On the automorphism group of a free group

Dedicated to Professor F. R. Cohen for his 60th birthday

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1 Twisted homology group of the automorphism group of a free group

Let F_n be a free group of rank n and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. Let $\Sigma_{g,1}$ be a connected oriented surface of genus g with one boundary and $\mathcal{M}_{g,1}$ the mapping class group of it. Nielsen showed that there is a natural inclusion map $\iota : \mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}\, F_{2g}$. We interested in the behavior of the homomorphisms on the twisted (co)homology groups induced from ι.

We denote by H and H^* the abelianized group of F_n and its dual group respectively. Let $\mathrm{Out}\, F_n$ be the outer automorphism group of F_n. The groups $\mathrm{Aut}\, F_n$ and $\mathrm{Out}\, F_n$ naturally act on these groups. Here we calculate the twisted first homology groups of these groups with coefficients in H and H^*. We have

Theorem 1 (S. 2002)

$$H_1(\mathrm{Aut}\, F_n, H) = \begin{cases} 0 & \text{if } n \geq 4, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 3, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2, \end{cases}$$

$$H_1(\mathrm{Out}\, F_n, H) = \begin{cases} 0 & \text{if } n \geq 4, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2,\ 3. \end{cases}$$

Theorem 2 (S. 2002)

$$H_1(\mathrm{Aut}\, F_n, H^*) = \begin{cases} \mathbb{Z} & \text{if } n \geq 4, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2,\ 3. \end{cases}$$

$$H_1(\mathrm{Out}\, F_n, H^*) = \begin{cases} \mathbb{Z}/(n-1)\mathbb{Z} & \text{if } n \geq 4, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 3, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2. \end{cases}$$

To compute these homology group, we use a finite presentation for $\mathrm{Aut}\, F_n$ obtained by Gersten.

S. Morita [3] showed that $H_1(\mathcal{M}_{g,1}, H^*) \simeq \mathbb{Z}$ for $g \geq 2$ and We showed that the induced homomorphism

$$\iota_* : H_1(\mathcal{M}_{g,1}, H^*) \xrightarrow{\simeq} H_1(\mathrm{Aut}\, F_n, H^*)$$

is an isomorphism.

For the ring $A = \mathbb{Z}[\frac{1}{2}]$, set $H_A := H \otimes_{\mathbb{Z}} A$. Recently, we obtained

Theorem 3 (S. 2005) *For $n \geq 6$,*

$$H_2(\mathrm{Aut}\, F_n, H_A) = 0.$$

To compute this second homology group, we also use the Gersten's presentation.

2 The Johnson homomorphism of the automorphism group of a free group

Let $\Gamma_n(1),\ \Gamma_n(2),\ldots$ be the lower central series of F_n. For each $k \geq 1$, set $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$. Let $\mathcal{A}_n(k)$ be the kernel of a natural homomorphism $\mathrm{Aut}\, F_n \longrightarrow \mathrm{Aut}(F_n/\Gamma_n(k+1))$ for each $k \geq 0$. Then we have a descending series

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots .$$

S. Andreadakis [1] showed that this series is central. So $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is defined for $k \geq 1$.

We consider the $GL(n,\mathbb{Z})$-equivariant injective homomorphism

$$\tau_n(k) : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1)$$

defined by $[\sigma] \mapsto ([x] \mapsto [x^{-1}x^{\sigma}])$. This map is called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$.

Problem *Determine the structure of the cokernel of* $\tau_n(k)$.

First, we obtained

Theorem 4 (S. 2004) *For $n \geq 3$, we have $GL(n,\mathbb{Z})$-equivariant exact sequences*

$$0 \to \mathrm{gr}^2(\mathcal{A}_n) \xrightarrow{\tau_n(2)} H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(3) \to S^2 H \to 0,$$

$$0 \to \mathrm{gr}^3_{\mathbb{Q}}(\mathcal{A}_n) \xrightarrow{\tau_n(3)_{\mathbb{Q}}} H^*_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathcal{L}^{\mathbb{Q}}_n(4) \to S^3 H_{\mathbb{Q}} \oplus \Lambda^3 H_{\mathbb{Q}} \to 0.$$

Here the subscript $\mathbb{Q}$ of a module means the tensor products with $\mathbb{Q}$ of the module over $\mathbb{Z}$.

Recently, S. Morita introdued a certain $GL(n, \mathbb{Z})$-equivariant surjective homomorphism $\mathrm{Tr}_{[k]} : H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1) \to S^k H$ which vanishes on the image of $\tau_n(k)$. Hence $S^k H_{\mathbb{Q}}$ appears in the cokernel of $\tau_n(k)_{\mathbb{Q}}$. The map $\mathrm{Tr}_{[k]}$ is called Morita's trace map.

Let $\mathcal{A}'_n(1)$, $\mathcal{A}'_n(2)$, $\ldots$ be the lower central series of $\mathcal{A}_n(1)$. Then we can also define a $\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1)$. and also call it the Johnson homomorphism of $\mathrm{Aut}\, F_n$. Since it is conjectured that $\mathcal{A}'_n(k)_{\mathbb{Q}} = \mathcal{A}_n(k)_{\mathbb{Q}}$ for all k, so it is important to consider the map τ'_k.

Recently, we introduced a certain $GL(n, \mathbb{Z})$-equivariant surjective homomorphism

$$\mathrm{Tr}_{[1^k]} : H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1) \to \Lambda^k H$$

such that $\mathrm{Tr}_{[1^k]} \circ \tau'_n(k) = 0$ for any odd k. Then we have

Theorem 5 (S. 2004) *For any odd k,*

$$\Lambda^k H_{\mathbb{Q}} \subset \mathrm{Cok}(\tau'_n(k)_{\mathbb{Q}}).$$

References

[1] S. Andreadakis; On the automorphisms of free groups and free nilpotent groups, Proc. London Math. Soc. (3) 15 (1965), 239-268.

[2] N. Kawazumi; Cohomological aspects of Magnus expansions, preprint UTMS, 2005-18.

[3] S. Morita; Families of Jacobian manifolds and characteristic classes of surface bundles I, Ann. Inst. Fourier 39 (1989), 777-810.