A Parametrisation of Teichmüller Space by Polygons

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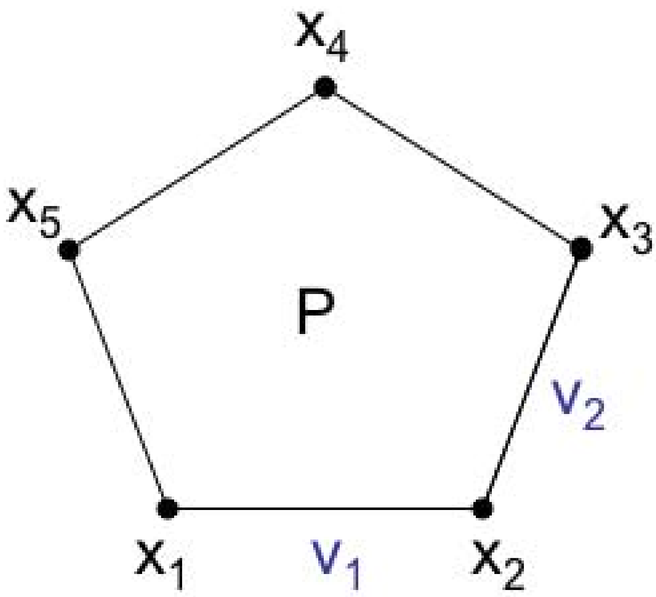
Euclidean Polygon



P consists of a finite sequence of points $x_1, \ldots, x_n$ in E^2 together with the oriented edges

$v_i = x_{i+1} - x_i \neq 0$, which can be regarded as a complex number for i =1, …, n. ($n \geq 3$)

Two polygons P and P' are congruent (resp. similar) $\Leftrightarrow$

$\exists$ an orientation preserving isometry (resp. similarity) of E^2 which sends the vertices and edges of P to those of P' preserving the indices.

Moduli space of Euclidean polygons

- The space of congruence classes of Euclidean n-gons is defined as

  $\{(v_1, \ldots, v_n) \in (\mathbf{C}^*)^n \mid v_1 + \cdot \ \cdot \ \cdot + v_n = 0\} / U(1)$,

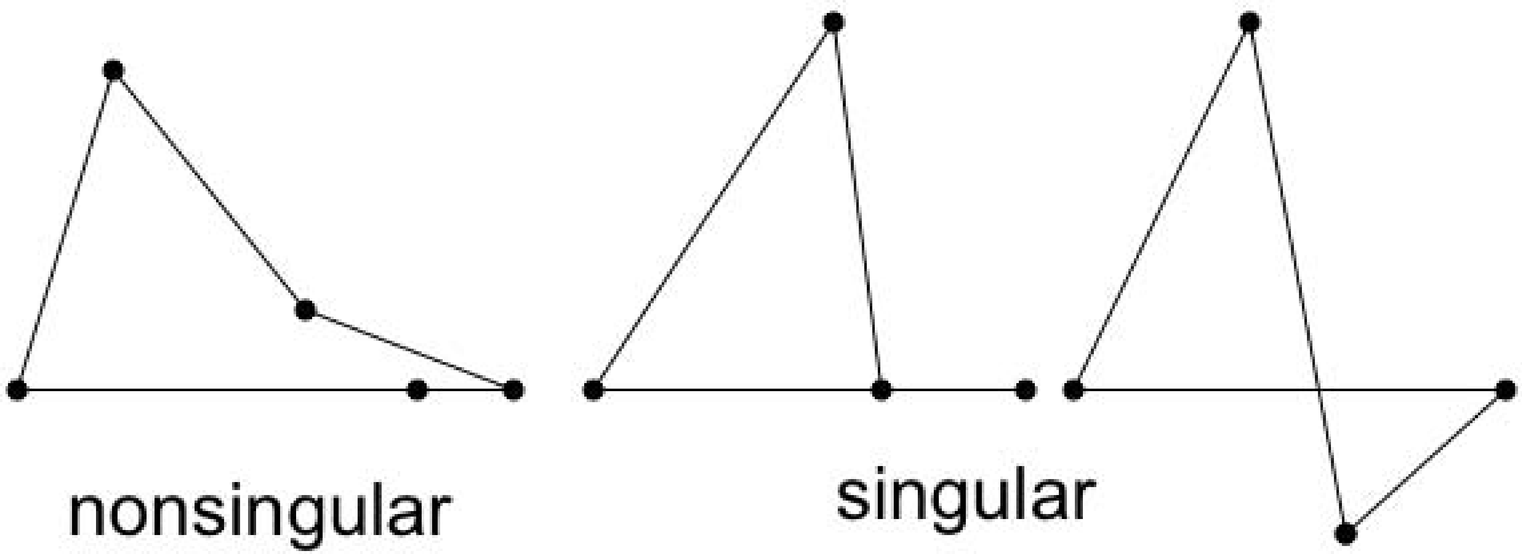
- The space of similarity classes of Euclidean n-gons is defined as

  $\{(v_1, \ldots, v_n) \in (\mathbf{C}^*)^n \mid v_1 + \cdot \ \cdot \ \cdot + v_n = 0\} / \mathbf{C}^*$,

  which has $\dim_{\mathbf{C}} = n-2$.

Non-singular and singular polygons

- We say that a polygon is nonsingular if there is no self-intersection of its edges, and singular otherwise.



