Moduli spaces of graphs and homology operations on loop spaces of manifolds

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String topology:= intersection theory in loop spaces (and spaces of paths) of a manifold. (Chas-Sullivan, C., Jones, Godin, Ramirez.....)

Get homology operations associated to a surface.

Let M be fixed n-dimensional closed manifold. $\Sigma_{g,p+q}$ a surface of genus g thought of a s a cobordism between p "incoming" circles to q "outgoing" circles:

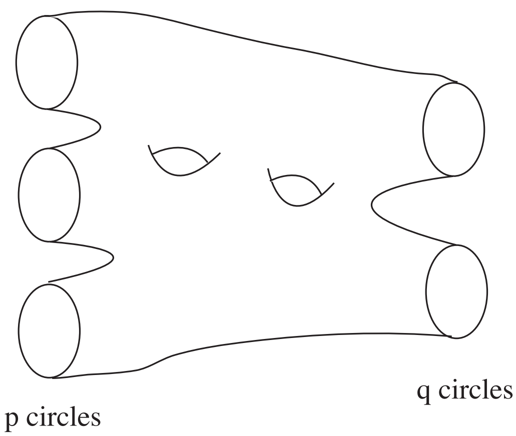


Figure 1: $\Sigma_{g,p+q}$

$$(LM)^q \xleftarrow{\rho_{out}} Map(\Sigma_{g,p+q}, M) \xrightarrow{\rho_{out}} (LM)^p$$

Using graphs, other techniques, one constructs an intersection or "umkehr map"

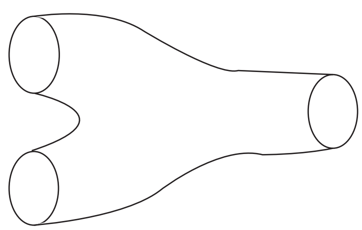
$$(\rho_{in})_! : h_*(LM^p) \to h_{*+n\cdot\chi(\Sigma)-np}(Map(\Sigma_{g,p+q}, M)$$

Here $h_* =$ generalized homology theory (H_*, K-homology, bordism) such that M^n is closed, h_*-oriented.

This defines a homology operation,

$$\mu_\Sigma = \rho_{out} \circ (\rho_{in})_! : h_*((LM)^p) \to h_{*+n\cdot\chi(\Sigma)-np}(Map(\Sigma_{g,p+q}, M)$$
$$\to h_{*+n\cdot\chi(\Sigma)-np}((LM)^q).$$

When Σ is the pair of pants, one gets the Chas-



Sullivan loop product,

$$\mu : H_p(LM) \otimes H_q(LM) \to H_{p+q-n}(LM)$$

making $\mathbb{H}_*(LM) = H_{*+n}(LM)$ an associative, commutative algebra.

These operations satisfy "gluing"

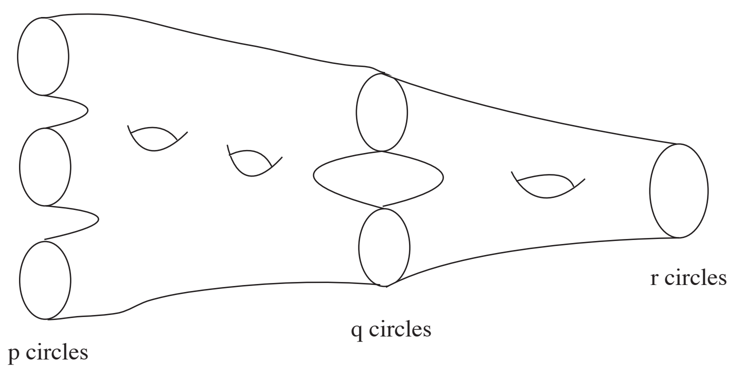


Figure 2: $\Sigma_1 \# \Sigma_2$

$$\mu_{\Sigma_1 \# \Sigma_2} = \mu_{\Sigma_2} \circ \mu_{\Sigma_1} \qquad \text{"Field theory"}$$

There is also a relative or "open" theory using $h_*(P_M(D_1, D_2))$ where $D_i \subset M$ and $P_M(D_1, D_2) = \{\gamma : [0,1] \to M,\ \gamma(0) \in D_1,\ \gamma(1) \in D_2\}$.

This is a *topological* theory.

