NK_2 DOES NOT REDUCE TO FINITES

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1. INTRODUCTION

Let Γ be a discrete group and $\mathbb{Z}\Gamma$ its integral group ring. The Farrell-Jones Isomorphism Conjecture predicts that the algebraic K-theory groups $K_i(\mathbb{Z}\Gamma)$ may be computed from the corresponding algebraic K-theory of its virtually cyclic group rings. In case this Conjecture holds there have been explicit examples like [], and it has been the case that these computations may even be reduced to finite subgroups []. The groups that allow such reductions are the Nil groups of the finite groups involved in Γ. In this paper we show that, in principle, such reduction cannot be achieved for $K_2(\mathbb{Z}\Gamma)$. Our Main result is the following:

Theorem 1. *Let n be natural number and C_n denote a finite cyclic group of order n. Then $NK_2(\mathbb{Z}C_n) \neq 0$.*

As a corollary,

Corollary 2. *$N^jK_i(\mathbb{Z}C_n) \neq 0$ for all $i \geq 2$ and $j \geq 1$.*

As a consequence, we get the following:

Theorem 3. *Let G be any finite group. Then $NK_i(\mathbb{Z}G) \neq 0$ for all $i \geq 2$.*

Theorem 4. *Let G be any finite group. Then $N^jK_i(\mathbb{Z}G) \neq 0$ for all $i, j \geq 2$.*

Corollary 5. *Higher K theory of integral group rings does not reduce to finites*

Let G be an hyperbolic group with torsion. If IC holds for G then $K_i(\mathbb{Z}G)$ is not finitely generated for all $i \geq 2$.

This contrasts with $NK_1(\mathbb{Z}C_n)$ where it is known [B-M] that

$$NK_1(\mathbb{Z}C_n) = 0 \text{ if and only if } n \text{ is square free.}$$

As the NK_2 terms are direct summands in the group $K_2(\mathbb{Z}[C_n \times T_s])$, where T_s is the free abelian group of rank s, $s \geq 1$, and it is known [Farr] that if NK_2 is nonzero then it is not finitely generated, we have the following corollary

Corollary 6. *Let C_n be a finite cyclic group and T_s a free abelian group of rank s, $s \geq 1$, then*

$$K_2(\mathbb{Z}[C_n \times T_s])$$

is not finitely generated.

On the other hand it was proved by Vorst that if $NK_2(R)$ is nonzero, then $NK_i(R) \neq 0$ for all $j \geq 2$. From this we have the following corollary

Corollary 7. *Let C_n and T_s be a finite cyclic group of order n, $n \geq 2$, and the free abelian group of rank $s \geq 1$, respectively. Then*

(1) $N^j K_i(\mathbb{Z}C_n) \neq 0$ *for all* $j \geq 1$, *and* $i \geq 2$,
(2) $K_i(\mathbb{Z}[C_n \times T_s]$ *is not finitely generated for all* $i \geq 2$.

Proposition 8. *Let $G = C \rtimes H$ be an hyperelementary group. Then*

$$NK_2(\mathbb{Z}[G]) \neq 0.$$

Proof. Use the isomorphism $\mathbb{Z}[G] \cong (\mathbb{Z}C_p)_\alpha[H]$ to get a split ring epimomorphism $\epsilon : (\mathbb{Z}C_p)_\alpha[H] \to \mathbb{Z}[C_p]$. By functoriality, this gives that $NK_2(\mathbb{Z}[C_p])$ injects in $NK_2(\mathbb{Z}[G])$. □

We have the following corollary

Corollary 9. *Let $G = C \rtimes H$ be an hyperelementary group. Then*

(1) $N^i K_2(\mathbb{Z}[G]) \neq 0$, *for all* $i \geq 1$.
(2) $N^i K_j(\mathbb{Z}[G]) \neq 0$, *for all* $i \geq 1, j > 2$.

The general results would follow from induction theory.

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