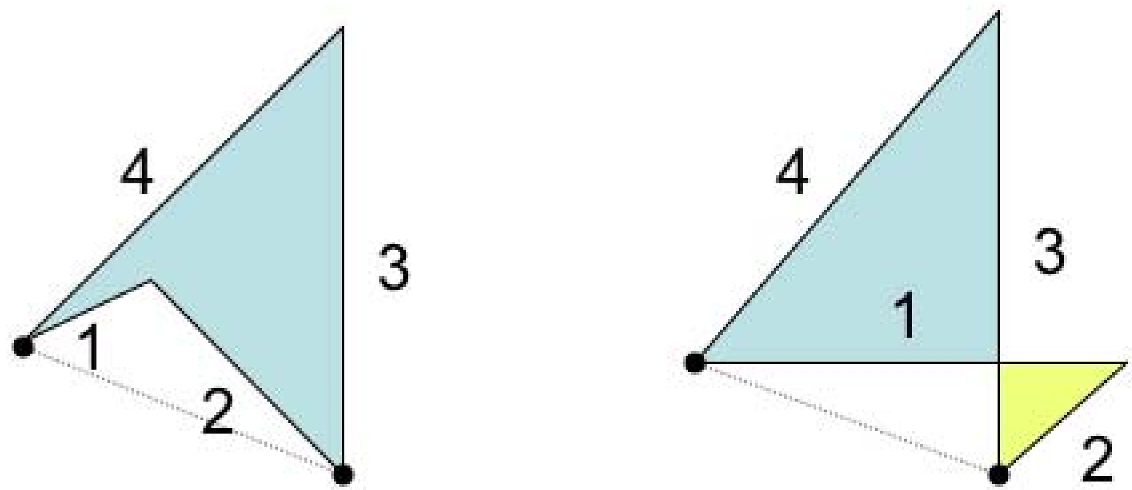
The area of polygons

The signed area Area(P) of a polygon $P = (v_1, \ldots, v_n)$ is given by

$$\mathrm{Area}(P) = \sum v_i \wedge v_j,$$

where $u_i \wedge u_j = (u_i \bar{u}_j - \bar{u}_i u_j)/2$, for $u_i, u_j \in \mathbf{C}$,

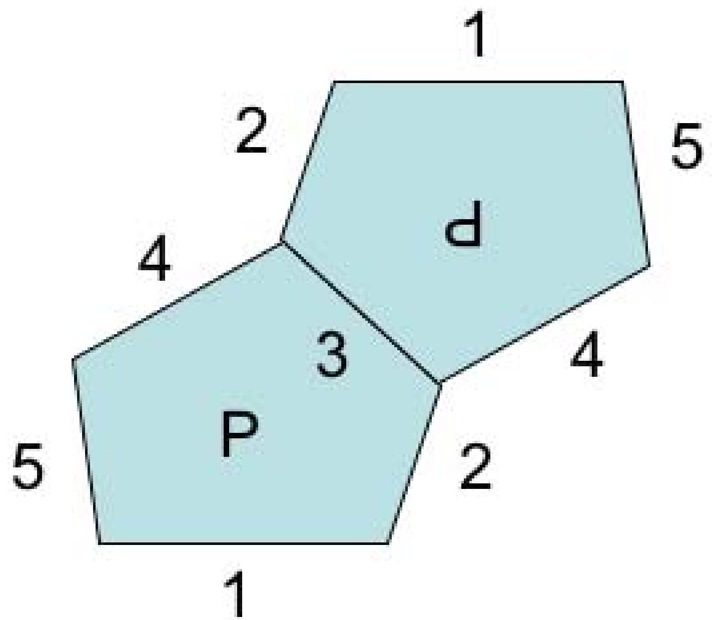
and the summation is taken over $0<i<j<n$.



‡ Each similarity class is represented by a polygon with Area(P)=1.

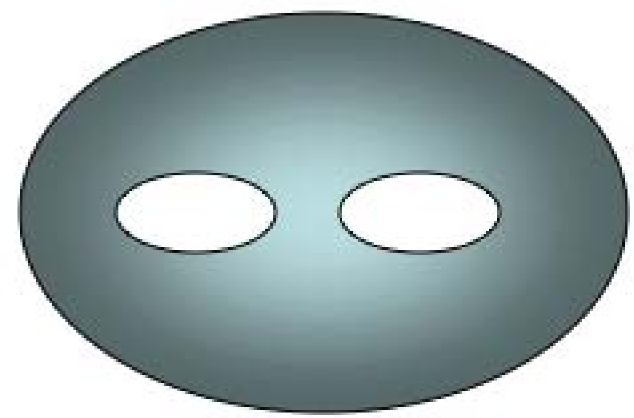
Riemann surface associated with a nonsingular polygon