Gromov-Witten theory: Count pseudo-holomorphic curves in a symplectic manifold.

This is a *geometric* theory.

$T^*M =$ cotangent bundle, has canonical symplectic structure:

$$p : T^*M \to M$$

$x \in M$, $v \in T_x^*M$, have

$$\alpha_{(x,v)} : T_{(x,v)}(T^*M) \xrightarrow{Dp} T_xM \xrightarrow{v} \mathbb{R}.$$

$\alpha \in \Omega^1(T^*M)$, $\omega = d\alpha \in \Omega^2(T^*M)$ is a symplectic 2-form.

There are many compatible almost $\mathbb{C}$-structures on T^*M, but given a Riemannian metric g on M one gets a canonical

$$J_g : T(T^*M) \to T(T^*M)$$

with $J_g^2 = -id$.

Goal: Understand the relationship between "String Topology" and Gromov-Witten theory.

A first step is to devise a Morse theoretic approach to string topology.

Strategy: Expand and generalize theory of graph flows of C. and Betz for constructing classical (co)homology invariants in algebraic topology. This is closely related to the work of K. Fukaya. Joint work with P. Norbury, U. Melbourne.

The basic idea is to make "toy model" of Gromov-Witten theory in which the role of surfaces are replaced by finite graphs. We develop a theory in which we make the following replacements from classical Gromov-Witten theory.

1. A smooth surface F is replaced by a finite, oriented graph Γ.
2. The role of the genus of F is replaced by the first Betti number, $b = b_1(\Gamma)$.

3. The role of marked points in F is replaced by the univalent vertices (or "leaves") of Γ.

4. The role of a complex structure Σ on F is replace by a metric on Γ.

5. The notion of a J-holomorphic map to a symplectic manifold with compatible almost complex structure, $\Sigma \to (N, \omega, J)$, is replaced by the notion of a "graph flow" $\gamma : \Gamma \to M$ which, when restricted to each edge is a gradient trajectory of a Morse function on M.

Definition 1. *Define $\mathcal{C}_{b,p+q}$ to be the category of oriented graphs (in $\mathbb{R}^\infty$) of first Betti number b, with*

1. Each edge of the graph Γ has an orientation.

2. Γ has $p+q$ univalent vertices, or "leaves". p of these are vertices of edges whose orientation points away from the vertex, = "incoming" .

The remaining q leaves are "outgoing".

3. Γ *comes equipped with a "basepoint", which is a nonunivalent vertex.*

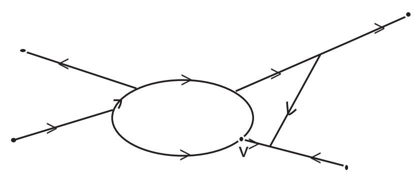


Figure 3: An object Γ in $\mathcal{C}_{2,2+2}$

A morphism between objects $\phi : \Gamma_1 \longrightarrow \Gamma_2$ *is combinatorial map of graphs (cellular map) that satisfies:*

1. *The inverse image of each vertex is a tree (i.e a contractible subgraph).*

2. *The inverse image of each open edge is an open edge.*

3. ϕ preserves the basepoints.

Notice a morphism $\phi : \Gamma_1 \to \Gamma_2$ is a combinatorial map of graphs which is a homotopy equivalence.

We now fix a graph Γ (an object in $\mathcal{C}_{b,p+q}$. Consider the category of "graphs over Γ", $\mathcal{C}_\Gamma$.

Definition 2. *Define $\mathcal{C}_\Gamma$ to be the category whose objects are morphisms in $\mathcal{C}_{b,p+q}$ with target Γ: $\phi : \Gamma_0 \to \Gamma$. A morphism from $\phi_0 : \Gamma_0 \to \Gamma$ to $\phi_1 : \Gamma_1 \to \Gamma$ is a morphism $\psi : \Gamma_0 \to \Gamma_1$ in $\mathcal{C}_{b,p+q}$ with the property that $\phi_0 = \phi_1 \circ \psi : \Gamma_0 \to \Gamma_1 \to \Gamma$.*

Notice the identity map $id : \Gamma \to \Gamma$ is a terminal object in $\mathcal{C}_\Gamma$. So $|\mathcal{C}_\Gamma|$ is contractible. But $Aut(\Gamma)$ acts freely (on objects by composition).

