On the space of long knots and loop spaces of configuration spaces

Keiichi Sakai

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1 Introduction

The aim of this article is to analize the space of knots by means of loop spaces of configuration spaces. The study of the topology of the space $\mathcal{K}_3$ of all (long) knots in $\mathbb{R}^3$ was initiated by V.Vassiliev [12] to obtain the Vassiliev invariants (or finite type invariants) for knots. Here a long knot is an embedding $\mathbb{R}^1 \hookrightarrow \mathbb{R}^3$ with fixed properties at infinity. This theory might be variously generalized. The matters become easier when considering the space $\mathcal{K}_n$, the space of embeddings $\mathbb{R}^1 \hookrightarrow \mathbb{R}^n$, $n \geq 4$ (for example, $\pi_0\mathcal{K}_n = \{0\}$).

Our approach is to generalize the notion of "closing braids," which is extremely important in the classical knot theory, to the higher dimensional case. Here we will regard $\Omega C_m(\mathbb{R}^{n-1})$, the loop space of configuration space of orderd m-tuples of distinguished points in $\mathbb{R}^{n-1}$, as the set of all geometric braids in $\mathbb{R}^n$. By closing pure braids suitably, we obtain a *closing map*

$$c : \Omega C \longrightarrow \mathcal{K}_n$$

here $\Omega C := \coprod_{m\geq 1} \Omega C_m(\mathbb{R}^{n-1})$. We would like to think of this space as an "approximation" of $\mathcal{K}_n$. In particular we expect that this map might shed light on the structure of (co)homology group of $\mathcal{K}_n$. Moreover, this should give some new information about Vassiliev invariants.

This article is organized as follows. In section 2 we recall the (co)homology of $\Omega C_m(\mathbb{R}^{n-1})$, introducing some new description. The properties of the above map c will be discussed in the third section. We will consider the case of framed long knots in section 4, there we will see another perspective of the famous theorem due to Kontsevich.

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2 Topology of $\Omega C_m(\mathbb{R}^{n-1})$

The Hopf algebra structure of $H_*(\Omega C_m(\mathbb{R}^l), \mathbb{Z})$, $l = n-1$, was studied by F.Cohen and S.Gitler [9]. It can be regarded as a graded algebra $A_m := D_m/\sim$, here D_m is an algebra generated by "horizontal chord diagrams" $\{X_{ij}\}_{1\leq i\neq j\leq m}$ (with degree $n-3$), and relations are so-called "4-term relations." As an approach to the cohomology, we can apply the singularity theoretic method which Vassiliev used to analize the space of knots [12]. The details are as follows.

Regard $\Omega C_m(\mathbb{R}^l)$ as a complement of the space of singular braids in the space of all maps $\mathbb{R}^1 \to C_m(\mathbb{R}^l)$. By the Alexander duality theorem (applied to the infinite dimensional case), it suffices to analize the topology of the space Σ of singular braids. This set posesses a natural stratification via the depth of the singularities. Hence we can construct a spectral sequence generated by the filtration and converging to $H^*(\Omega C_m(\mathbb{R}^l), \mathbb{Z})$. Using homological results, we can see that this spectral sequence behaves quite well.

Theorem 2.1. *Suppose $n \geq 3$. Then there is a second quadrant spectral sequence $E_r^{p,q}$ which converges to $H^*(\Omega C_m(\mathbb{R}^{n-1}))$ and degenerates at E_1 term. Moreover, only the "diagonal part" $E_1^{-i,(n-2)i}$, whose total degree is $(n-3)i$, survive at E_∞. The group $E_1^{-i,(n-2)i}$ is isomorphic to certain kind of graph homology group. Since the degeneration also holds when $n = 3$, we have an isomorphism $\tilde{V}_i/\tilde{V}_{i-1} \cong$*

$H^{(n-3)i}(\Omega C_m(\mathbb{R}^{n-1}))$, *where* $\tilde{V}_i$ *is the space of Vassiliev invariants of degree* $\leq i$ *for pure braids.* □

The statement about Vassiliev invariants for pure braids in the theorem first appeared in [10]. It is known [2] that all Vassiliev invariants for pure braids distinguish all pure braids.

Since $C_m(\mathbb{R}^l)$ is $(l-2)$-connected, $H^*(\Omega C_m(\mathbb{R}^l))$ can be computed by using Chen's iterated integrals [8] if $n > 3$. Let B^* be the bar complex of $\mathcal{M}$, the minimal model of the de Rham complex $\Omega^*(C_m(\mathbb{R}^l))$. $\mathcal{M}$ is generated by the forms $\{\omega_{ij}\}$ and $\{\phi_{ijk}\}$, here $\{\omega_{ij}\}$ give the generating system of $H^*(C_m(\mathbb{R}^l))$, and $\{\phi_{ijk}\}$ are the relations, $\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = d\phi_{ijk}$. The bar complex is bigraded, $B = \bigoplus_{p \geq (n-2)q > 0} B^{p,-q}$, here p is the sum of the degree of forms, and q is the length of the tensor products. The following is a restatement of a theorem of Kohno [10].

