Poincaré maps and Bol's theorem

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Abstract

The usual Poincaré map is defined for a four-web admitting three independent Abelian relations, and gives a natural projective model of the web where the leaves are straight lines. This model thus proves that the web is associated with a quartic curve in the dual projective plane.

We shall present a generalization of this situation to the case of five-webs for which every extracted three-web carries an Abelian relation. This generalization takes into account not only the Abelian relations of the web, but also the spaces of Abelian relations of the extracted sub-webs. This way we prove the following theorem of G.Bol:

If all extracted three-webs of a five-web admit an Abelian relation, then the web is diffeomorphic either to five pencils of straight lines, or to Bol's exceptional example.

The main tool used in the separation of cases is a careful investigation of the configuration formed by the spaces of relations of the extracted sub-webs in the space of relations of the whole web.

The author is extremely grateful to the organizers of this conference, and especially Toshitake Kohno, for giving him the opportunity to explain the relations that exist between webs and configuration spaces. These will be explained through a proof of the following theorem:

Theorem 1 *(G.Bol, 1936)*
If a five-web (a family of five foliations in an open subset $U \subset \mathbf{C}^2$) satisfies the property that every extracted three-web admits a (nonzero) Abelian relation, then either it is locally diffeomorphic to five pencils of lines, or it is locally diffeomorphic to Bol's exceptional example.

Abelian relations will be defined, as well as many other elementary tools concerning webs, in the second part of this talk. The first part is a complete description of Bol's web including its relations to the space of configurations of five points in the projective line. The third part generalizes some of the tools of part two, and explains the connections between totally hexagonal five-webs and the description of Bol's web obtained at the end of part one.

1 Bol's web

1.1 Historical background

The starting point is in the beginning of the nineteenth century, where people found the five-term relation satisfied by Euler's dilogarithm

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \log(1-t)\frac{dt}{t}.$$

This equation can be written in many different ways (cf. [4]). The following one is due to Abel (1830) and holds for $x, y < 1$

$$\begin{aligned} Li_2(\frac{x}{1-x}\frac{y}{1-y}) &= Li_2(\frac{x}{1-y}) + Li_2(\frac{y}{1-x}) \\ &\quad - Li_2(x) - Li_2(y) - \log(1-x)\log(1-y). \end{aligned}$$

The technique used to prove such a relation consists in differentiating it: one obtains a sum of expressions which are logarithms multiplied by rational functions; one needs only gather the terms corresponding to the same logarithm and check that the rational functions cancel.

An improvement of this result has been found by Rogers in 1907: in fact he proved that, replacing the function Li_2 in the equation by

$$L(z) = \frac{1}{2}(Li_2(z) - Li_2(1-z))$$

gives equations involving only dilogarithms. He thus obtains for $0 < x, y < 1$

$$L(x) + L(y) = L(xy) + L(\frac{x(1-y)}{1-xy}) + L(\frac{y(1-x)}{1-xy}).$$

The holomorphic setting is chosen here in order to get rid of assumptions such as $x, y < 1$, so the equations will always be supposed to hold somewhere, and we'll use analytic continuation to obtain equations elsewhere.

1.2 Bol's presentation

In 1936, in the same paper [2] in which he proves theorem 1, Bol remarks that the level sets of the five functions occurring as arguments of the dilogarithm in Abel's equation (after changing y with $1-y$)

$$U_1 = x, \qquad U_2 = y, \qquad U_3 = x/y, \qquad U_4 = \frac{1-x}{1-y}, \qquad U_5 = \frac{y(1-x)}{x(1-y)},$$

correspond to four pencils of lines passing through four points of $\mathbf{CP}^2$ in general position and a pencil of conics passing through these four points, as shown in figure 1.