identify the pairs of sides of $P \cup -P$ with same indices by translations



P : n-gon

$\approx$

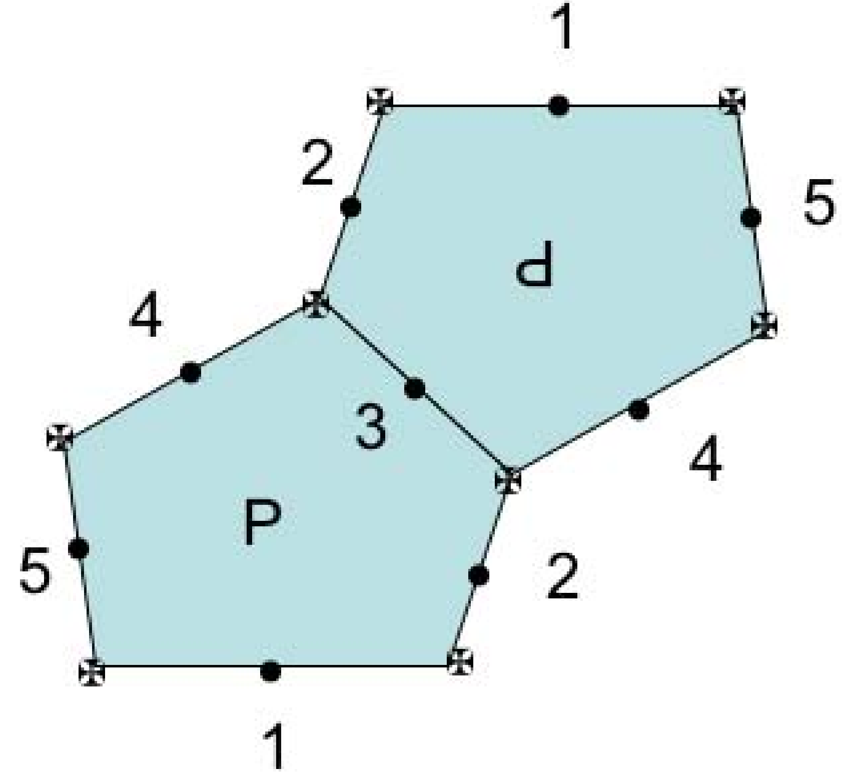


Σ_P : holomorphic curve of genus [(n-1)/2] (with one distinguished point if n is even).

‡ The holomorphic structure of Σ_P does not depend on the choice of P in its similarity class.

Properties of Σ_P

- Σ_P admits a Euclidean metric with cone singularities at
  - one point of angle $2(n-2)\pi$ when n is odd,
  - at two points of angle $(n-2)\pi$ when n is even.
- Moreover, Σ_P admits a holomorphic involution induced from the π rotation around the midpoint of an edge of P so that it is a hyperelliptic curve.
- The edges of P determine a marking of Σ_P.



✠, • : fixed points of the involution

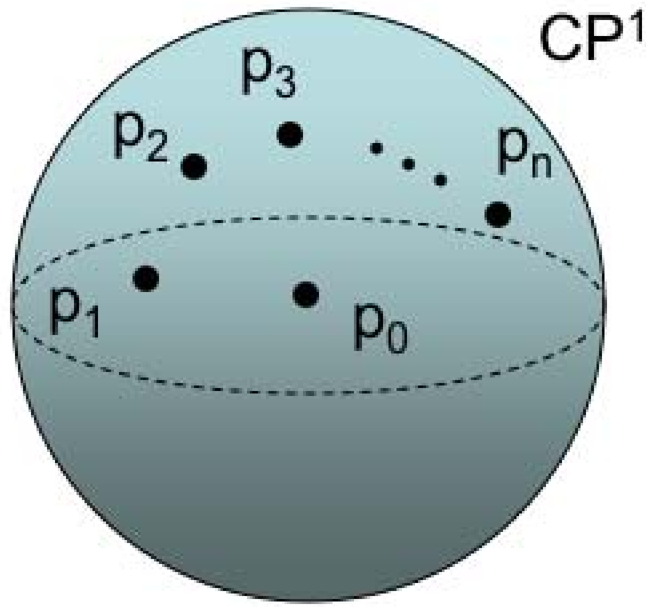
Moduli space of hyperelliptic curves

The set of isomorphism classes of hyperelliptic curves of genus [(n-1)/2] (with one distinct point if n is even)} with marked branching points

$\cong$

the set of sequences of distinct n points on CP^1 modulo PGL(2,C), which is the configuration space X(n+1) of n+1 points on $\mathbf{C}P^1$

- $\dim_C X(n+1) = n-2$



The quotient of hyperelliptic curve Σ_P by the holomorphic involution

-p_0 corresponds to the vertices of P

-p_i corresponds to the midpoints of the sides of P

Isomorphism we expect to obtain

Moduli space of polygons/polygons with angles

$\Rightarrow$

Moduli/Teichmüller space of Hyperelliptic curve

locally homeo at nonsingular polygons P

How about globally?

i.e. To what extent can we generalise the construction to get an isomorphism?

