Conformal geometry of knots

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  - The meanings of its real and imaginary parts $\Longleftarrow$ symplectic form and area elements
  - Tools: Minkowski space, semi-Riemannian structure of the space of 0-spheres in S^3, and pencils of codimension 1 spheres.

Motivation of energy of knots

- **Problem** (Fukuhara, Sakuma)
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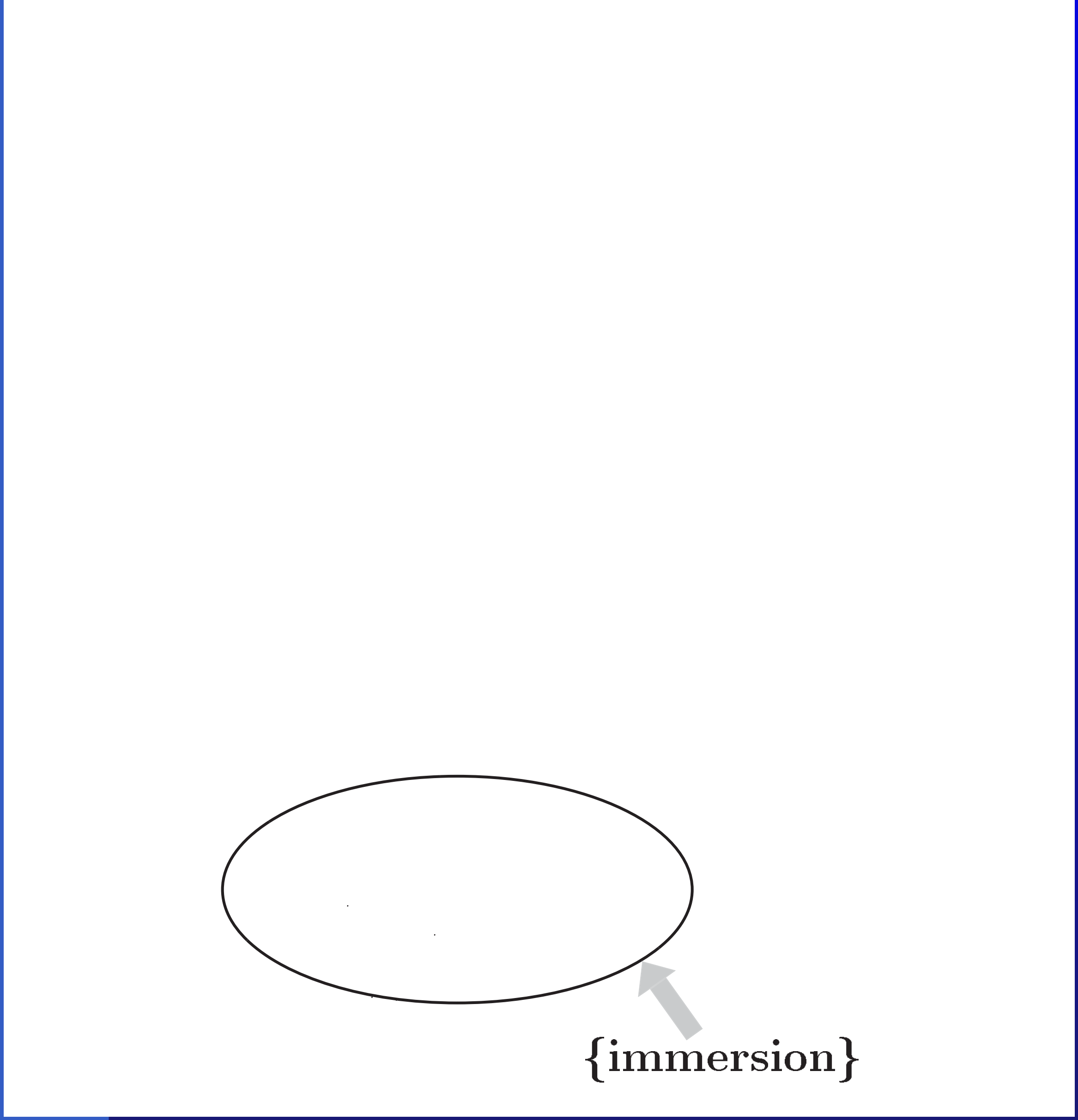
  - We call it an <u>e-*minimizer*</u>.

Our strategy

- A conceptual illustration



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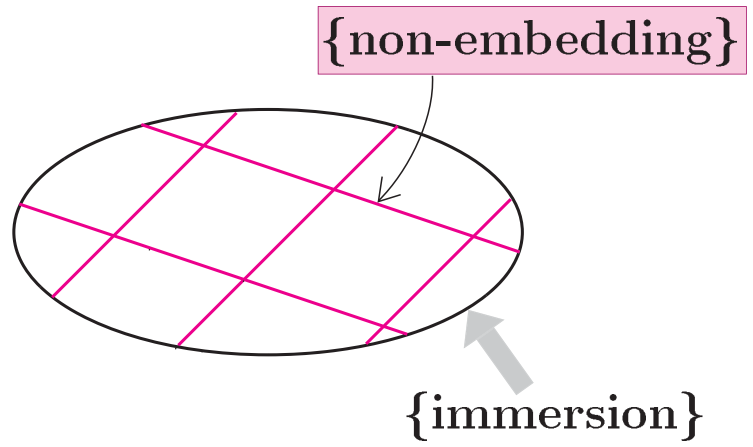

The complement is the set of embeddings, i.e. the space of knots.

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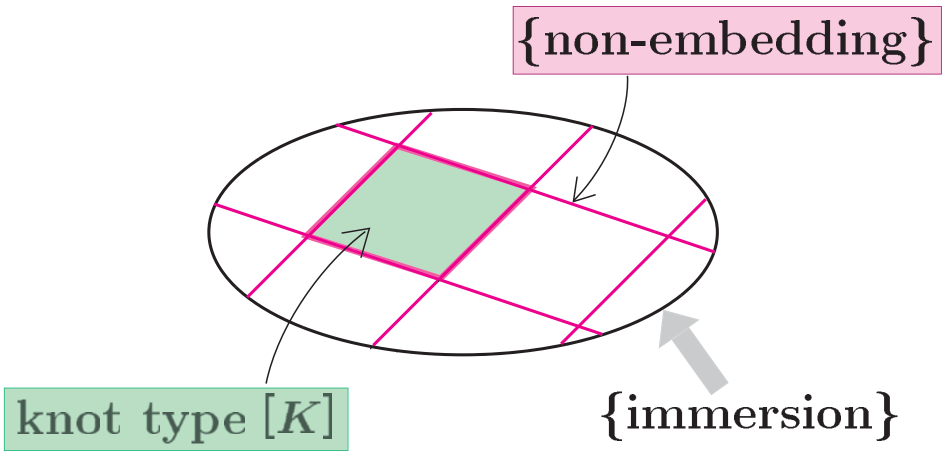

Each "cell" corresponds to a knot type, as two points in the space of knots can be connected by a path if and only if two corresponding knots are ambient isotopic.