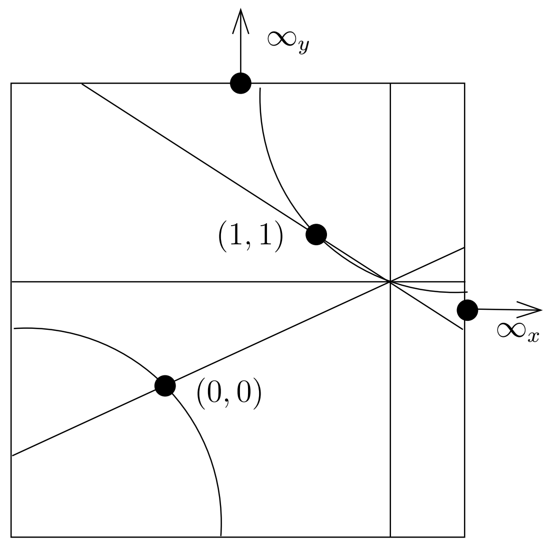

Figure 1: Four pencils of lines and a pencil of conics

Since the group $PGL_(3, \mathbf{C})$ of projective transformations of $\mathbf{CP}^2$ acts transitively on configurations of four points in general position, we obtain a group of projective transformations (and hence diffeomorphisms) which is isomorphic to S_4, and which sends the web to itself.

Moreover, the action of the Cremona transformation $(x, y) \mapsto (1/x, 1/y)$ sends

$$U_1 \leftrightarrow 1/U_1, \qquad U_2 \leftrightarrow 1/U_2, \qquad U_3 \leftrightarrow 1/U_3, \qquad U_4 \leftrightarrow U_5.$$

Therefore the automorphism group of the web is the extension of S_4 by this Cremona transformation, acting as the transposition t_{45}. This group is therefore isomorphic to S_5 and the whole picture consists of five projective planes, with four distinguished points on each, and transition maps which are Cremona transformations, as shown in figure 2.

1.3 Relations with a space of configurations

Then, in 1982, Gelfand and MacPherson [3] remark that these five functions represent the cross-ratios of four points out of $(\infty, 0, 1, x, y)$ in $\mathbf{CP}^1$. Therefore, these functions can be interpreted as functions on the space $X(5)$ of non-degenerate configurations of five points in a projective line.

The cross-ratio can be extended by setting (for x, y and z distinct)

$$\begin{array}{lll} cr(x,y,z,x) = \infty, & cr(x,y,z,y) = 0, & cr(x,y,z,z) = 1, \\ cr(x,y,y,z) = \infty, & cr(x,y,x,z) = 0, & cr(x,x,y,z) = 1, \\ cr(x,y,y,x) = \infty, & cr(x,y,x,y) = 0, & cr(x,x,y,y) = 1. \end{array}$$