$\dim_{\mathbb{C}} = n-2$

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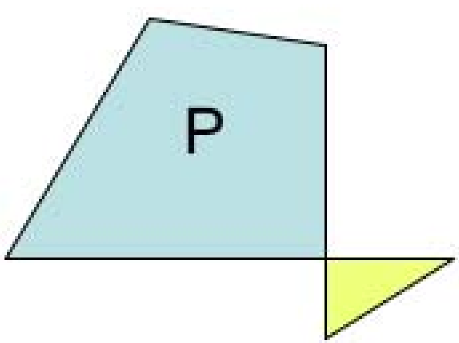
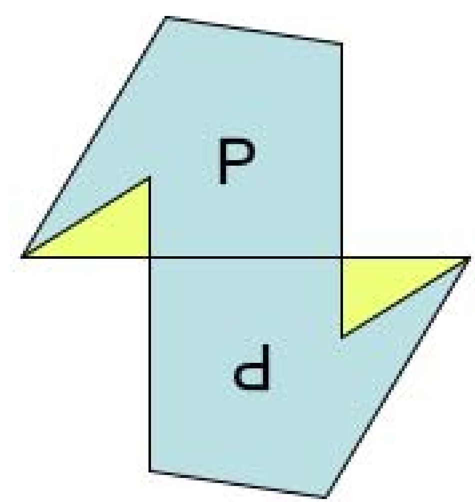
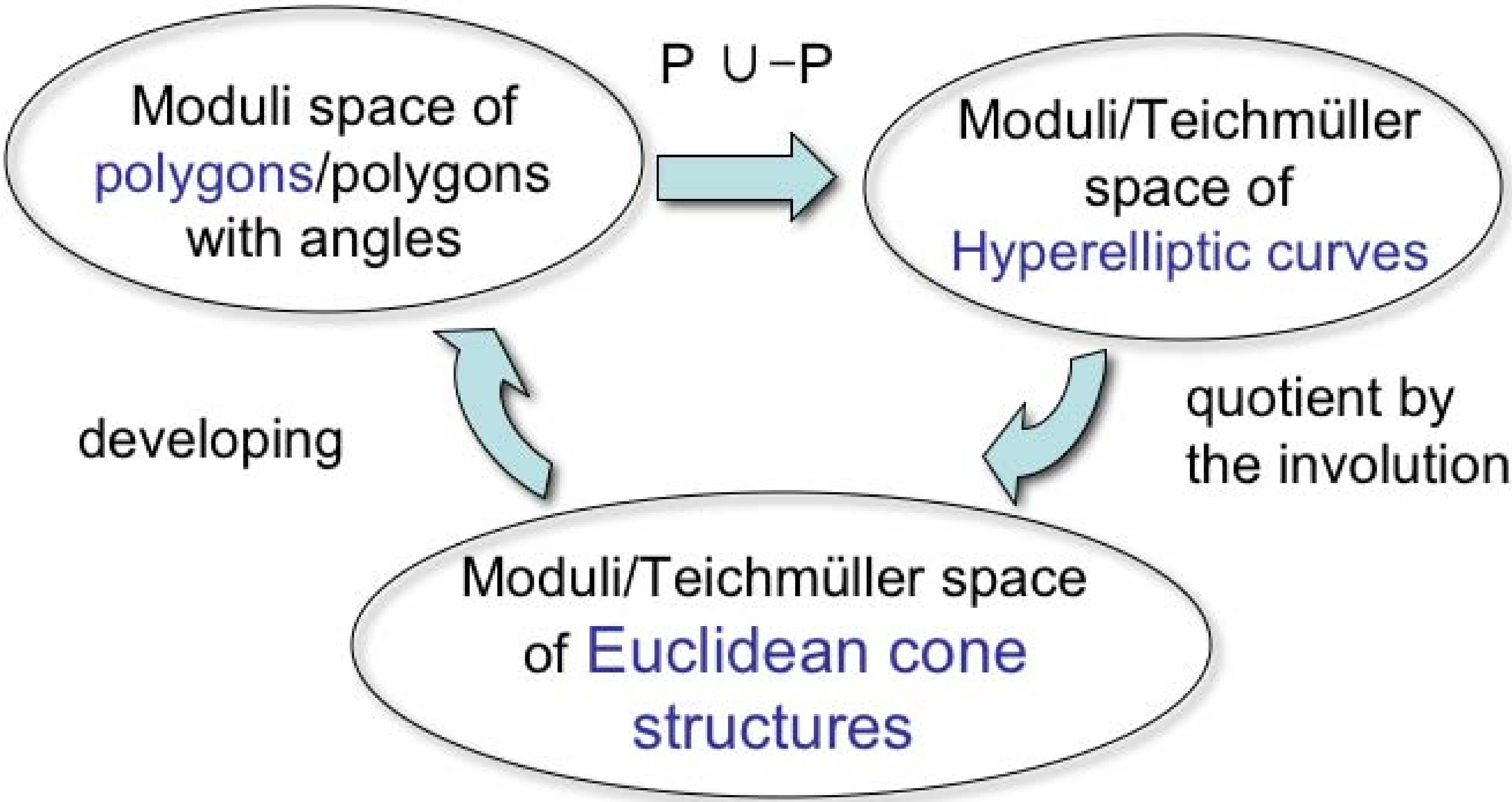




Isomorphisms we expect to obtain

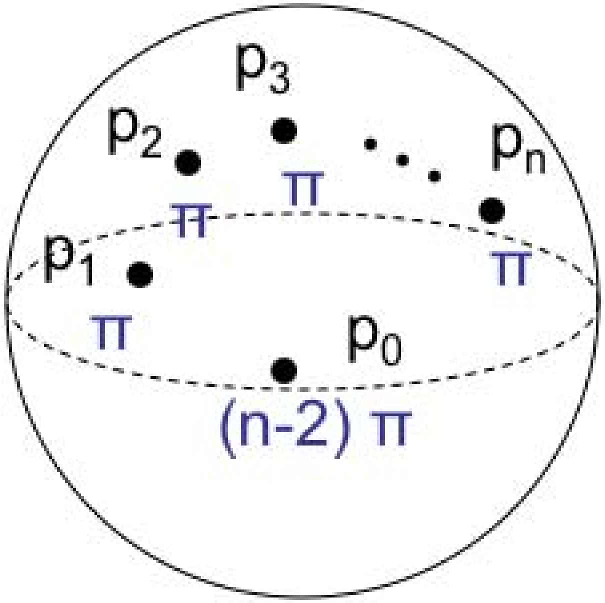


Euclidean cone structures on CP^1

The quotient of hyperelliptic curve Σ_P by the holomorphic involution has a Euclidean cone structure on CP^1 induced from that of Σ_P with cone points

- p_0 with cone angle $(n-2)\pi$
- p_i with cone angle π, for $i=1,\ldots,n$.