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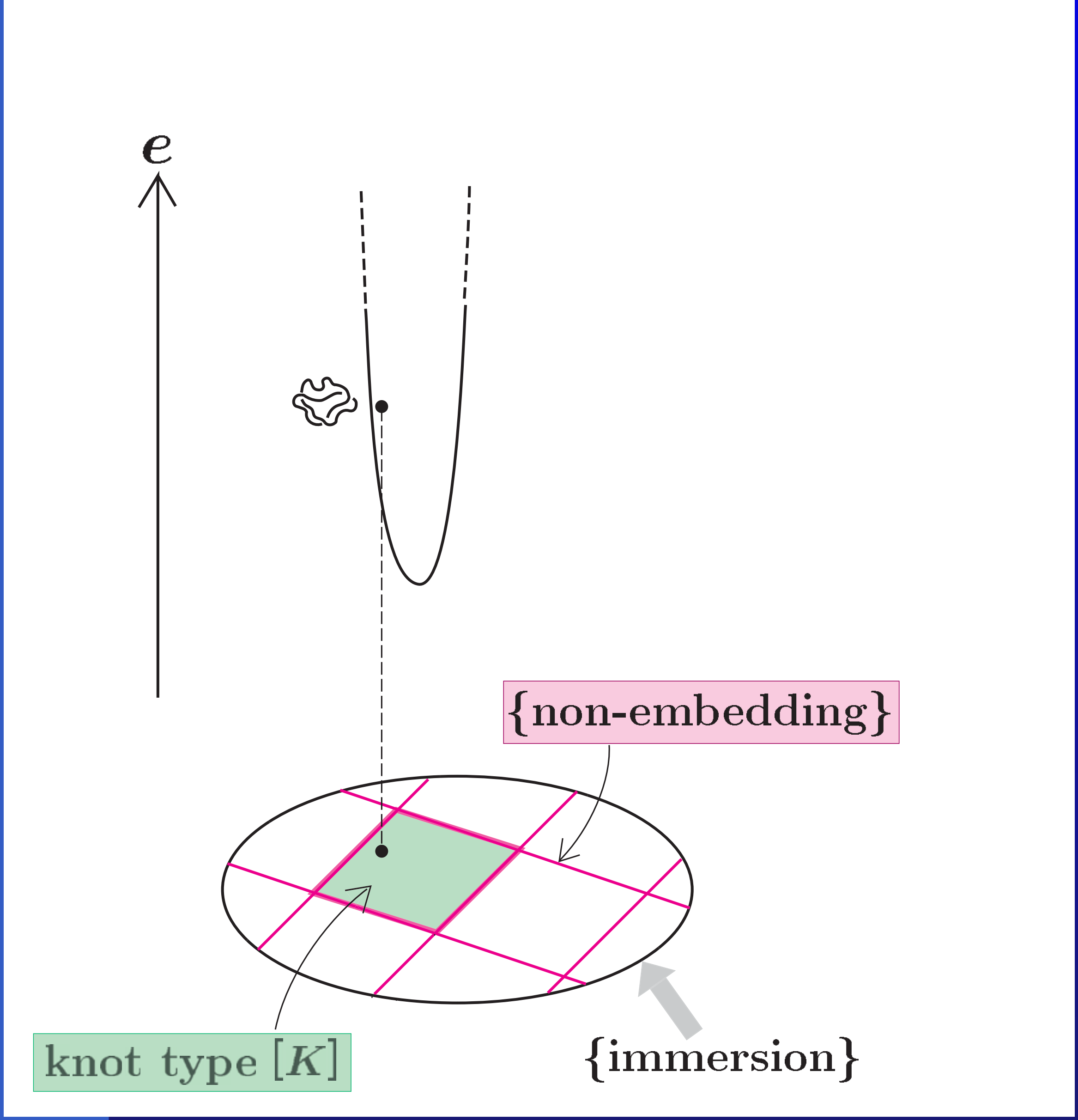

Given a knot.

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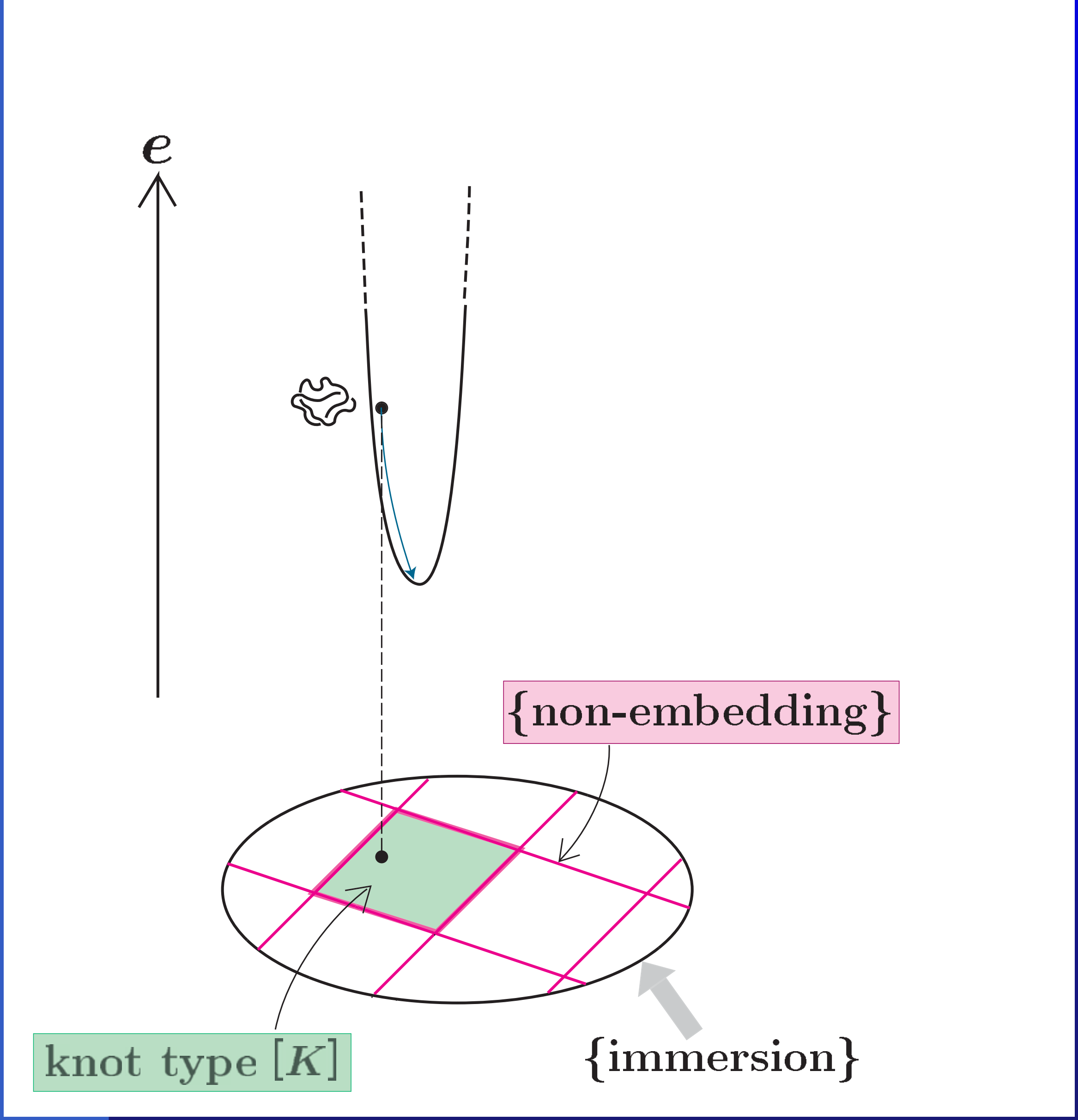

Deform it along the gradient flow of the "energy" $e : \{\text{knots}\} \to \mathbb{R}$.