Corollary 1. *$|\mathcal{C}_\Gamma|/Aut(\Gamma) \simeq BAut(\Gamma)$.*

The following ideas of Culler-Vogtman, Igusa associates to a point in $|\mathcal{C}_\Gamma|$ a metric on a graph over Γ.

Recall that

$$|\mathcal{C}_\Gamma| = \bigcup_k \Delta^k \times \{\Gamma_k \xrightarrow{\psi_k} \Gamma_{k-1} \xrightarrow{\psi_{k-1}} \Gamma_{k-2} \to \cdots \xrightarrow{\psi_1} \Gamma_0 \xrightarrow{\phi} \Gamma\}/\sim$$

where the identifications come from the face and degeneracy operations.

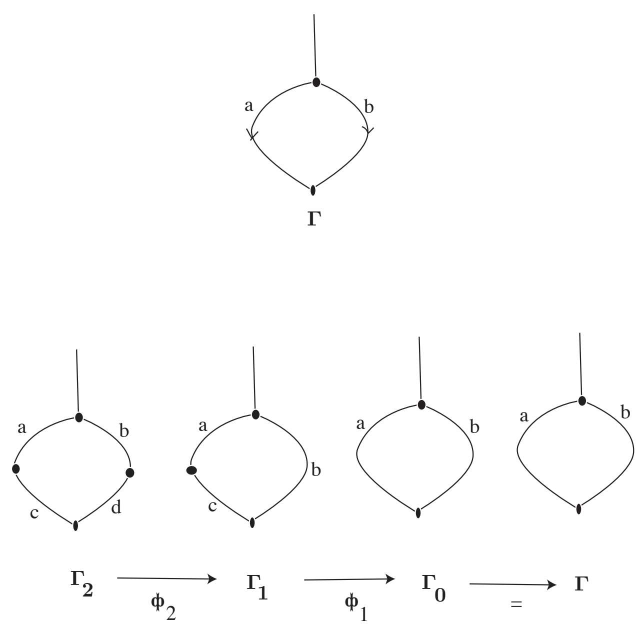


Figure 4: A 2-simplex in $|\mathcal{C}_\Gamma|$.

Γ_k is in a sense a generalized subdivision of Γ, in that Γ is obtained from Γ_k by collapsing various edges. We use the coordinates of Δ^k of the simplex Δ^k to define a metric on Γ_k

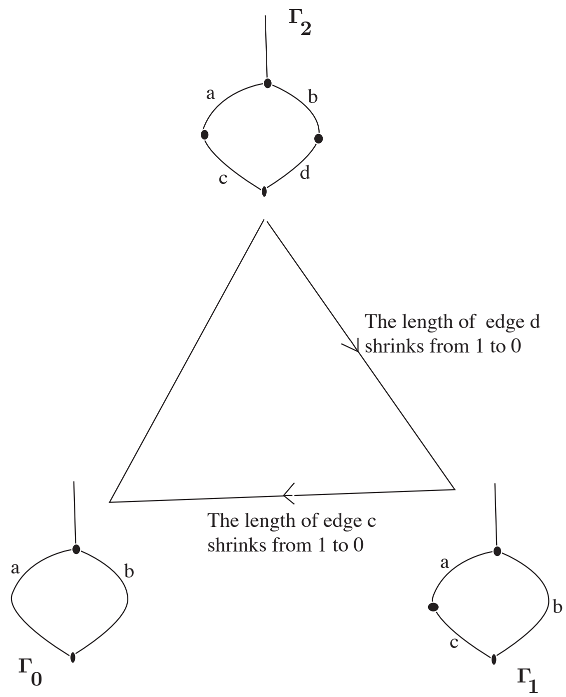


Figure 5: A 2-simplex of metrics.

Define

$$\mathcal{S}(\Gamma, M) = \{(\vec{t}, \vec{\psi}) \in |\mathcal{C}_\Gamma|,\ \mu = \text{labeling of each edge of } \Gamma_k \\ \text{by a distinct } f_i : M \to \mathbb{R}\}.$$

Note: $\mathcal{S}(\Gamma, M)$ is contractible with free action of $Aut(\Gamma)$ (finite group). Define

$$\mathcal{M}(\Gamma) = \mathcal{S}(\Gamma, M)/Aut(\Gamma) \simeq BAut(\Gamma)$$

= moduli space of metric graphs with Morse labelings = "*structures*".