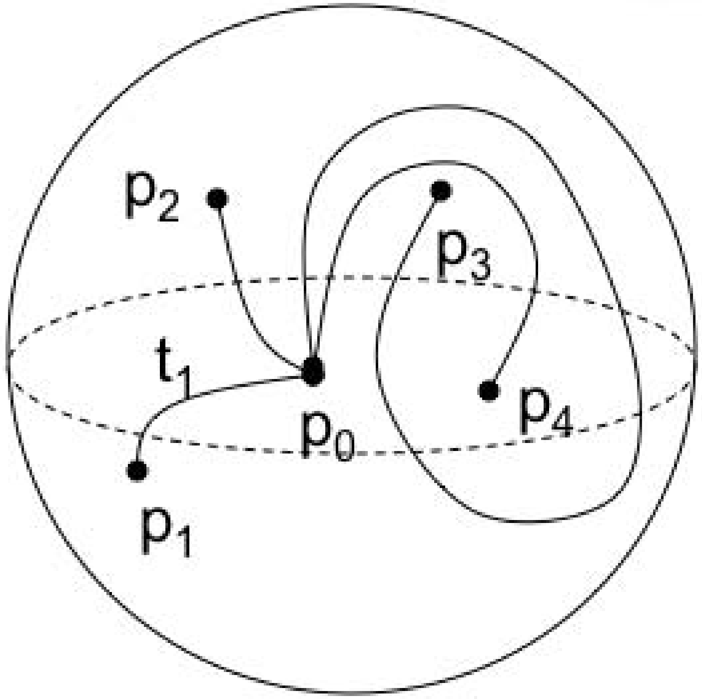
$C((n-2)\pi, \pi \times n)$:= the set of all Euclidean cone structures on CP^1 of this type up to orientation and label preserving similarities.



<u>Theorem</u> (Troyanov, 1991)

There is a natural homeomorphism from $C((n-2)\pi, \pi\times n)$ to the configuration space $X(n+1)$ of $(n+1)$ points on CP^1.

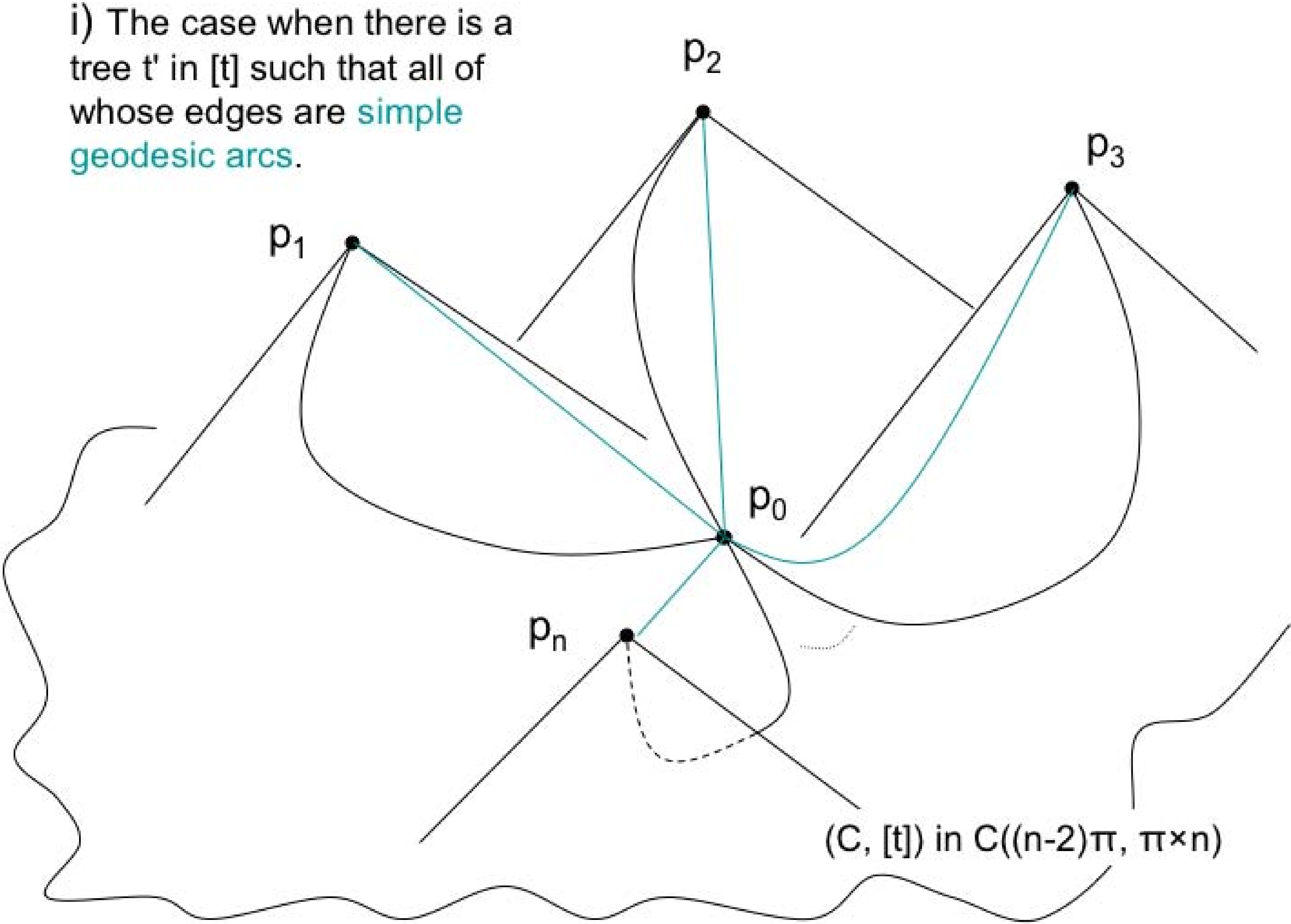
Marked Euclidean cone structures



$C \in C((n-2)\pi, \pi \times n)$
$t = (t_1, \ldots, t_n)$: a collection of disjoint smooth paths in C where t_i starts from p_0 and ends at p_i, for $i=1, \ldots, n$.
[t] : its ambient isotopy class
marking
(C, [t]) : a marked Euclidean cone structure.