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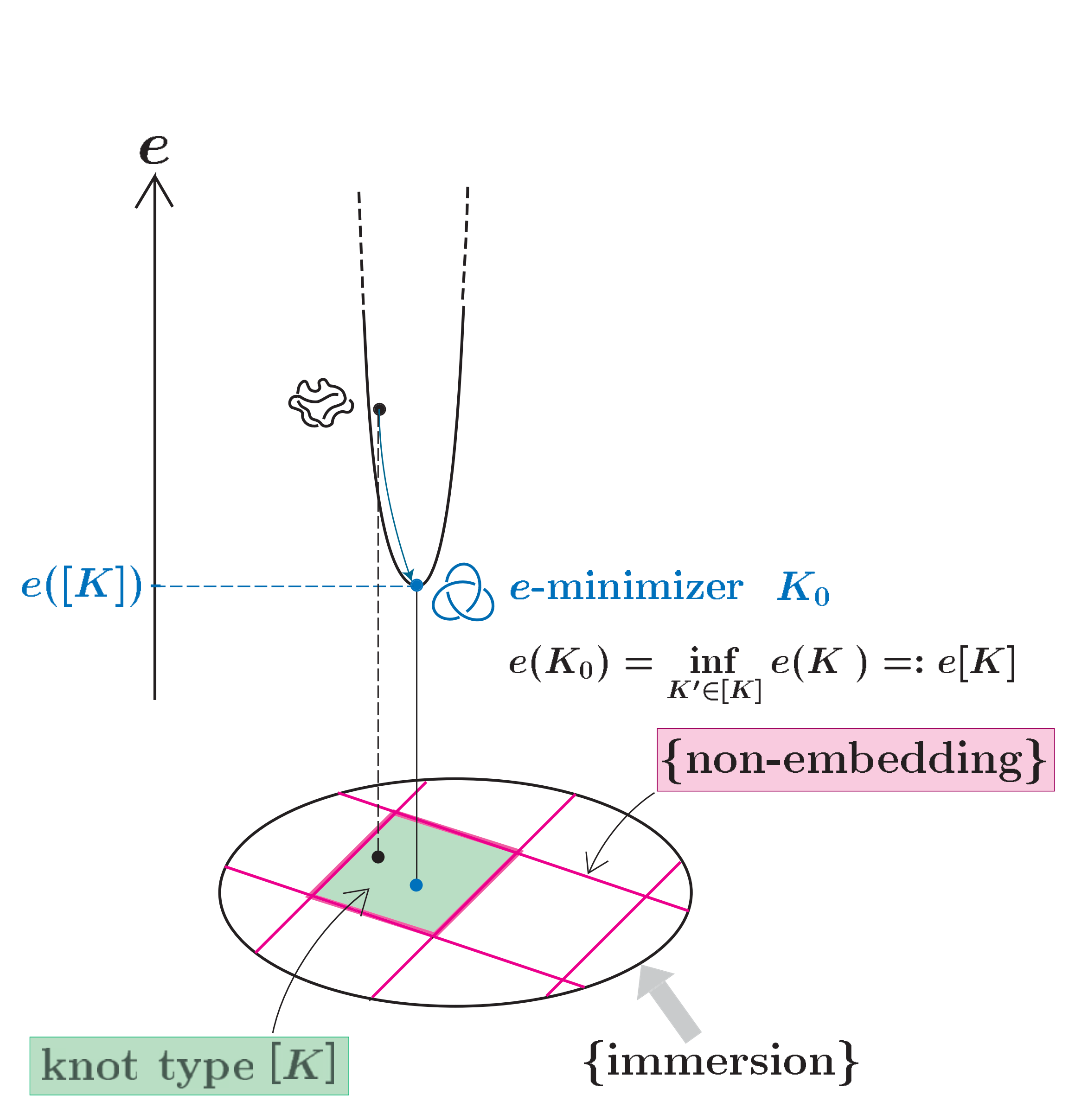
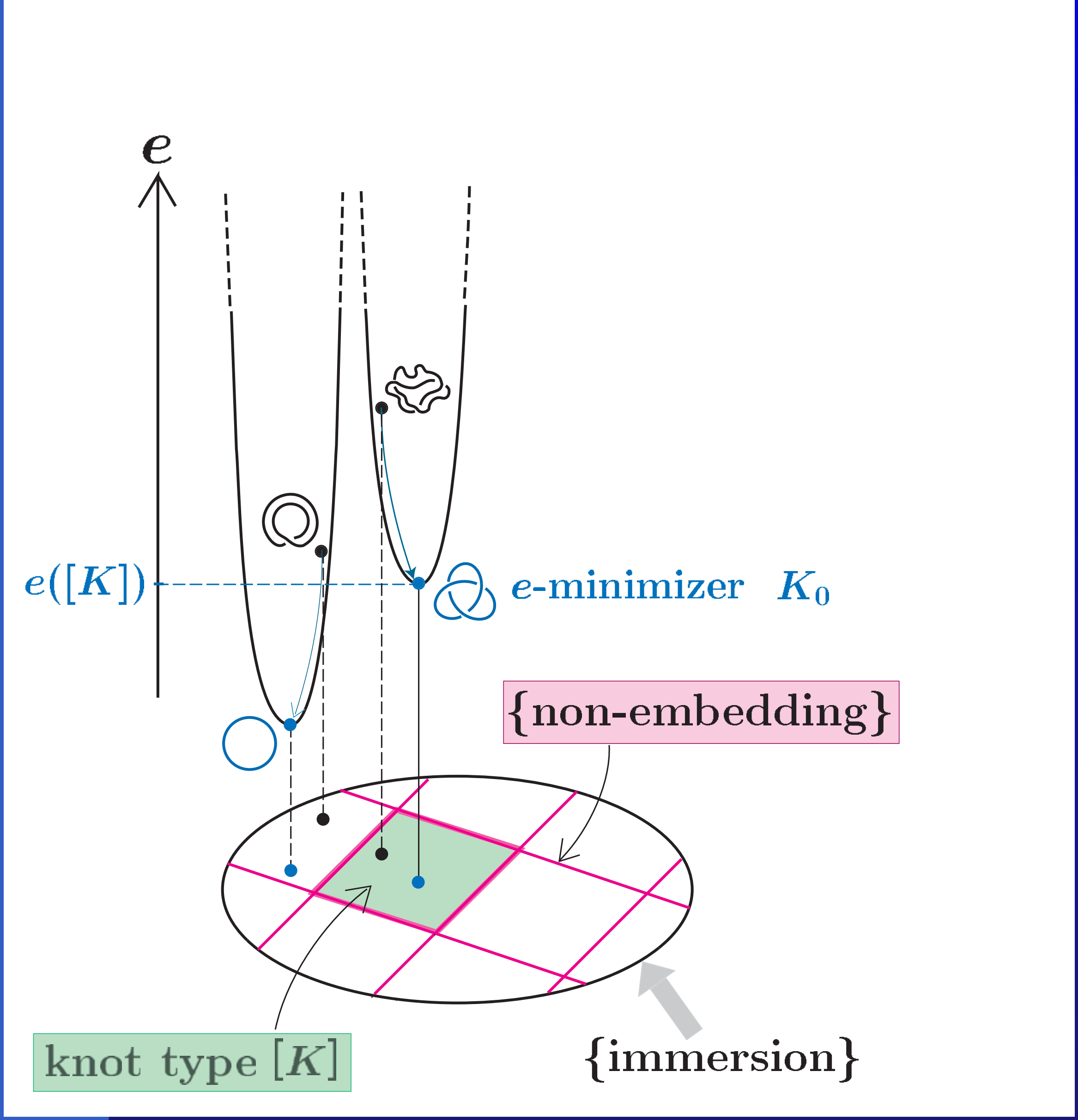


If we are lucky the knot might reach an e-minimizer, which is a "canonical position" of that knot type.