Graph flows in M:

$$\tilde{\mathcal{M}}(\Gamma, M) = \{(\sigma, \gamma) : \sigma \in \mathcal{S}(\Gamma, M),\ \gamma : \Gamma_0 \to M \\ \frac{d\gamma_i}{dt} + \nabla f_i = 0, \text{ for all } i\}$$

$$\mathcal{M}(\Gamma, M) = \tilde{\mathcal{M}}(\Gamma, M)/Aut(\Gamma)$$

Theorem 2. *When Γ is a tree,*

$$\mathcal{M}(\Gamma, M) \cong \mathcal{M}(\Gamma) \times M$$
$$(\sigma, \gamma) \to \sigma \times \gamma(v)$$

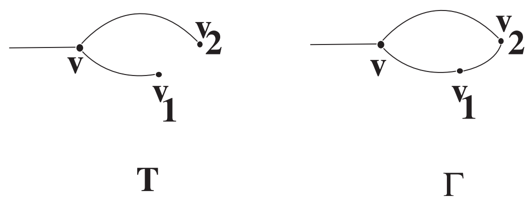
Proof. Existence and uniqueness of solutions to ODE's.

$\square$

Theorem 3. *For general* Γ*, there is a homotopy pull-back diagram*

$$\begin{array}{ccc} \tilde{\mathcal{M}}(\Gamma, M) & \xrightarrow{\hookrightarrow} & \mathcal{S}(\Gamma, M) \times M \\ \downarrow & & \downarrow \\ M^b & \xrightarrow[\Delta^b]{} & (M^2)^b. \end{array}$$

Idea when $b = 1$: Pick a maximal tree $T \subset \Gamma$ (remove edge)



$(\sigma, \gamma) \in \mathcal{M}(\Gamma) \times M$ determines a flow $\tilde{\gamma} : T \to M$ by existence and uniqueness. Have

$$(\tilde{\gamma}(v_1), \tilde{\gamma}(v_2)) \in M^2$$

$\tilde{\gamma}$ extends to $\gamma : \Gamma \to M$ iff $\tilde{\gamma}(v_1)$ and $\tilde{\gamma}(v_2)$ are connected by a flow. This will imply that the above diagram is homotopy cartesian.

This allows the construction of a Thom collapse map

$$\tau : \mathcal{M}(\Gamma) \times M \simeq BAut(\Gamma) \times M \longrightarrow \mathcal{M}(\Gamma, M)^{\nu}$$

$\nu =$ pull back of normal bundle of Δ^b. Apply homology and the Thom isomorphism, get umkehr map

$$e_! : H_*(BAut(\Gamma)) \otimes H_*(M) \to H_{*-nb}\mathcal{M}(\Gamma, M)$$

Given $[N] \in H_k(BAut(\Gamma))$ can define *virtual fundamental class*

$$[\mathcal{M}^{[N]}(\Gamma, M)] = e_!([N]\times[M]) \in H_{k+n\chi(\Gamma)}(\mathcal{M}(\Gamma, M)).$$

Idea. Suppose have $Aut(\Gamma)$-equivariant, compact manifold $\tilde{N} \subset \mathcal{S}(\Gamma, M)$, define

$$N = \tilde{N}/Aut(\Gamma) \subset \mathcal{M}(\Gamma).$$

Let

$$\mathcal{M}^{[N]}(\Gamma, M) = \{(\sigma, \gamma) \in \tilde{\mathcal{M}}(\Gamma, M) \, : \, \sigma \in \tilde{N}\}/Aut(\Gamma).$$

Smoothness???? Compactness??? If so one has fundamental class

$$[\mathcal{M}^{[N]}(\Gamma, M)] \in H_*(\mathcal{M}(\Gamma, M))$$

and it is equal to $e_!([N] \times [M])$.

Play Gromov-Witten game:

$$ev : \tilde{\mathcal{M}}(\Gamma, M) \to M^p \times M^q$$

evaluation of the graph flow at the marked points (univalent vertices)

Pull back cohomology classes, evaluate ("integrate") on the virtual fundamental class $[\mathcal{M}^{[N]}(\Gamma, M)]$. Using Poincare duality this defines operations

$$q_\Gamma : H_*(BAut(\Gamma)) \otimes H_*^{\Sigma_p}(M^p) \to H_{*+\chi(\Gamma)n-np}^{\Sigma_q}(M^q)$$

respecting gluing, natural w.r.t morphisms of graphs.