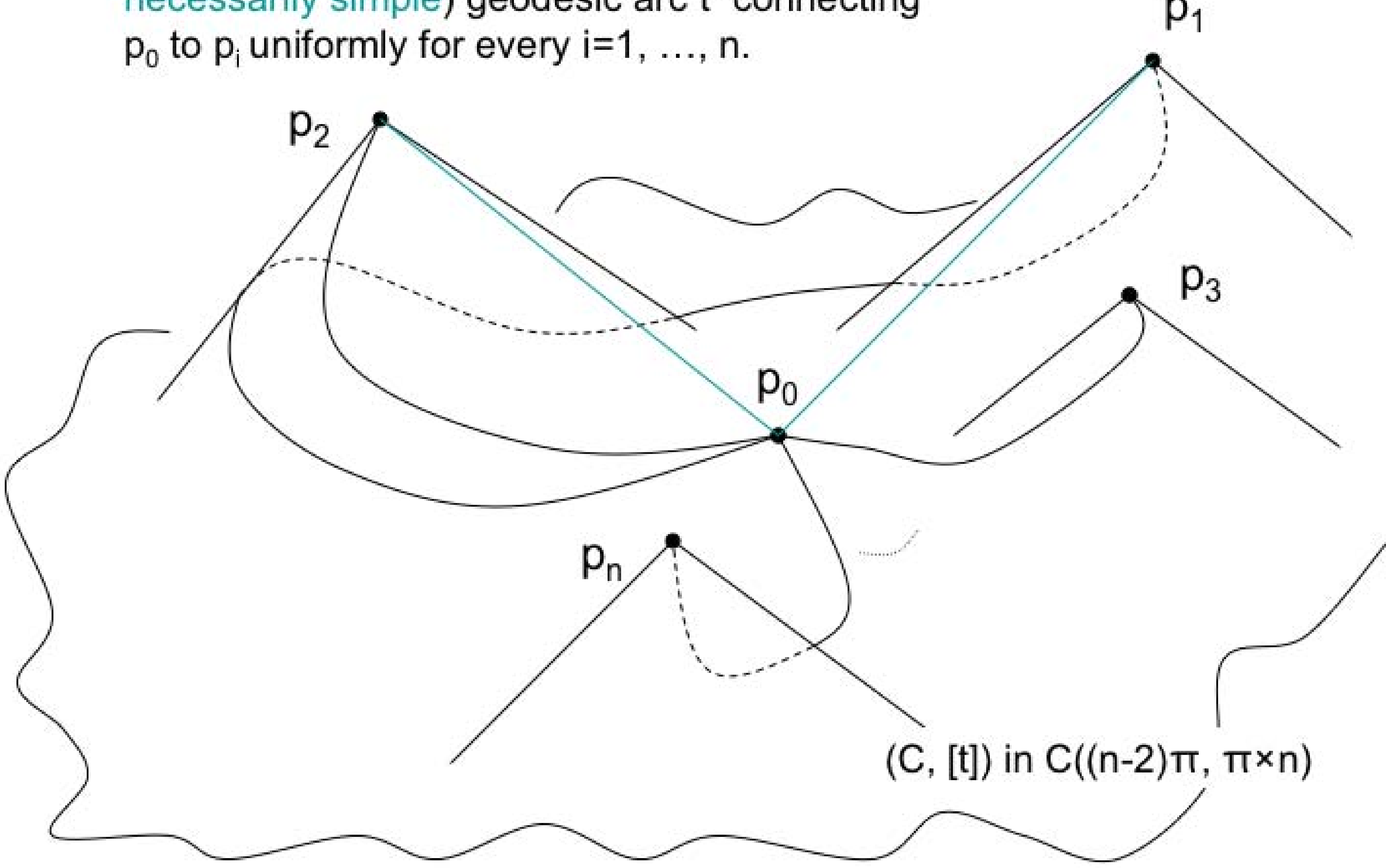
$\hat{C}((n-2)\pi, \pi \times n)$ = the set of all marked Euclidean cone structures up to similarities.

$\hat{C}_0((n-2)\pi, \pi \times n)$ = the subspace of $\hat{C}((n-2)\pi, \pi\ n)$ consisting of the elements with a marking t which has edges issuing from p_0 in the order of $t_1, \ldots, t_n$ clockwise.



i) The case when there is a tree t' in [t] such that all of whose edges are simple geodesic arcs.
p2
p3
p1
p0
pn
(C, [t]) in C((n-2)π, π×n)

ii) The case when there is a family of trees t_s in [t] such that the i-th edge of t_s converges to a (not necessarily simple) geodesic arc t' connecting p_0 to p_i uniformly for every i=1, …, n.



The Teichmüller space of Euclidean cone structures

For a Euclidean cone structure C, we define the Teichmuller space T(C) of C by

$T(C) = \{(X, \Phi) \mid X$ is a Euclidean cone structure,

$\Phi : C \to X$: orientation preserving diffeomorphism which maps the cone points of C to the cone points of X$\}$ with label preserving$\} / \sim$