Our strategy

- Required property of our functional



In order to keep the knot type unchanged during the deformation process,

crossing changes should be avoided!

Our strategy

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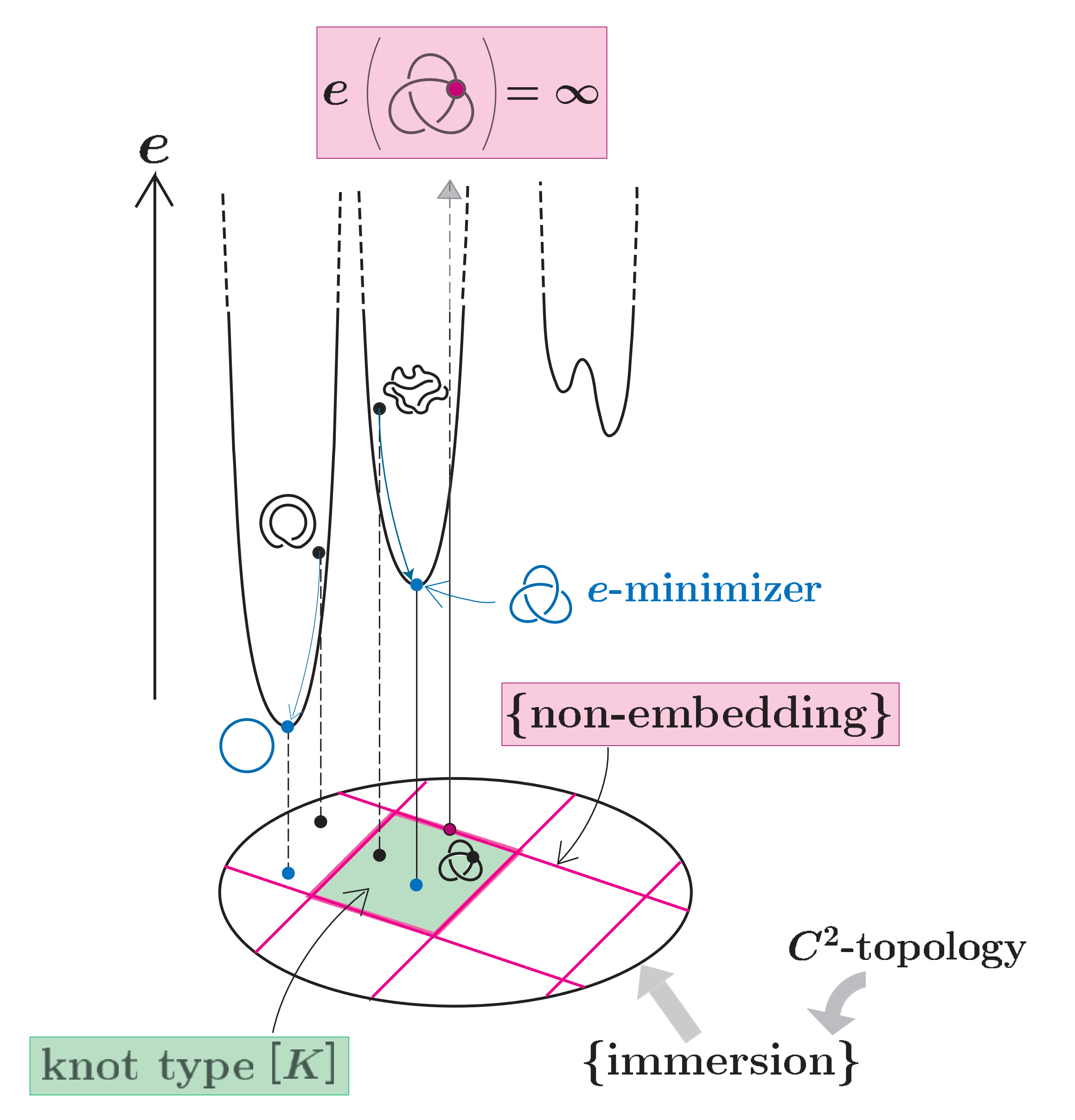

Definition.

e is "*self-repulsive*"

def. $\Updownarrow$

$e(K)$ blows up as a knot K degenerates to a singular knot with double points.

We say that e is an *energy of knots* if it is self-repulsive.

Electrostatic energy of charged knots

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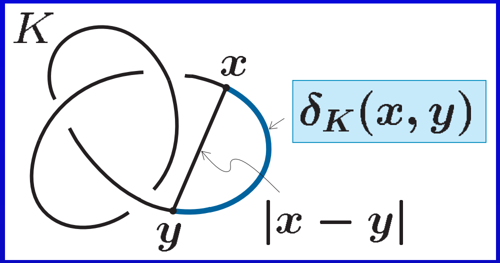
- But $E^{(1)}(\widehat{K}) < \infty$ for a singular knot $\widehat{K}$ with double points.
- Increase the power of $|x-y|$ in the integrand to produce a self-repulsive energy.
- $2 \leq$ the power $< 3 \Longrightarrow$ a well-defined energy.

Definition of $E_{\circ}^{(2)}$

- Let $\delta_K(x, y)$ denote the arc-length between x and y.



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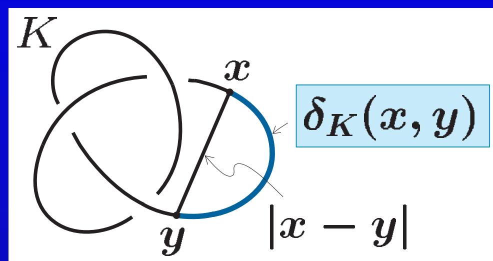

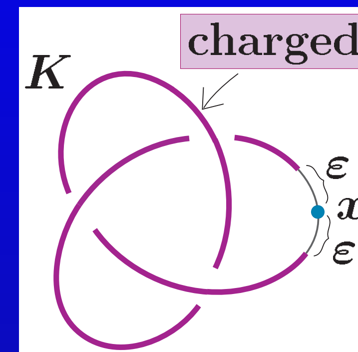

- $$E_{\circ}^{(2)}(K) = \lim_{\varepsilon \to +0} \left(\iint_{\{\delta_K(x,y) \geq \varepsilon\} \subset K \times K} \frac{dxdy}{|x-y|^2} - \frac{2}{\varepsilon} \right)$$

We assumed that $Length(K) = 1$ in the above.