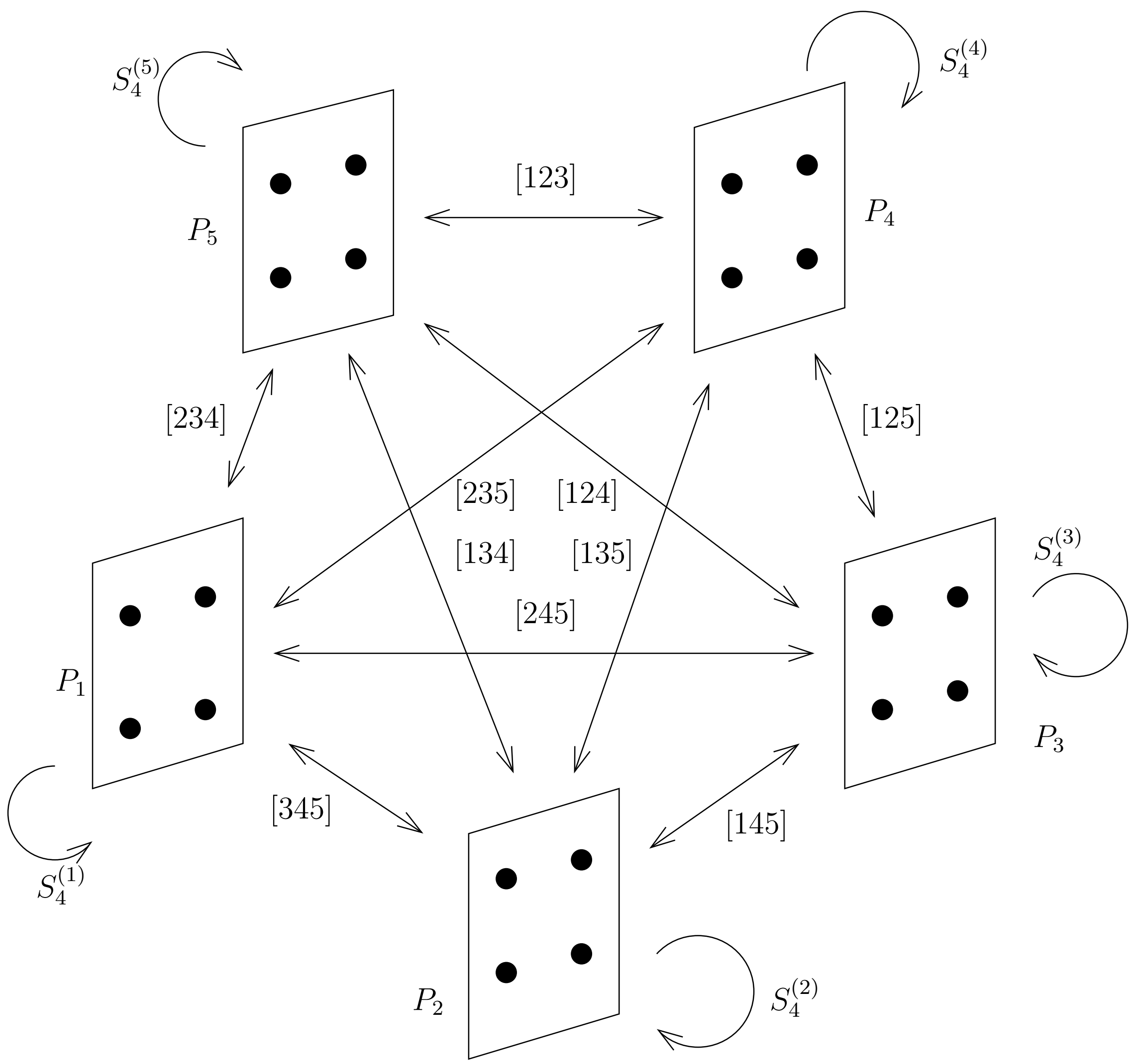

Figure 2: Five projective planes representing Bol's web

This gives an extension of the functions, and hence of the web, to the compactification $\overline{X(5)}$ of $X(5)$ obtained by adding the degenerate configurations with one or two double points, but no triple point. This is the same as blowing-up the four basis points of the pencils in figure 1.

Another relation between Bol's web and the configuration space $X(5)$ is the surprising fact that the configuration in the projectified tangent space at point (x, y) formed by the five tangents to leaves of the foliations at that point is equivalent to the configuration $(\infty, 0, 1, x, y)$.

2 Webs, Abelian relations, Poincaré map

This part is intended to provide material of elementary web theory. The main reference here is Blaschke and Bol's book [1].

2.1 Definitions

Definition 1 *The main definitions are the following:*

1. *A d-web $\mathcal{W}$ in an open subset $U \subset \mathbf{C}^2$ is a family of d nonsingular foliations $(\mathcal{F}_1, \ldots, \mathcal{F}_d)$ of U that intersect transversally at every point in U.*

2. *An Abelian relation of the web $\mathcal{W}$ is a family of d one-forms $(\alpha_1, \ldots, \alpha_d)$ satisfying*

   - *the tangent space of the leaf of the foliation $\mathcal{F}_i$ at point $(x, y) \in U$ is in the kernel of $\alpha_i(x, y)$,*
   - *the one-forms α_i are closed, and*
   - *the sum $\alpha_1 + \cdots + \alpha_d$ is zero.*

3. *If $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a Abelian relation of the web $\mathcal{W}$, then the support of α is the extracted sub-web $\{\mathcal{F}_i, 1 \leq i \leq d \mid \alpha_i \neq 0\}$.*

The space $\mathcal{A}b(\mathcal{W})$ of Abelian relations of $\mathcal{W}$ is a finite dimensional subspace of $\Omega^1(U)^d$, and its dimension is controlled by the following result

Theorem 2 *(W.Blaschke, G.Bol, 1932)*
The dimension of $\mathcal{A}b(\mathcal{W})$, also called the rank of $\mathcal{W}$, is less than or equal to $(d-1)(d-2)/2$.

