The Magnus representation and higher-order degrees for homology cylinders

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Definitions

- $\Sigma_{g,1}$: a compact connected oriented surface of genus $g \geq 1$
  with one boundary component.

- A homology cylinder (M, i_+, i_-) over $\Sigma_{g,1}$ ([2], [1], [6]) consists of

$$\begin{cases} M & : \text{ a compact oriented 3-manifold,} \\ i_+, i_- & : \text{ two embeddings } \Sigma_{g,1} \hookrightarrow \partial M \end{cases} \quad \text{satisfying that}$$

1. i_+ is orientation-preserving and i_- is orientation-reversing,
2. $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$ and $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$,
3. $i_+|_{\partial\Sigma_{g,1}} = i_-|_{\partial\Sigma_{g,1}}$,
4. $i_+, i_- : H_*\Sigma_{g,1} \to H_*M$ are isomorphisms.

Namely, it is a homology cobordism of $\Sigma_{g,1}$ with markings of its boundary.

- $\mathcal{C}_{g,1}$: the set of all diffeomorphism classes of homology cylinders
  over $\Sigma_{g,1}$.
  $\Longrightarrow \mathcal{C}_{g,1}$ has a natural monoid structure.
  unit : $(\Sigma_{g,1} \times I, \mathrm{id} \times 1, \mathrm{id} \times 0)$

- The mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_{g,1}$ is embedded in $\mathcal{C}_{g,1}$ by assigning $(\Sigma_{g,1} \times I, \mathrm{id} \times 1, \varphi \times 0)$ to $\varphi \in \mathcal{M}_{g,1}$.

- By Stallings' theorem, for every $(M, i_+, i_-) \in \mathcal{C}_{g,1}$,

$$i_+, i_- \ : \ N_k(\pi_1\Sigma_{g,1}) \xrightarrow{\cong} N_k(\pi_1 M), \quad \text{for } k \geq 2$$

where $N_k(\pi_1\Sigma_{g,1})$, $N_k(\pi_1 M)$ are the k-th nilpotent quotients of $\pi_1\Sigma_{g,1}$ and $\pi_1 M$. This gives a monoid homomorphism

$$\sigma_k : \mathcal{C}_{g,1} \longrightarrow \mathrm{Aut} N_k, \quad \big((M, i_+, i_-) \mapsto (i_+)^{-1} \circ i_-\big)$$

for each $k \geq 2$. Then

$$\mathcal{C}_{g,1}[1] := \mathcal{C}_{g,1}, \quad \mathcal{C}_{g,1}[k] := \mathrm{Ker}\, \sigma_k \ \text{ for } k \geq 2$$

defines a filtration of $\mathcal{C}_{g,1}$.

The Magnus representation For every $(M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$,

Lemma 1. $H_*(M, i_\pm(\Sigma_{g,1}); \mathcal{K}_{N_k}) = 0$,

$$\begin{aligned} \text{where } N_k &:= N_k(\pi_1\Sigma_{g,1}) \xrightarrow[i_+=i_-]{\cong} N_k(\pi_1 M), \\ \mathcal{K}_{N_k} &:= \mathbb{Z}N_k(\mathbb{Z}N_k - \{0\})^{-1}. \ \text{(the right quotient field)} \end{aligned}$$

- The Magnus matrix $r_k(M) \in GL(2g, \mathcal{K}_{N_k})$ of $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$ is the representation matrix of the composite of isomorphisms

$$\mathcal{K}_{N_k}^{2g} \cong H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k}) \xrightarrow[i_+^{-1} \circ i_-]{\cong} H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k}) \cong \mathcal{K}_{N_k}^{2g}.$$

$\Longrightarrow$ $r_k : \mathcal{C}_{g,1}[k] \longrightarrow GL(2g, \mathcal{K}_{N_k})$ becomes a monoid homomorphism, and we call it the Magnus representation.

Rem. r_k factors through the homology cobordism group $\mathcal{H}_{g,1}$ of homology cylinders, namely it is homology cobordism invariant.

- By taking the Dieudonné determinant of $(2g-1)$-minor of $I_{2g} - r_k(M)$ (with some adjustment), we can define the N_k-Alexander rational function

$$\Delta_{N_k}(M) \in (\mathcal{K}_{N_k}^{\times})_{\mathrm{ab}} \cup \{\overline{0}\}.$$

These invariants generalize those for string links in [5].