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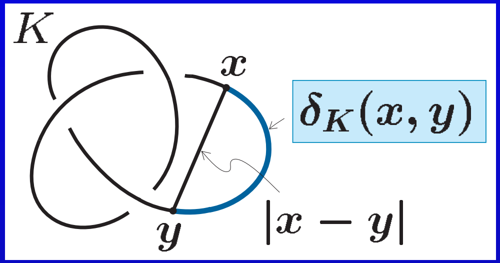

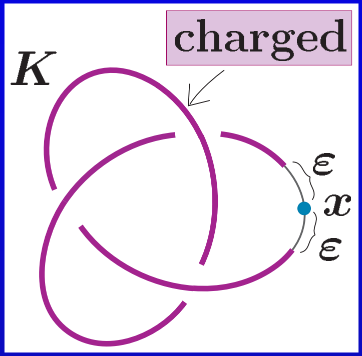

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- $$E_{\circ}^{(2)}(K) = -4 + \iint_{K \times K \setminus \triangle} \left(\frac{1}{|x-y|^2} - \frac{1}{\delta_K(x,y)^2} \right) dxdy.$$

Properties of $E_\circ^{(2)}$

- **Theorem** (Freedman-He-Wang) $E_\circ^{(2)}$ is conformally invariant, i.e. if T is a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$ then $E_\circ^{(2)}(T(K)) = E_\circ^{(2)}(K) \quad \forall K.$

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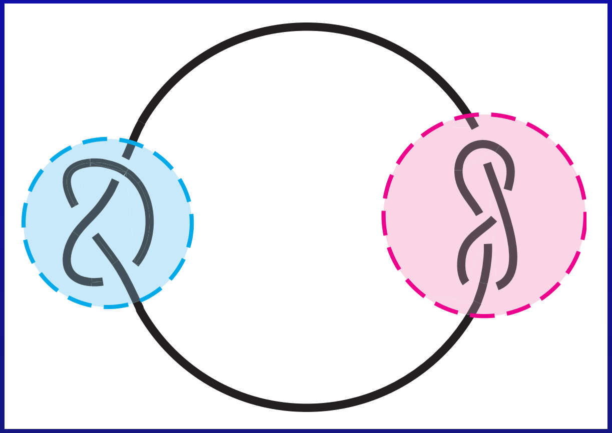
- Prime $=$ Not composite. A composite knot:



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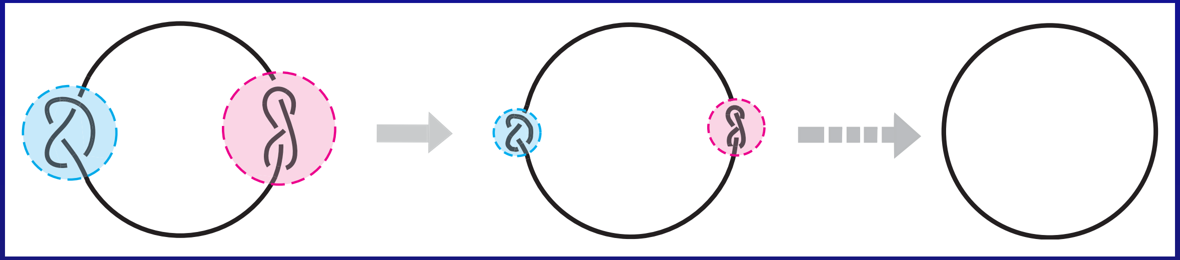
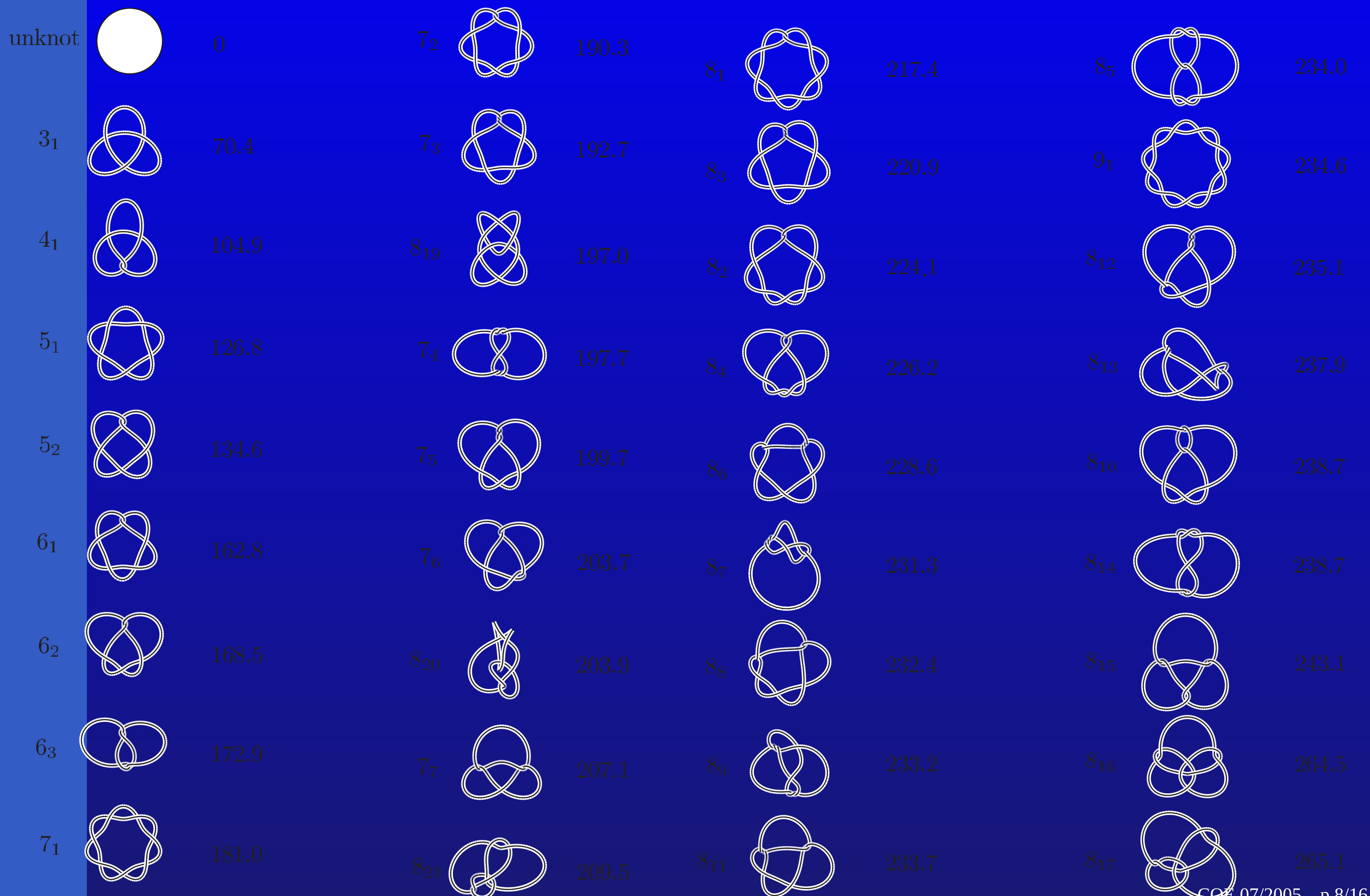
- **Theorem** (Freedman-He-Wang) There exists an $E_\circ^{(2)}$-minimizer for any *prime* knot type.
- **Conjecture** (Kusner-Sullivan) There are no $E_\circ^{(2)}$-minimizers for any *composite* knot types. Numerical experiments imply