In particular, the rank of a three-web is 0 or 1, the rank of a four-web is less than 3, and that of a five-web is less than 6. A three-web admitting a nonzero Abelian relation is said to be hexagonal, and a d-web whose extracted three-webs are all hexagonal is said to be totally hexagonal.

If $\mathcal{W}'$ is an extracted sub-web of $\mathcal{W}$, then every Abelian relation of $\mathcal{W}'$ extends to a relation of $\mathcal{W}$, so that we have an inclusion of $\mathcal{A}b(\mathcal{W}')$ in $\mathcal{A}b(\mathcal{W})$ [this is the same as considering relations whose support is a sub-web of $\mathcal{W}'$].

Let $\mathcal{A}b_k(\mathcal{W})$ be the subspace of $\mathcal{A}b(\mathcal{W})$ spanned by all the $\mathcal{A}b(\mathcal{W}')$, where $\mathcal{W}'$ varies over all the extracted k-webs of $\mathcal{W}$. This defines a filtration

$$0 = \mathcal{A}b_1(\mathcal{W}) = \mathcal{A}b_2(\mathcal{W}) \subset \mathcal{A}b_3(\mathcal{W}) \subset \cdots \subset \mathcal{A}b_d(\mathcal{W}) = \mathcal{A}b(\mathcal{W}).$$

Define $\rho_k(\mathcal{W}) = \dim \mathcal{A}b_k(\mathcal{W}) - \dim \mathcal{A}b_{k-1}(\mathcal{W})$.

Definition 2 *The weave of $\mathcal{W}$ is the family $(\rho_3(\mathcal{W}), \ldots, \rho_d(\mathcal{W}))$. Its rank is $\rho_3(\mathcal{W}) + \cdots + \rho_d(\mathcal{W})$.*

2.2 Poincaré map for four-webs

Assume that $\mathcal{W}$ is a four-web in U admitting three linearly independent Abelian relations, so that it has maximal rank 3.

For a point $(x, y) \in U$, define $P(x, y)$ as the subspace of Abelian relations vanishing at that point

$$P(x, y) = \{\alpha = (\alpha_1, \ldots, \alpha_4) \in \mathcal{A}b(\mathcal{W}) \mid \alpha_1(x, y) = \cdots = \alpha_4(x, y) = 0\}.$$

This subspace is defined by four equations, but since the sum of the one-forms is zero, two of them are redundant, hence the codimension of $P(x, y)$ in $\mathcal{A}b(\mathcal{W})$ is $4 - 2 = 2$; therefore, the dimension of $P(x, y)$ is $3 - 2 = 1$.

Definition 3 *The Poincaré map is the map defined by*

$$P : \left\{ \begin{array}{ccl} U & \longrightarrow & \mathbf{P}\mathcal{A}b(\mathcal{W}) \simeq \mathbf{CP}^2 \\ (x, y) & \longmapsto & P(x, y). \end{array} \right.$$

The local inversion theorem proves that this map is a local diffeomorphism. Its main interest is the following proposition:

Proposition 1 *The map P sends leaves of the foliations of $\mathcal{W}$ into straight lines.*

Proof: Let (α, β, γ) be a basis of $\mathcal{A}b(\mathcal{W})$, (x_0, y_0) a point in U, and $1 \leq i \leq 4$.

We can assume that the one-form α_i is nonzero; since it is closed, we can integrate it in a neighborhood of (x_0, y_0) to obtain a function U_i which is constant over the leaves of the foliation $\mathcal{F}_i$.