Theorem 2.2 ([10]). *Define the subcomplex of* B^* *by*

$$B' := \bigoplus_{k \geq 0} \bigoplus_{l \geq 1} B^{(n-2)k-l,-k+l}.$$

Then we can show $H^{(n-3)k}(B/B') \cong E_1^{-k,(n-2)k}$, *hence* B' *is acyclic. Any element of the homology of* $(B/B')^*$ *can be written as*

$$\sum a_{i_1 j_1 \dots i_k j_k} \omega_{i_1 j_1} \otimes \cdots \otimes \omega_{i_k j_k},$$

where $a_{i_1 j_1 \dots i_k j_k} \in \mathbb{Z}$, $1 \leq i_p \neq j_p \leq m$ *satisfy the "4-term relations."* □

3 closing braids, and the space of long knots

First we discuss the homological properties of the map c mentioned in the introduction.

Let $\tilde{\mathcal{A}}_k$ be the vector space spanned by all chord diagrams with k chords. Corresponding to $\Gamma \in \tilde{\mathcal{A}}_k$, we can construct an $(n-3)k$-cycle $s_\Gamma : (S^{n-3})^k \to \mathcal{K}_n$, $k \geq 1$. It is defined by blowing up the self-intersections of the singular knot determined by the chord diagrams Γ. Such cycles are nontrivial [7]. In a similar way, given a horizontal chord diagram σ, we have an $(n-3)k$-cycle $B_\sigma : (S^{n-3})^k \to \Omega C_m(\mathbb{R}^l)$ $(l = n-1)$. The cycles B_σ gives an generators of the homology of $\Omega C_m(\mathbb{R}^l)$.

Consider a "closing map" $\tilde{c} : \bigoplus_m D_m^k \to \tilde{\mathcal{A}}_k$ which is compatible with c. Then it is clear that, if $\Gamma = \tilde{c}(\sigma)$, then $s_\Gamma = c_* B_\sigma$ as a cycle of $\mathcal{K}_n$.

Thus we relate the cycles s_Γ to those of $\Omega C_m(\mathbb{R}^l)$. Denote by $\mathcal{A}_k$ the space of chord diagrams with k chords modulo the (ordinary) 4-term relations.

Proposition 3.1. *Suppose that four chord diagrams* Γ_i $(1 \leq i \leq 4)$ *with* k *chords satisfy* $\sum \Gamma_i = 0$ *by the* $4T$*-relation in* $\mathcal{A}_k$. *Then the four cycles* s_{Γ_i}, $1 \leq i \leq 4$, *also satisfy the "4-term relation"* $\sum s_{\Gamma_i} = 0$. *In other words, we have a linear map*

$$s : \mathcal{A}_k \longrightarrow H_{(n-3)k}(\mathcal{K}_n)$$

for each $k \geq 1$.

Moreover, the map s *preserves the Hopf algebra structure. That is, for any* $\Gamma, \Gamma_1, \Gamma_2 \in \mathcal{A} := \bigoplus_k \mathcal{A}_k$,

$$s_{\Gamma_1 \cdot \Gamma_2} = s_{\Gamma_1} s_{\Gamma_2},$$
$$\Delta(s_\Gamma) = (s \otimes s)_{\Delta(\Gamma)},$$

here the Hopf algebra structure of $\mathcal{A}$ *is defined as explained in [1].* □

Next we treat with the cohomology classes. The following notions are introduced by Kohno [10]. Recall that the homology of $\Omega C_m(\mathbb{R}^l)$ is isomorphic to an algebra generated by horizontal chord diagrams. Since $\tilde{c}$ preserves the 4-term relations, we can regard $\tilde{c}$ as a surjection $\tilde{c} : \bigoplus_m H_*(\Omega C_m(\mathbb{R}^l)) \to \mathcal{A}$. Dually, we have an injection

$$\tilde{c}^* : W_k \longrightarrow H^{(n-3)k}(\Omega C, \mathbb{R}),$$

where $W_k = \mathcal{A}_k^*$ is the space of the weight systems of degree k. This means that the space of Vassiliev invariants for (framed) knots can be embedded into the space of those for pure braids. Kohno asked in [10] how we can characterize the image of $\tilde{c}^*$ geometrically. For example,

Proposition 3.2. *The space* $\tilde{c}^*(W_1) \subset H^{n-3}(\Omega C)$ *is one dimensional. A generator* $v := \sum_{i<j} \mathrm{link}_{ij}$ *is a Vassiliev invariant for pure braids, where* link_{ij} *is the linking number of* i*-th and* j*-th strings.* □