Numerical experiments by Kusner and Sullivan

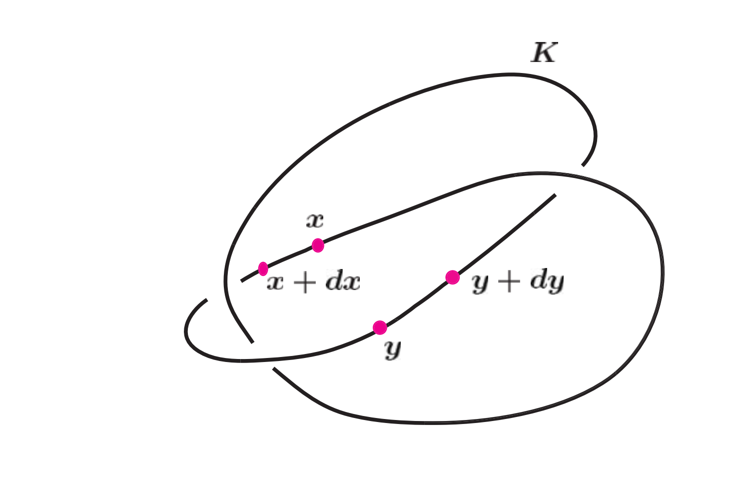


Part II. Conformal geometry

- Joint work with Rémi Langevin

Infinitesimal cross ratio

- Geometric definition. Let $x, x + dx, y, y + dy \in K$.



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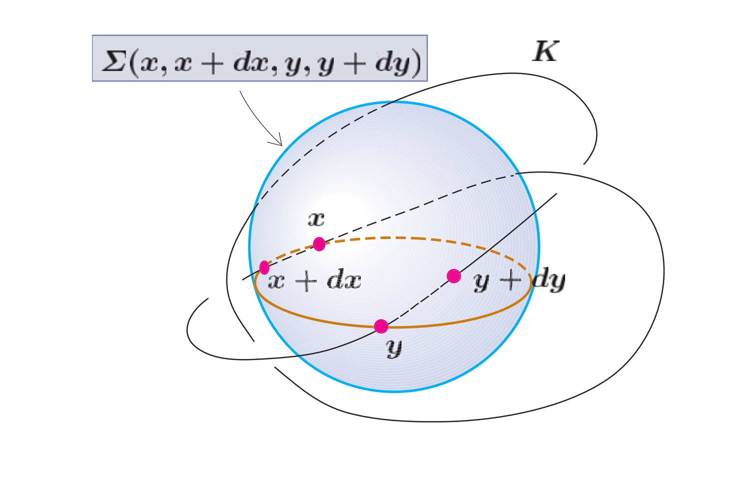

Let $\Sigma = \Sigma(x, x+dx, y, y+dy)$ be a 2-sphere through $x, x+dx, y, y+dy$.

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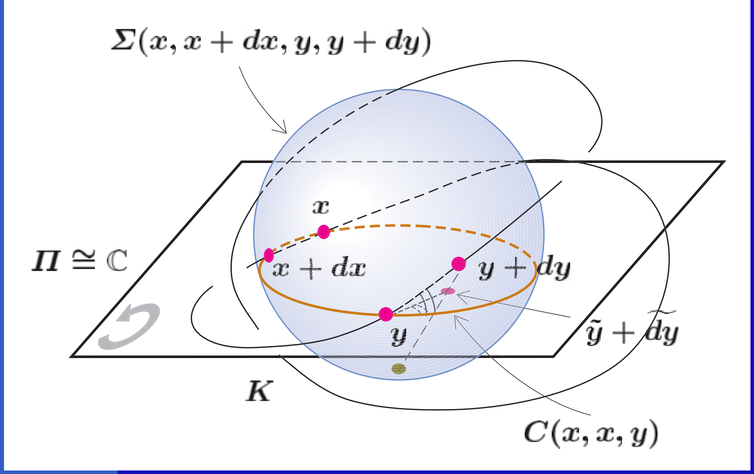

Let $\Sigma = \Sigma(x, x+dx, y, y+dy)$ be a 2-sphere through $x, x+dx, y, y+dy$. Identify $\Sigma \cong \mathbb{C} \cup \{\infty\}$ through a stereographic projection. These four points can be identified with four complex numbers $\tilde{x}, \tilde{x}+\widetilde{dx}, \tilde{y}, \tilde{y}+\widetilde{dy}$.

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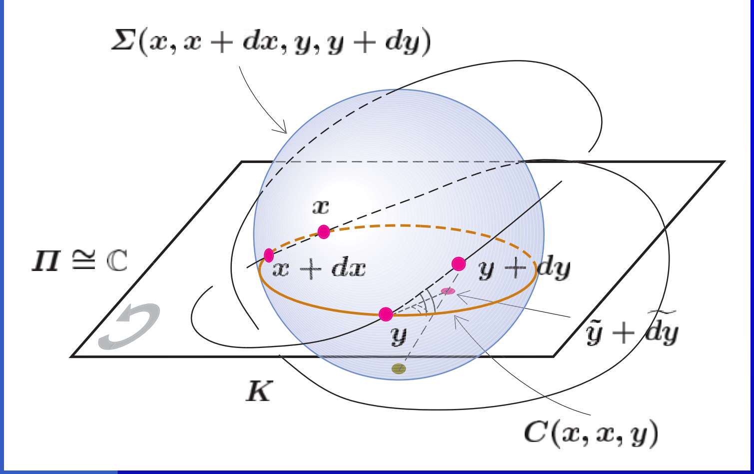

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- **Definition.** Let the *infinitesimal cross ratio* of a knot, $\Omega_{CR}(x, y)$, be the cross ratio

$$\frac{(\tilde{x}+\widetilde{dx})-\tilde{x}}{(\tilde{x}+\widetilde{dx})-(\tilde{y}+\widetilde{dy})} : \frac{\tilde{y}-\tilde{x}}{\tilde{y}-(\tilde{y}+\widetilde{dy})} \sim \frac{\widetilde{dx}\widetilde{dy}}{(\tilde{x}-\tilde{y})^2}.$$

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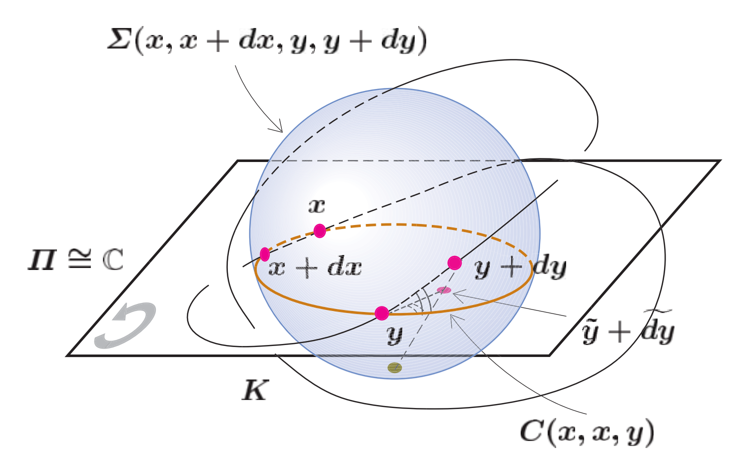