N_k-torsion Lemma 1 admits us to define the Reidemeister torsion

$$\tau_{N_k}(M) \in K_1(\mathcal{K}_{N_k})/(\pm N_k)$$

of the acyclic complex $C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_k})$ for every $(M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$. We call it the N_k-torsion of (M, i_+, i_-).

Rem. For $M = (M, i_+, i_-) \in \mathcal{M}_{g,1} \cap \mathcal{C}_{g,1}[k]$, we have $\tau_{N_k}(M) = 1$.

Harvey's higher-order degree [3], [4] For each finitely presentable group G (or a finite CW-complex X with $\pi_1 X = G$), Harvey's higher-order degree is defined by

$$\overline{\delta}^{\psi}_{\Gamma}(G) = \overline{\delta}^{\psi}_{\Gamma}(X) := \mathrm{rank}_{\mathbb{K}^{\psi}_{\Gamma}} H_1(G; \mathbb{K}^{\psi}_{\Gamma}[t^{\pm}]) \ \in \mathbb{Z}_{\geq 0} \cup \{\infty\},$$

where (Γ, ψ) is a pair of a group and a primitive element of $H^1 G$ with certain conditions, and $\mathbb{K}^{\psi}_{\Gamma}[t^{\pm}]$ is the skew Laurent polynomial ring of the skew field $\mathbb{K}^{\psi}_{\Gamma}$ defined by the pair.

Rem. When $\Gamma = H_1 G/(\text{torsion})$, this invariant is nothing other than the degree of the Alexander polynomial w.r.t. ψ.

Main results For $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$, we define

$$C_M := M/(i_+(x) = i_-(x)), \quad x \in \Sigma_{g,1},$$

and call it the closing of M. Note that $N_i(\pi_1 \Sigma_{g,1}) = N_i(\pi_1 C_M)$ for $i \leq k$.

Theorem 1. For $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$ and a primitive element $\psi \in H^1 C_M = H^1 \Sigma_{g,1}$,

$$\overline{\delta}^{\psi}_{N_k}(C_M) = d^{\psi}\big(\det(\tau_{N_k}(M))\big) + d^{\psi}(\Delta_{N_k}(M)),$$

where det is the Dieudonné determinant, and d^{ψ} is the map assigning the degree w.r.t. ψ.

This decomposition consists of

$$\begin{array}{ccl} d^{\psi}\big(\det(\tau_{N_k}(M))\big) & \cdots & \text{monoid homomorphism part} \\ d^{\psi}(\Delta_{N_k}(M)) & \cdots & \text{homology cobordism invariant part} \end{array}$$

Comparing with Harvey's Realization Theorem in [3], we have the following result for the homomorphism part.

Theorem 2. For each primitive element $\psi \in H^1\Sigma_{g,1}$,

$$d^{\psi}\big(\det(\tau_{N_k}(\cdot))\big)\colon \mathcal{C}_{g,1}[2] \to \mathbb{Z}_{\geq 0}, \quad (k = 2, 3, \dots)$$

are defined on $\mathcal{C}_{g,1}[2]$. Moreover they are all non-trivial monoid homomorphisms, independent of each other, and trivial on $\mathcal{M}_{g,1} \cap \mathcal{C}_{g,1}[2]$.

This theorem is proved by constructing homology cylinders that are homology cobordant to the unit of $\mathcal{C}_{g,1}$, and we see that the above decomposition gives a new insight of higher-order degrees for C_M.

References

[1] S. Garoufalidis, J. Levine, *Tree-level invariants of three-manifolds, Massey products and the Johnson homomorphism*, Graphs and patterns in mathematics and theoretical physics, Proc. Sympos. Pure Math. 73 (2005), 173–205

[2] K. Habiro, *Claspers and finite type invariants of links*, Geom. Topol. 4 (2000), 1–83

[3] S. Harvey, *Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm*, to appear in Topology

[4] S. Harvey, *Monotonicity of degrees of generalized Alexander polynomials of groups and 3-manifolds*, to appear in Math. Proc. Cambridge Philos. Soc.

[5] P. Kirk, C. Livingston, Z. Wang, *The Gassner representation for string links*, Commun. Contemp. Math. 1(3) (2001), 87–136

[6] J. Levine, *Homology cylinders: an enlargement of the mapping class group*, Algebr. Geom. Topol. 1 (2001), 243–270

[7] T. Sakasai, *The Magnus representation for homology cylinders*, preprint

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