Below we will consider the question for general $k \geq 1$. Recall that A.Cattaneo, P.Cotta-Ramusino and R.Longoni [7] generalized the method of R.Bott and C.Taubes [3] and others to obtain some de Rham cohomology classes of $\mathcal{K}_n$. According to whether n is even or odd, there are bigraded graph complices $(\mathcal{D}^*_o, \delta)$ and $(\mathcal{D}^*_e, \delta)$ generated by Feynman diagrams, and if $n > 3$, cochain maps $I : \mathcal{D}^{k,q} \to \Omega^{(n-3)k+q}(\mathcal{K}_n)$ inducing injections $I : H^{k,0}(\mathcal{D}) \to H^{(n-3)k}_{DR}(\mathcal{K}_n)$. This map is defined as the iterated integrals of $\{\omega_{ij}\}$ associated to graph cocycles. In the case of $n = 3$, such integrals (together with some correction terms) give integral expressions of finite type invariants for knots. In such a sense the subspace $V_k := I(H^{k,0}(\mathcal{D}^*)) \subset H^{(n-3)k}(\mathcal{K}_n)$ is a generalization of the space of invariants of order k.

Under these preliminaries, we can state the following proposition.

Proposition 3.3. *The map* $c^* : H^{(n-3)k}(\mathcal{K}_n) \to H^{(n-3)k}(\Omega C)$ *is injective when restricted onto the subspace* V_k*;* $\ker \tilde{c}^*|_{V_k} = \{0\}$*. Moreover,* $\operatorname{im} c^*$ *is a proper subspace of* $\tilde{c}^*(W)$*.* □

At the beginning the author guessed that $\operatorname{im} c^*$ would coincide with $\tilde{c}^*(W)$, but this turns out to be wrong. This is because the map s of Proposition 3.1 is not injective (for example, $s|_{\mathcal{A}_1} = 0$), or, another way of saying, we have not considered so-called *framing independence*. So our space of weight systems should correspond to the framed (long) knots which is the subject of the next section.

4 framed case

The framed knots in $\mathbb{R}^n$ are defined in [7]. A *framing* of a knot γ is a map $w : \mathbb{R}^1 \to SO(n)$ such that the last column of $w(t)$ is equal to the normalized derivative $\gamma'(t)/|\gamma'(t)|$. The space $\widetilde{\mathcal{K}}_n$ of all framed long knot is a set of pairs (γ, w) with w being a framing of $\gamma \in \mathcal{K}_n$. Hence $\widetilde{\mathcal{K}}_n \subset \mathcal{K}_n \times \Omega SO(n)$. Another formulation has been done by R. Budney [4] and the topology of $\widetilde{K}_3$ is studied by Budney and Cohen [4, 6]. As in the unframed case, we can construct a closing map

$$c : \Omega C \longrightarrow \widetilde{\mathcal{K}}_n$$

by attaching some fixed “canonical framing” to the original map c, and we can show $\operatorname{im} c^* \subset \tilde{c}^*(W)$.

Conjecture. *In the framed case,* $\operatorname{im} c^* = \tilde{c}^*(W)$*.*

It seems that this conjecture is true, but at the moment the author has no proof for this. The difficulty comes from the fact that we do not know whether the cochain map $\tilde{I} : \widetilde{\mathcal{D}}^{k,q} \to \Omega^{(n-3)k+q}(\widetilde{\mathcal{K}}_n)$, constructed when n is odd, similarly as in the unframed case [7], induces an injection $\tilde{I} : H^{k,0}(\widetilde{\mathcal{D}}^*) \hookrightarrow H^{(n-3)k}(\widetilde{\mathcal{K}}_n)$ or not. But by combinatorial arguments, we can see

Lemma 4.1 ([7]). *The group* $H^{k,0}(\widetilde{\mathcal{D}}^*)$ *is isomorphic to* W_k*,* $k \geq 1$*.* □

This lemma allows us to state the following.

Lemma 4.2. *The above conjecture is true when* n *is odd, if the injectivity of* $\tilde{I}$ *on* $H^{k,0}(\widetilde{\mathcal{D}})$ *is proved similarly as in the unframed case.* □

In the case $n = 3$, the set V of Vassiliev invariants for framed knots are in one to one correspondence to the space W of all weight systems, which is a result due to Kontsevich [11]. The Bott-Taubes construction gives a formulation of the map $W_k \to V_k$ by configuration space integrals. The inverse correspondence $V_k \to W_k$ is easily constructed.

The subspace $\tilde{I}(H^{k,0}(\widetilde{\mathcal{D}}^*))$ is a “higher dimensional generalization” of the Vassiliev invariants. In higher dimensional case, the inverse correspondence $V_k \ni v \mapsto f_v \in W_k$ becomes slightly subtle. The closing map c should be such a inverse which is easily and *geometrically* defined.

Problems. One of the aims of this research is to pick up new cohomology classes of $\mathcal{K}_n$ which may not correspond to finite type invarinats for knots. The author does not know whole properties of the map c right now. Beside proving the above conjecture, it is important and interesting to determine $\ker c^*$.

Recently R.Budney studies $\mathcal{K}_3$ in detail [5]. It is also interesting to know to what extent we can do similar investigations in the higher dimensional case.

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Graduate school of Mathematical Science,
University of Tokyo, Tokyo 153-8914 Japan
E-Mail: ksakai@ms.u-tokyo.ac.jp