Since β_i and γ_i are closed and have the same kernel as α_i, they can be expressed as $\lambda_i(U_i)\, dU_i$ and $\mu_i(U_i)\, dU_i$. Therefore, the i-th equation becomes (if (a, b, c) designate the coordinates in $\mathcal{A}b(\mathcal{W})$ with respect to the basis (α, β, γ))

$$a + b\lambda_i(U_i(x, y)) + c\mu_i(U_i(x, y)) = 0.$$

Now, if the point (x, y) is made to vary on a leaf of the foliation $\mathcal{F}_i$, then the function $U_i(x, y)$ is constant, so also $\lambda_i(U_i(x, y))$ and $\mu_i(U_i(x, y))$. Therefore the previous equation is the equation of a fixed plane in $\mathcal{A}b(\mathcal{W})$, i.e. of a fixed projective line in $\mathbf{P}\mathcal{A}b(\mathcal{W})$.

2.3 Totally hexagonal four-webs

The previous proposition can be used to prove that every four-web with maximal rank 3 is associated with a quartic curve in the dual projective plane. We will not prove this result but a more elementary one

Theorem 3 *(K.Mayrhofer, K.Reidemeister, 1928)*
If a four-web is such that each of its extracted three-webs admits a (nonzero) Abelian relation, then it is locally diffeomorphic to four pencils of lines.

Proof: If the space $\mathcal{A}b(\mathcal{W})$ has dimension 3, then we can use the previous result; moreover, if we choose γ to be a relation of the extracted three-web $\mathcal{W}_i$ obtained by removing the foliation $\mathcal{F}_i$, then we have $\gamma_i = 0$ and the equation becomes

$$a + b\lambda_i(U_i(x, y)) = 0.$$

As (x, y) varies, this equation varies over equations of the form $a + \lambda b = 0$, i.e. equations of planes containing the c-axis, or equations of projective lines passing through the point $\mathbf{P}\mathcal{A}b(\mathcal{W}_i)$. This finishes the proof if $\dim \mathcal{A}b_3(\mathcal{W}) = 3$.

The only remaining case is when $\dim \mathcal{A}b_3(\mathcal{W}) = 2$; in that case, the equations of the four three-webs can be written as

$$\begin{aligned} \alpha_2 + \alpha_3 + \alpha_4 &= 0, & \alpha_1 - \alpha_2 + (t-1)\alpha_4 &= 0, \\ \alpha_1 + \alpha_3 + t\alpha_4 &= 0, & \alpha_1 - t\alpha_2 + (1-t)\alpha_3 &= 0. \end{aligned}$$

If $(U_1, \dots, U_4)$ are local primitives of $(\alpha_1, \dots, \alpha_4)$ vanishing at (x_0, y_0), they satisfy $U_2 = -U_3 - U_4$ and $U_1 = -U_3 - tU_4$. Then we also have the equality $U_1^2 - tU_2^2 = (1-t)U_3^2 + t(1-t)U_4^2$, so that there is another Abelian relation

$$U_1\, dU_1 - tU_2\, dU_2 + (t-1)U_3\, dU_3 + t(1-t)U_4\, dU_4 = 0.$$

This last equation proves that we always have $\dim \mathcal{A}b(\mathcal{W}) = 3$, and hence finishes the proof.

We can thus distinguish three types of such four-webs according to alignment properties of the configuration of the points $\mathbf{P}\mathcal{A}b(\mathcal{W}_i)$ in $\mathbf{P}\mathcal{A}b(\mathcal{W})$ (cf. figure 3):

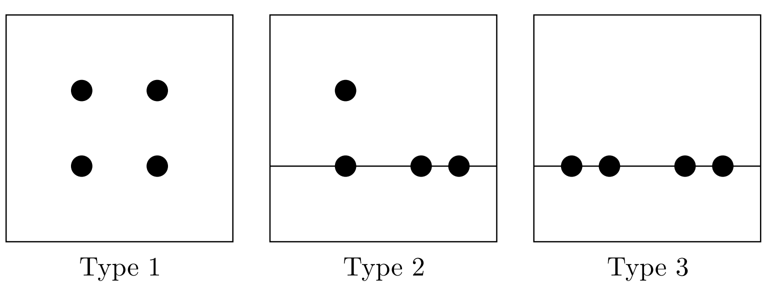

Figure 3: Three types of configurations of four points in the plane

- type 1 if the points are in general position,
- type 2 if three points are on the same line and the other isn't,
- type 3 if all four points are on the same line.