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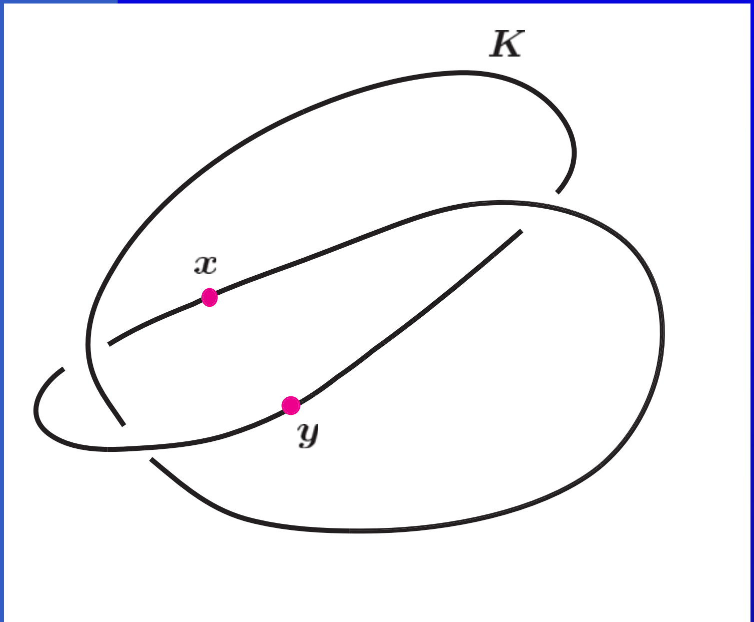
- The four complex numbers are not uniquely determined. But the cross ratio is well-defined. We need the orientation of Σ.

Conformal angle and Ω

- **Definition.** (Doyle and Schramm)



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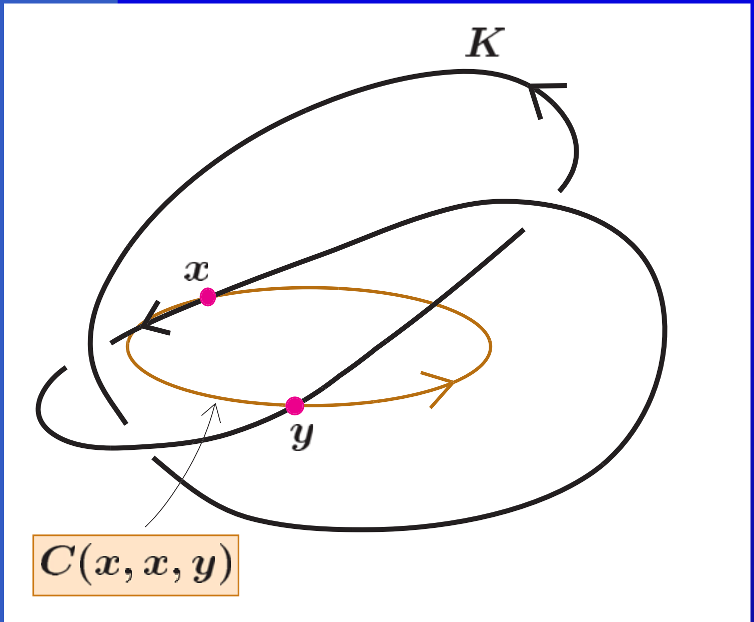

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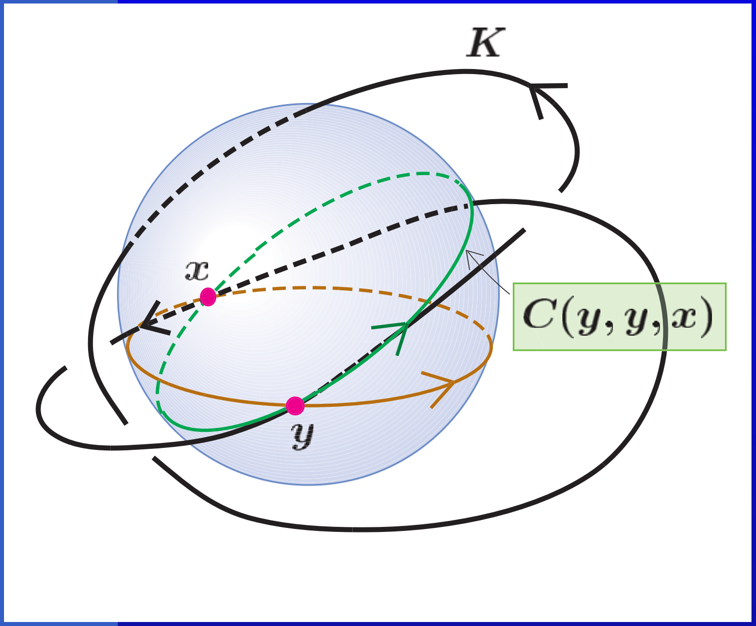

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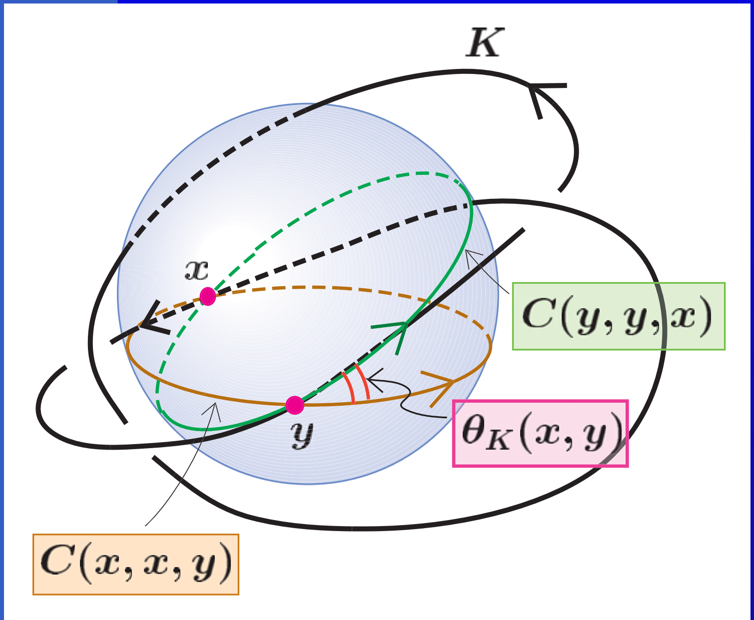

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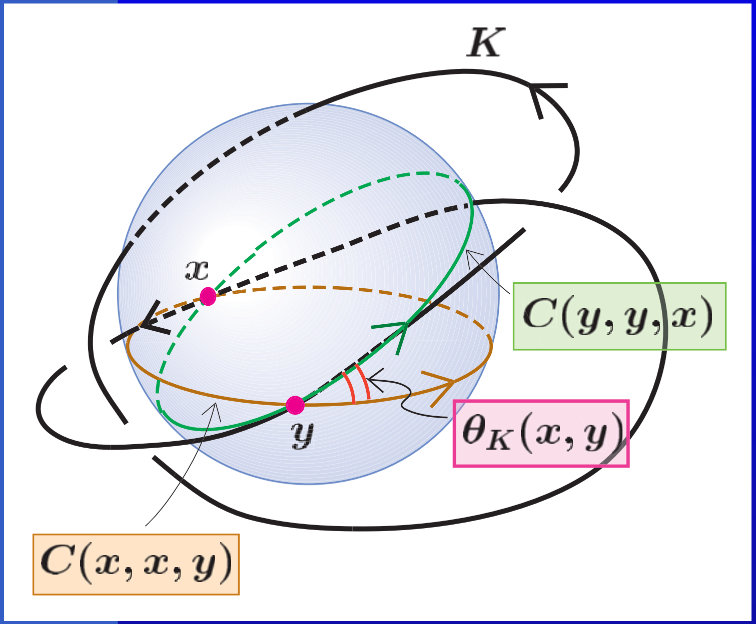