"Morse Field Theory"

Examples.

1. Consider the graph $\Gamma =$

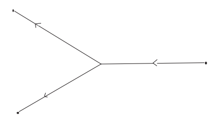


Figure 6: The “Y-graph”

$Aut(\Gamma) = \mathbb{Z}/2$. So the operation is a map

$$q_\Gamma : H_*(B\mathbb{Z}/2) \otimes H_*(M) \to H_*^{\Sigma_2}(M \times M).$$

“Equivariant diagonal”

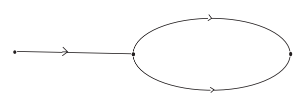
$$B\mathbb{Z}/2 \times M \to E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (M \times M)$$

Use $\mathbb{Z}/2$ coefficients. The Steenrod squares are defined if we take the dual map in cohomology. Namely, if $\alpha \in H^k(M)$, then

$$(q_\Gamma)^*(\alpha \otimes \alpha) = \sum_{i=0}^{k} a^k \otimes Sq^{k-i}(\alpha).$$

$a \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ is the generator. The Cartan and Adem relations for the Steenrod squares follow from the homotopy invariance and gluing properties.

2. Now consider the graph Γ



$Aut(\Gamma) = \mathbb{Z}/2$.

$$q_\Gamma : H_*(B\mathbb{Z}/2) \otimes H_*(M) \to \mathbb{Z}/2,$$

or equivalently, $q_\Gamma \in H^*(B\mathbb{Z}/2) \otimes H^*(M)$.

$$q_\Gamma = \sum_{i=0}^{d} a^i \otimes w_{d-i}(M)$$

where $d = dim(M)$, and $w_j(M) \in H^j(M; \mathbb{Z}/2)$ is the j^{th} Stiefel-Whitney class of the tangent bundle.

We now study the loop space, LM.

Let $V : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ be a smooth map. "Potential function". Give M a Riemannian metric. Define energy function

$$\begin{aligned} \mathcal{S}_V : LM &\longrightarrow \mathbb{R} \\ \gamma &\longrightarrow \int_0^1 \left(\frac{1}{2} |\frac{d\gamma}{dt}|^2 + V(t, \gamma(t)) \right) dt. \end{aligned} \tag{1}$$

For generic choice of V, $\mathcal{S}_V$ is a Morse function (J. Weber). Its critical points are those $\gamma \in LM$ satisfying the ODE

$$\nabla_t \frac{d\gamma}{dt} = -\nabla V_t(x) \tag{2}$$

where $\nabla V_t(x)$ is the gradient of the function $V_t(x) =$

$V(t,x)$, and $\nabla_t \frac{d\gamma}{dt}$ is the Levi-Civita covariant derivative.

There is CW-complex with one cell for each critical point $\simeq LM$, yielding Morse chain complex,

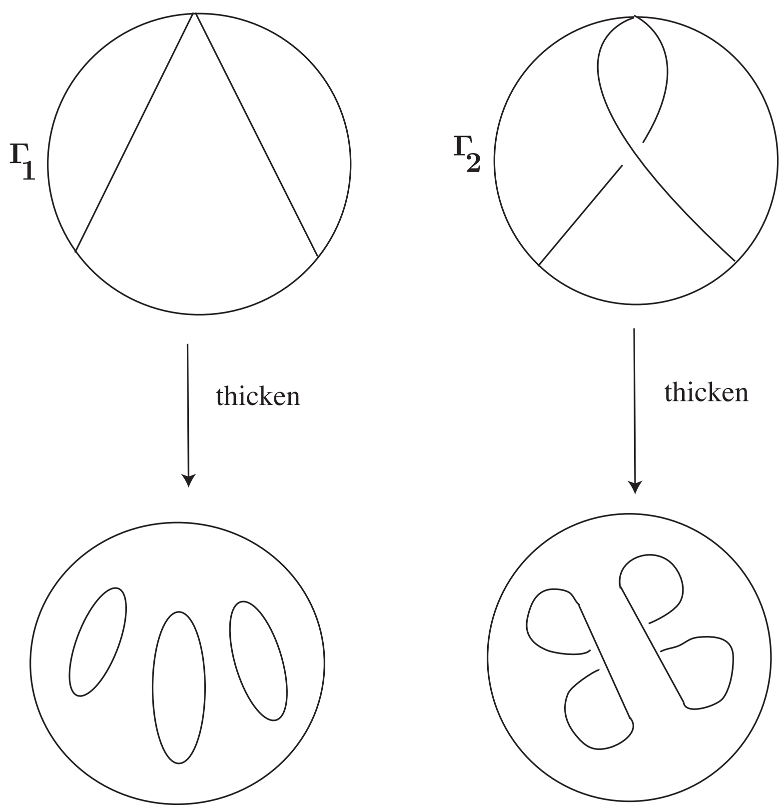
$$\cdots \longrightarrow C_q^V(LM) \xrightarrow{\partial} C_{q-1}^V(LM) \xrightarrow{\partial} \cdots$$