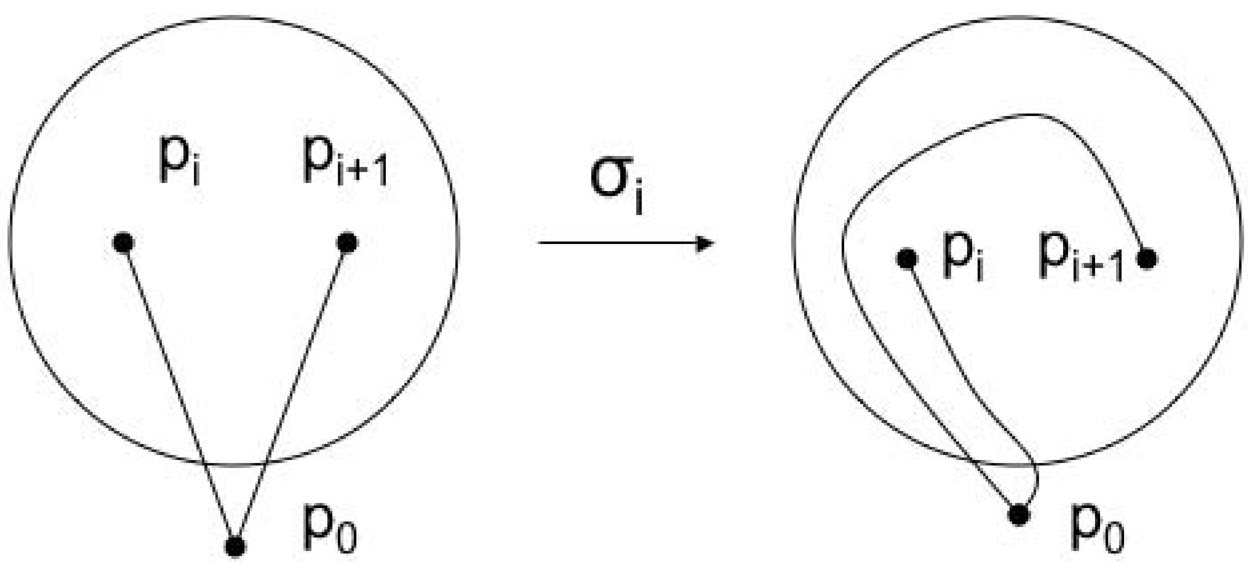
Theorem

Let $C \in \mathcal{C}((n-2)\pi, \pi\times n)$ and $t = (t_1, \ldots, t_n)$ a tree of C which issues at the point p_0 in the order of $t_1, \ldots, t_n$ clockwise.

Then the map from T(C) to $\hat{C}_0((n-2)\pi, \pi\times n)$ sending (X, Φ) to $(X, \Phi_*(t))$ is bijective.

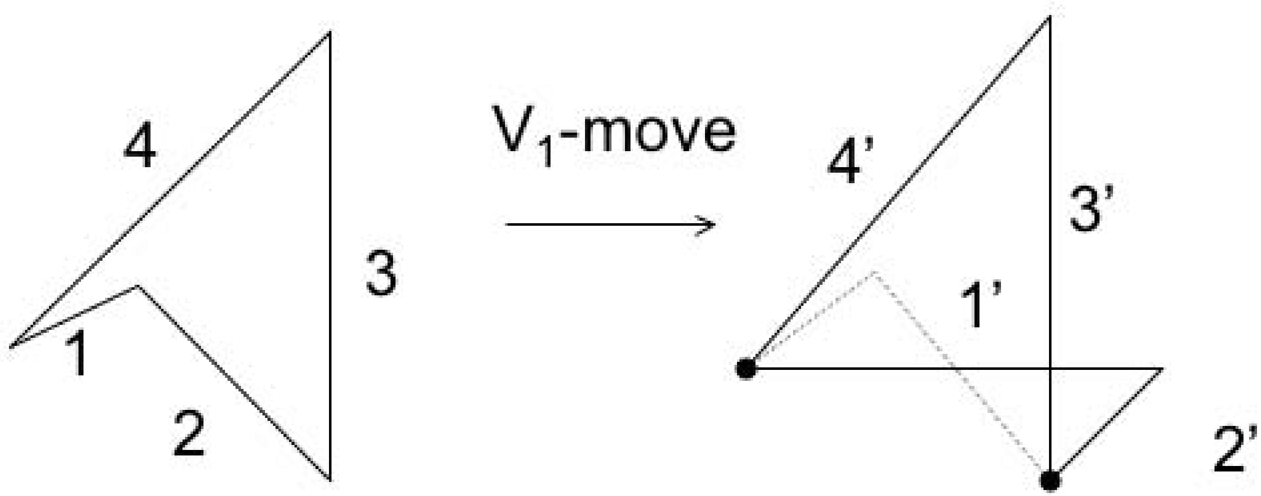
Mapping class group

- The subgroup Stab(p_0) $\subset \pi_0 \mathrm{Diff}^+(S^2, \{p_0, \ldots, p_n\})$ acts on $\hat{C}((n-2)\pi, \pi \times n)$ by $\alpha \cdot (C, [t]) = (C, [\alpha_*(t)])$, for $(C, [t]) \in \hat{C}((n-2)\pi, \pi \times n)$, $\alpha \in$ Stab(p_0) .
  ‡ This action is compatible with the action of the mapping class group on T(C).
- Half Dehn twists σ_i (i=1 … n) generate Stab(p_0).



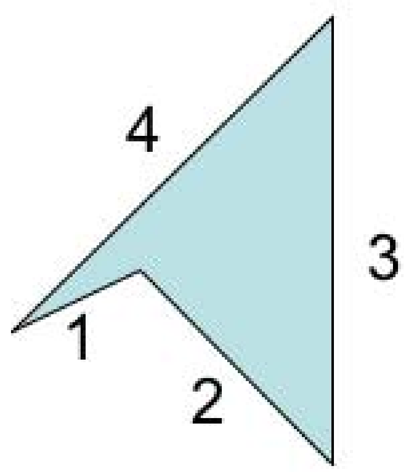
An action on the space of polygons

- For a polygon $P=(v_1,\dots, v_n)$ and $P'=(v_{1'},\dots, v_{n'})$, we say that P' is obtained from P by a V_k-move, if
  - $v_k' = 2v_k + v_{k+1}$, $v_{k+1}' = -v_k$.
  - $v_i'=v_i$ for $i \neq k, k+1$.
- We note that the areas of polygons are invariant under V_k-moves.