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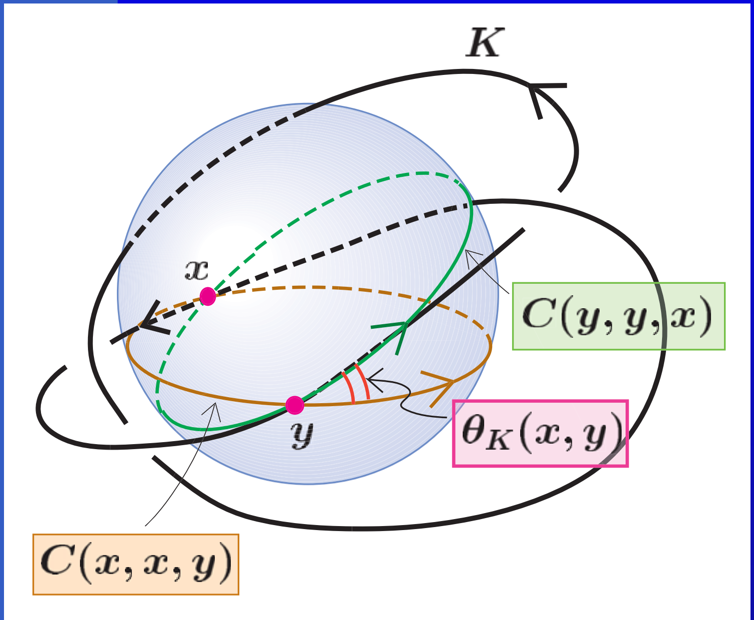

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- **Proposition.** $\boxed{\Omega(x,y) = e^{i\theta_K(x,y)} \dfrac{dxdy}{|x-y|^2}.}$

- **Proposition** (Doyle and Schramm's cosine formula)

$$E_{\circ}^{(2)}(K) = \iint_{K\times K\setminus\triangle} \frac{1-\cos\theta_K(x,y)}{|x-y|^2}\,dxdy.$$

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- **Proposition** $\boxed{E_\circ^{(2)}(K) = \iint_{K\times K\setminus\triangle} (|\Omega_{CR}| - \mathfrak{Re}\,\Omega_{CR}).}$

$\Re\mathfrak{e}\,\Omega$ and a canonical symplectic form

- Recall every T^*M admits a "*canonical symplectic form*" (locally $\omega_M = \sum dq_i \wedge dp_i$), which is exact ($\omega_M = -d \sum p_i dq_i$).

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- Inclusion $\iota : K \times K \setminus \triangle \hookrightarrow S^3 \times S^3 \setminus \triangle$.
- **Proposition** $\boxed{\Re\mathfrak{e}\,\Omega(x,y) = -\frac{1}{2}\iota^*\omega_{S^3}.}$

Semi-Riemannian str. of $S^3 \times S^3 \setminus \triangle$

- Put $\mathcal{S}(3,0) := \{S^0 \subset S^3\} \cong S^3 \times S^3 \setminus \triangle$.

Semi-Riemannian str. of $S^3 \times S^3 \setminus \triangle$

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  - Homogeneous space
    $\mathcal{S}(3,0) \cong SO(4,1)/SO(3) \times SO(1,1)$.

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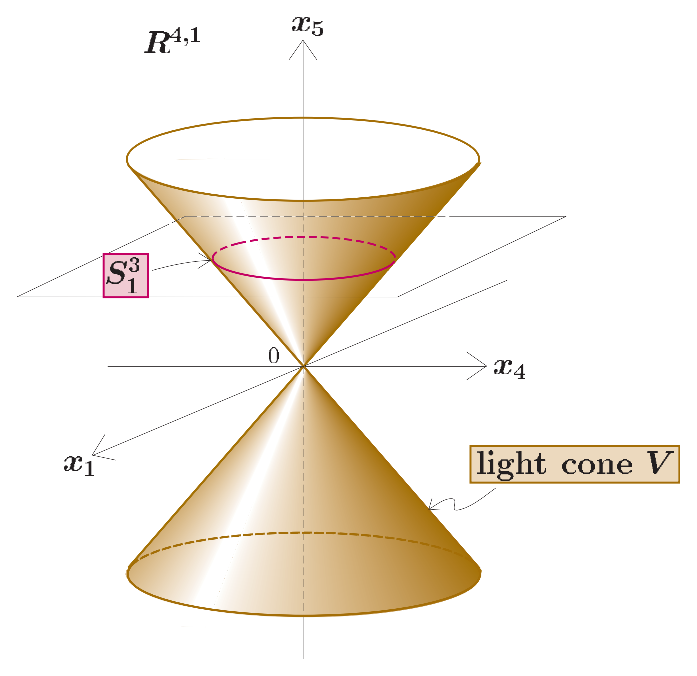

Plücker coordinates.
Let $\mathbb{R}^{4,1}$ be the Minkowski space and V be the light cone.
$\mathcal{S}(3,0) \cong \{$2-plane $\Pi \subset \mathbb{R}^{4,1} \mid \mathbf{0} \in \Pi, \Pi \cap V$ transversely$\} \subset \mathbb{R}^{4,1} \wedge \mathbb{R}^{4,1} \cong \mathbb{R}^{4,6}$.

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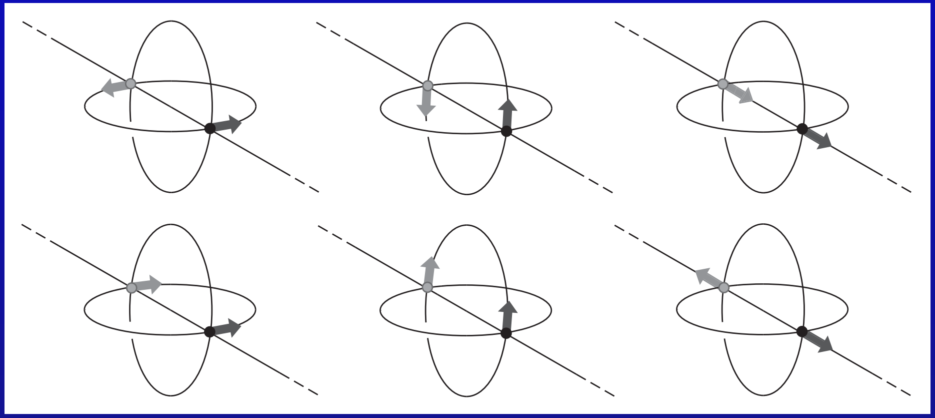
  - 6 pencils of $S^0 \subset S^1$, 3 space-like and 3 time-like.



$\Re\Omega$ as area form

- Let $\boldsymbol{v}$ be a composite $\boldsymbol{v} : K \times K \setminus \triangle \hookrightarrow S^3 \times S^3 \setminus \triangle \xrightarrow{\cong} \mathcal{S}(3,0)$. Then $\boldsymbol{v}(K \times K \setminus \triangle)$ is a surface in $\mathcal{S}(3,0)$.

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- **Theorem.** Its "area element" w.r.t. the semi-Riem. str. is given by

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- **Corollary.** Let $\gamma_1 \cup \gamma_2$ be a 2-component link. Then the area of $\boldsymbol{v}(\gamma_1 \times \gamma_2) \subset \mathcal{S}(3,0)$ is equal to 0.

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