Note that the second part of the proof is needed only in the type 3 case, since in the other two cases, we have $\dim \mathcal{A}b_3(\mathcal{W}) = 3$.

3 Generalizations

3.1 Generalization of the Poincaré map

Assume $\mathcal{W}$ is a d-web for which we know a space $\mathcal{A}b^0(\mathcal{W})$ of Abelian relations of dimension $r \geq d-1$. For a point $(x,y) \in U$, define as before $P(x,y)$ to be the subspace of Abelian relations in $\mathcal{A}b^0(\mathcal{W})$ vanishing at that point

$$P(x,y) = \{\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathcal{A}b^0(\mathcal{W}) \mid \alpha_1(x,y) = \cdots = \alpha_d(x,y) = 0\}.$$

This subspace is defined by d equations, but two of them are redundant, hence the codimension of $P(x,y)$ in $\mathcal{A}b^0(\mathcal{W})$ is $d-2$; therefore, the dimension of $P(x,y)$ is $r-d+2 \geq 1$. It is thus (in general) not in a projective space, but in a Grassmanian manifold.

Definition 4 *The Poincaré map is defined by*

$$P : \left\{ \begin{array}{ccl} U & \longrightarrow & G(r-d+2, \mathcal{A}b^0(\mathcal{W})) \simeq G(r-d+2, r) \\ (x,y) & \longmapsto & P(x,y). \end{array} \right.$$

The local inversion theorem proves that this map is an immersion.

Example: Bol's web

The five functions defining the web are

$$U_1 = x, \qquad U_2 = y, \qquad U_3 = x/y, \qquad U_4 = \frac{1-y}{1-x}, \qquad U_5 = \frac{x(1-y)}{y(1-x)}.$$

The relations of the extracted three-webs can be written explicitly using logarithmic differentials. The following five form a basis

$$\begin{array}{rrrrrr}
\frac{dU_1}{1-U_1} & -\frac{dU_2}{1-U_2} & & -\frac{dU_4}{U_4} & & =0, \\
-\frac{dU_1}{U_1(1-U_1)} & & +\frac{dU_3}{U_3(1-U_3)} & -\frac{dU_4}{1-U_4} & & =0, \\
\frac{dU_1}{U_1} & & & +\frac{dU_4}{U_4(1-U_4)} & -\frac{dU_5}{U_5(1-U_5)} & =0, \\
 & \frac{dU_2}{U_2(1-U_2)} & -\frac{dU_3}{1-U_3} & +\frac{dU_4}{U_4(1-U_4)} & & =0, \\
 & -\frac{dU_2}{U_2} & & -\frac{dU_4}{1-U_4} & +\frac{dU_5}{1-U_5} & =0.
\end{array}$$

Therefore, if (a, b, c, d, e) are the coordinates in this basis, the equations are

$$\begin{aligned} ax - b + c(1-x) = -ay + d - e(1-y) = by - dx = \\ -a(y-x) - (b+e)(1-y) + (c+d)(1-x) = -cy(1-x) + ex(1-y) = 0. \end{aligned}$$

We get therefore $b = \lambda x$, $d = \lambda y$, $c = \mu x/(x-1)$, $e = \mu y/(y-1)$ and $a = \lambda + \mu$, so that the plane $P(x, y)$ is

$$P(x, y) = \mathbf{C}(1, x, 0, y, 0) \oplus \mathbf{C}(1, 0, x/(x-1), 0, y/(y-1)).$$

3.2 Proof of Bol's theorem

We can now proceed with the proof of Bol's theorem. Assume that $\mathcal{W}$ is a five-web for which every extracted three-web admits a (nonzero) Abelian relation, then certainly every extracted four-web satisfies the hypotheses of theorem 3. Hence each four-web is locally diffeomorphic to four pencils of lines and can be assigned a type according to figure 3.

Therefore, there are numerous cases to be taken into account, most of them giving rise to five pencils of lines, so we will add two extra hypotheses to get rid of those cases, and prove only

Theorem 4 *If a five-web $\mathcal{W}$ satisfies the properties that*

- *every extracted three-web admits a (nonzero) Abelian relation,*
- *every extracted four-web has type 1,*
- *the dimension of $\mathcal{A}b_3(\mathcal{W}) = \mathcal{A}b_4(\mathcal{W})$ is 5,*

then $\mathcal{W}$ is locally diffeomorphic to Bol's web.