To study graph flows in LM, we use "fat (ribbon)" graphs.

A *fat graph* is a finite, combinatorial graph (one dimensional CW complex - no "leaves") such that

1. Vertices are at least trivalent
2. Each vertex has a cyclic order of the (half) edges.

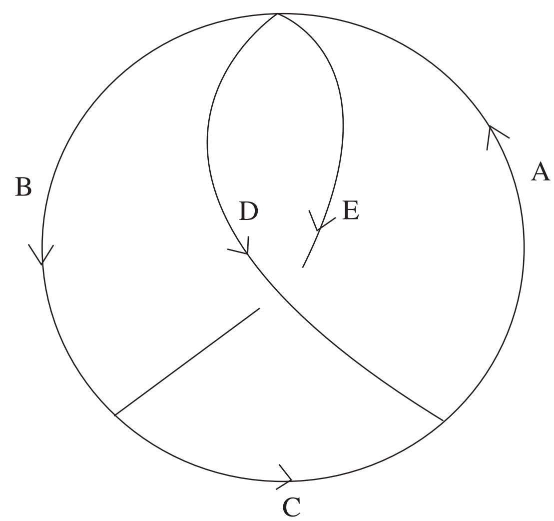
ORDER IS IMPORTANT!



Do the combinatorics more precisely:

Let E_G = set of edges of G, $\tilde{E}_G$ = set of oriented edges. So each edge of G appears twice in $\tilde{E}_G$: $e, \bar{e}$

Have a partition of $\tilde{E}_G$:



$$(A, B, C), \quad (\bar{A}, \bar{D}, E, \bar{B}, D, \bar{C}, \bar{E})$$

Theorem 4. *(Penner, Strebel) The space of metric fat graphs of topological type* $(g, n) \simeq \mathcal{M}_{g,n}$.

Given a fat graph G of type (g, n), designate p boundary cycles as "incoming" and $q = n - p$ boundary cycles as "outgoing".

Let Γ be a metric fat graph. Have parameterizations of the boundary cycles

$$\alpha^- : \coprod_p S^1 \longrightarrow \Gamma, \quad \alpha^+ : \coprod_q S^1 \longrightarrow \Gamma.$$

By taking the circles to have circumference equal to the sum of the lengths of the edges making up the boundary cycle it parameterizes, each component of α^+ and α^- is a local isometry.

Define the surface Σ_Γ to be the mapping cylinder of these parameterizations,

$$\Sigma_\Gamma = \left(\coprod_p S^1 \times (-\infty, 0] \right) \sqcup \left(\coprod_q S^1 \times [0, +\infty) \right) \bigcup \Gamma / \sim \tag{3}$$

where $(t, 0) \in S^1 \times (-\infty, 0] \sim \alpha^-(t) \in \Gamma$, and $(t, 0) \in S^1 \times [0, +\infty) \sim \alpha^+(t) \in \Gamma$

