $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$.

If there exists a finite group G having an $\mathcal{F}_1$ or an $\mathcal{F}_2$-block then there exists a quasi-simple group L with $Z(L)$ a p'-group having an $\mathcal{F}_1$ or an $\mathcal{F}_2$-block.

Ruiz and Viruel [2000] classified all possible fusion systems on extra-special p-groups of order p^3 for p odd.

They found three *exotic* fusion systems on the extraspecial 7-group of order 7^3 and exponent 7.

exotic = not from the p-local structure of a finite group.

Theorem 2 [Kessar, S. 05]. Let $\mathcal{F}$ be an exotic fusion system on the extra-special group of order 7^3 and exponent 7. Then $\mathcal{F}$ is not a fusion system of a 7 block of any finite group.

Settings for the proof of Theorem 1:
- k be an algebraically closed field of characteristic p,
- G a finite group,
- N a normal subgroup of G,
- c a G-stable block of kN (i.e. fixed by G-conjugation).

Definition. A (c, G)-*Brauer pair* is a pair (Q, e_Q)
- Q is a p-subgroup of G, $\mathrm{Br}_Q^N(c) \neq 0$
- e_Q is a block of $kC_N(Q)$, $\mathrm{Br}_Q^N(c)e_Q \neq 0$.

When $G = N$, a (c, G)-Brauer pair is a c-Brauer pair.

Let (P, e_P) a maximal (c, G)-Brauer pair. Similarly to the case of the Brauer category we define the *generalized Brauer category* $\mathcal{F}_{(P,e_P)}(G, N, c)$. This is a fusion system on P.

Proof of the Theorem 1 (sketch).

Let G be a minimal order group having an $\mathcal{F}_1$ or an $\mathcal{F}_2$-block b.

P is a b-defect group and let $H := < {}^gP \,|\, g \in G >$.

Step 1: prove that $G = H$.

Let d be a block of kH covered by b.

Intermediate step: Consider $N := \ker(G \longrightarrow \mathrm{Out}(kHd))$

Step 1a: prove that $N = G$. Let c be a block of kN covered by b.

$$\begin{tikzcd}
 & \begin{array}{c}\mathcal{F}_{(P,e'_P)}(G,N,c)\\ \mathcal{F}_1 \text{ or } \mathcal{F}_2\end{array} \arrow[dl, Rightarrow, "b)"] \arrow[dr, Rightarrow, "c)"] & \\
\begin{array}{c}\mathcal{F}_1 \text{ or } \mathcal{F}_2\\ \mathcal{F}_{(P,e_P)}(G,b)\end{array} & & \begin{array}{c}\mathcal{F}_1 \text{ or } \mathcal{F}_2\\ \mathcal{F}_{(P,e'_P)}(N,c)\end{array}
\end{tikzcd}$$

As b and d have the same defect group P and G acts on kHd by inner automorphisms, using a result of Külshammer, we have that kGb and kHd have isomorphic source algebras, so d is also a $\mathcal{F}_1$ or $\mathcal{F}_2$-block.

Step 2. Now $G = <{}^gP \mid g \in G>$. Let M be a proper normal subgroup of G. Then $P \cap M$ is a strongly $\mathcal{F}_1$ or $\mathcal{F}_2$-closed subgroup.

a) $P \cap M = P$. Then $G = M$ as all the G-conjugated of P are contained in M; contradiction.

b) $P \cap M = 1$. A variation of Fong reduction allows us to deduce that there is a central p'-extension G' of G/M having an $\mathcal{F}_1$ or $\mathcal{F}_2$-block.

$\square$