Proof: Let $\mathcal{W}_i$ denote the extracted four-web obtained by omitting foliation $\mathcal{F}_i$ and $\mathcal{W}_{ij}$ denote the extracted three-web obtained by omitting the foliations $\mathcal{F}_i$ and $\mathcal{F}_j$.

The projective space $\mathbf{P}\mathcal{A}b_3(\mathcal{W}) \simeq \mathbf{CP}^4$ contains five planes $P_i = \mathbf{P}\mathcal{A}b(\mathcal{W}_i)$. The intersections $P_i \cap P_j = \mathbf{P}\mathcal{A}b(\mathcal{W}_{ij}) = A_{ij}$ are points, so that each of the five planes contains four distinguished points.

Therefore, the situation is almost the same as in figure 2. We only have to prove that the transition maps from one plane to another are Cremona transformations.

In order to prove this, we will use Poincaré maps: for $1 \leq i \leq 5$, the Poincaré map p_i of the sub-web $\mathcal{W}_i$ is a local diffeomorphism from U to P_i.

Using the generalization of the Poincaré map to the space $\mathcal{A}b_3(\mathcal{W})$ of dimension $r = 5$, we obtain a family of projective lines $D(x, y) = \mathbf{P}P(x, y)$ intersecting all five planes P_i in the points $p_i(x, y)$. We will see that there is only one configuration of five planes satisfying this requirement.

Indeed, the assumptions we made ensure that the points A_{12}, A_{13}, A_{14}, A_{23}, A_{24} and A_{34} are in general position, and hence form a projective basis. Using this projective basis, we obtain a system of coordinates (x, y, u, v) in which the first four planes have the equations

$$P_1 : u = v = 0, \qquad P_2 : x = y = 0, \qquad P_3 : x = u = 1, \qquad P_4 : y = v = 1.$$

Therefore, the family of lines must be of the form

$$(\lambda x, \lambda y, (1-\lambda)u(x, y), (1-\lambda)v(x, y)).$$

The intersections with P_3 and P_4 thus correspond to $\lambda_3 = 1/x$ and $\lambda_4 = 1/y$, and we must have

$$u(x, y) = \frac{1}{1-\lambda_3} = \frac{x}{x-1}, \qquad v(x, y) = \frac{1}{1-\lambda_4} = \frac{y}{y-1}.$$

The transition map from P_1 to P_2 is therefore $(x, y) \mapsto (u(x, y), v(x, y)) = (x/(x-1), y/(y-1))$ which is a Cremona transformation.

Further, the fifth plane has a pair of equations of the form

$$u = ax + by + c, \qquad v = a'x + b'y + c'.$$

We must have λ_5 satisfying

$$(1-\lambda_5)\frac{x}{x-1} = \lambda_5(ax + by) + c, \qquad (1-\lambda_5)\frac{y}{y-1} = \lambda_5(a'x + b'y) + c'.$$

This means that

$$\lambda_5 = \frac{x - c(x-1)}{(ax + by)(x-1) + x} = \frac{y - c'(y-1)}{(a'x + b'y)(y-1) + y}.$$

The only solution is $c = c' = 1$, $a = b' = 0$ and $b = a' = -1$, so that a pair of equations of the fifth plane is $x + v = y + u = 1$ and the corresponding value of the parameter is $\lambda_5 = 1/(x + y - xy)$.

Thus, we obtained a map into the Grassmanian manifold $G(2, 5)$, but moreover to every point $(x, y) \in U$, we can associate the configuration in $X(5)$ of the five points $p_i(x, y)$ in the line $D(x, y)$.

Choosing coordinates as above, this configuration is equivalent to

$$(\lambda_1, \ldots, \lambda_5) = (1, 0, \frac{1}{x}, \frac{1}{y}, \frac{1}{x + y - xy}) \equiv (\infty, 1, y, x, 0).$$

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