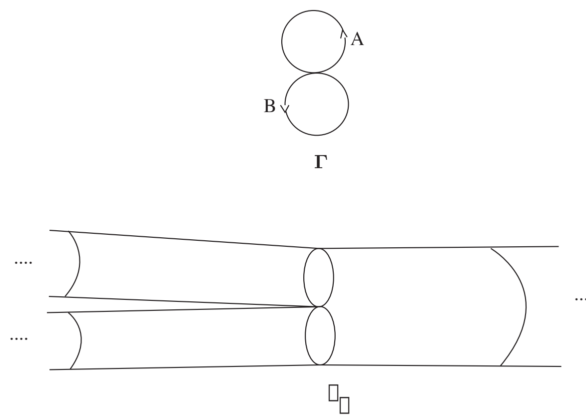


Figure 7: Σ_Γ

Definition 3. *An LM-structure σ on Γ is a metric and labeling of each boundary cylinder by a distinct $V_i : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ (or $\mathcal{S}_{V_i}$).*

$$\mathcal{M}^\sigma(\Gamma, LM) = \{\gamma : \Sigma_\Gamma \to M \; : \; \frac{d\gamma_i}{dt} + \nabla \mathcal{S}_{V_i} = 0\}$$

Using work of Kaufmann, these define operations

$$q_\Gamma : H_*(\mathcal{M}_{g,p+q}) \otimes H_*(LM)^{\otimes p} \to H_*(LM)^{\otimes q}$$

Theorem 5. *(C.) These are the string topology operations.*

(C., Godin) They respect gluing of surfaces and define a positive boundary homological CFT.

Back to T^*M, with its symplectic form ω:

Have a symplectic action functional

$$\mathcal{A} : L(T^*M) \to \mathbb{R}$$

Given $\gamma \in LM$ and $\eta(t) \in T^*_{\gamma(t)}M$, then

$$\mathcal{A}(\gamma, \eta) = \int_0^1 \langle \eta(t), \frac{d\gamma}{dt}(t) \rangle dt.$$

J_g-holomorphic cylinders $S^1 \times \mathbb{R} \to T^*M$ are gradient trajectories.

Floer: Do Morse theory with $\mathcal{A}$. Perturb using potential $V : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$.

Define Hamiltonian,

$$H_V : \mathbb{R}/\mathbb{Z} \times T^*M \to \mathbb{R}$$
$$(t, (x, v)) \to \frac{1}{2}||v||^2 - V(t, x)$$

Perturbed symplectic action:

$$\mathcal{A}_V(\gamma,\eta) = \mathcal{A}(\gamma,\eta) + \int_0^1 H_V(t,\gamma(t),\eta(t)dt.$$

Critical points of $\mathcal{A}_V =$

$$\{(\gamma,\eta) \in L(T^*M) \ : \gamma \text{ is a critical point of } \mathcal{S}_V,$$
$$\eta_(t) = \langle \frac{d\gamma}{dt}, - \rangle\}$$

So the critical points of $\mathcal{A}_V$ are in bijective correspondence with the critical points of $\mathcal{S}_V$.

Have Floer complex, $CF_*^V(T^*M)$

$$\cdots \longrightarrow CF_q^V(LM) \xrightarrow{\partial} CF_{q-1}^V(LM) \xrightarrow{\partial} \cdots$$

generated by the critical points of $\mathcal{A}_V$ with boundary maps computed by counting the trajectories between critical points. (cylinders $S^1 \times \mathbb{R} \to T^*M$ satisfying perturbed Cauchy-Riemann equations- "J_g -holomorphic cylinders with respect to V")

Theorem 6. *(Viterbo, Salamon-Weber) The Morse complex of $\mathcal{S}_V$ is chain homotopy equivalent to the*

Floer complex of $\mathcal{A}_V$*. Therefore*

$$HF_*(T^*M) \cong H_*(LM).$$

Theorem 7. *(C.) Can replace moduli space* $\mathcal{M}^\sigma(\Gamma, LM)$ *of graph flows in* LM *by*

$$\mathcal{M}^\sigma_{hol}(\Sigma_\Gamma; T^*M) = \{\phi : \Sigma_\Gamma \to T^*M,\ J_g \text{ holomorphic on the } i^{th} \text{ cylinder with respect to } V_i.\}$$

to define the string topology operations q_Γ*.*

Moral of Story: String topology = Gromov-Witten theory on T^*M with very thin Riemann surfaces with cylindrical ends.

Consequence. (Abondondolo-Schwarz) The pair of pants (quantum) product in $HF_*(T^*M)$ corresponds to the Chas-Sullivan loop product in H_*LM.