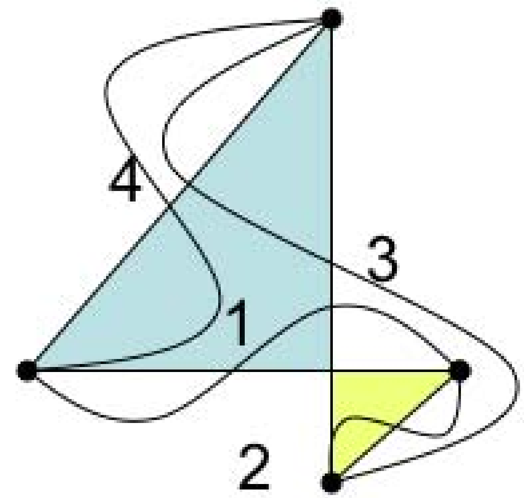


Developing a Euclidean cone structure into E^2

- For a marked Euclidean cone structure (C, [t]), let P(C) be the result of cutting C along the tree t and developed into the Euclidean plane E^2.
  - ‡ In general, P(C) is an immersed region. Straightening its boundary, we get a polygon with angles, which is uniquely determined by (C,[t]).

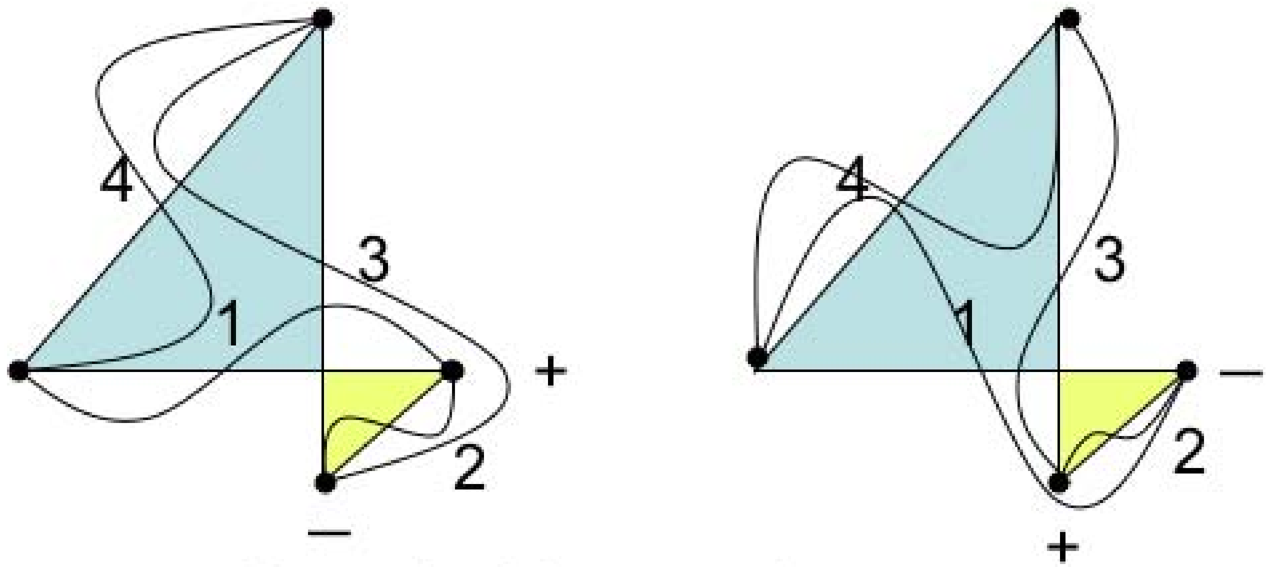


i) when there is a simple geodesic representative t.



ii) when there is no simple geodesic representative t.

Q_4 : The space of polygons with angles
is defined as a cover of P_4 such that each vertex is allowed to have angles not necessarily in $[0,2\pi)$.



Example of the same 4-gons
with different angles

Main theorem (the case of n=4)

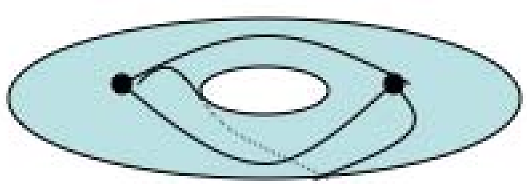
Let $P_4 = \{P=(v_1, v_2, v_3, v_4) : \text{4-gon with Area}(P)=1,$
$v_i \wedge v_{i+1} \notin \mathbf{Z}$ for some $i\}$ / $U(1)$,

Q_4 a space of quadrilaterals in P_4 with angles.

$D : \hat{C}_0(2\pi, \pi\times 4) \to Q_4$ be a map which sends each marked Euclidean cone structure $(C,[t])$ in $\hat{C}_0(2\pi, \pi\times 4)$ to a 4-gon with angles in Q_4 by straightening its developed image obtained by cutting C along t. Let $p : Q_4 \to P_4$ be the projection obtained by forgetting the angles.

<u>Theorem</u>. D is injective, and $p \circ D$ is surjective.

<u>Corollary</u>. The Teichmuller space of 1-pointed tori is in 1-to-1 correspondence with the space of quadrilaterals in P_4 with angles (a subspace of Q_4) .



Key idea of the proof of Theorem

Lemma

Any 4-gon with area=1 satisfying $v_i \wedge v_{i+1} \notin \mathbf{Z}$ for some i can be transformed into a non-singular quadrilateral by a finite sequence of V-moves.