HARMONIC MAGNUS EXPANSION ON THE UNIVERSAL FAMILY OF RIEMANN SURFACES

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in honor of Professor Fred Cohen

$g \geq 1$
$\mathbb{M}_{g,1} := \{(C, P_0, v);$ "triple" of genus $g\}/$biholo.,
where C is a compact Riemann surface of genus g, $P_0 \in C$, and $0 \neq v \in T_{P_0}C$.
$\mathcal{T}_{g,1} := \widetilde{\mathbb{M}_{g,1}}$, Teichmüller space,
$\mathcal{M}_{g,1} := \pi_1(\mathbb{M}_{g,1})$, Mapping Class group. (Nielsen) $\mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}(F_{2g})$.

The purpose of this study is to find **"canonical" differential forms representing the Morita-Mumford classes**

$$e_i = (-1)^{i+1}\kappa_i \in H^{2i}(\mathbb{M}_{g,1}), \quad i \geq 1,$$

on the moduli space $\mathbb{M}_{g,1}$.

Hyperbolic Approach. S. Wolpert, Invent. math. **85**(1986), 119–145.

Classical Approach. (Grothendieck-Riemann-Roch formula)

$$\begin{array}{ccc} \mathcal{T}_{g,1} & \xrightarrow{\text{Period Matrix}} & \mathfrak{H}_g \textit{ (Siegel upper space)} \\ \underline{\textit{odd}}\ \textit{Morita-Mumford classes} & \overset{\text{pullback}}{\dashleftarrow} & \textit{Chern forms} \end{array}$$

Here we can find **no** *even Morita-Mumford classes.*

Harmonic Magnus Expansion. (Johnson maps)

$$\begin{array}{ccc} \mathcal{T}_{g,1} & \xrightarrow{\text{H.M.E.}} & \Theta_{2g} \textit{ (Space of Magnus expansions)} \\ \underline{\textit{all}}\ \textit{Morita-Mumford classes} & \overset{\text{pullback}}{\dashleftarrow} & \textit{twisted Morita-Mumford forms} \end{array}$$

§1. Magnus Expansions and Johnson Maps

$n \geq 2$,
$F_n = \langle x_1, \dots, x_n \rangle$, free group of rank n,
$H = H_{\mathbb{R}} := H_1(F_n; \mathbb{R}) = {F_n}^{\mathrm{abel}} \otimes_{\mathbb{Z}} \mathbb{R}, \quad X_i := [x_i] \in H$,
$\otimes = \otimes_{\mathbb{R}}$,
$\widehat{T} := \prod_{m=0}^{\infty} H^{\otimes m} = \mathbb{R}\langle\langle X_1, X_2, \dots, X_n \rangle\rangle$, non-commutative formal power series,
$\widehat{T}_p := \prod_{m \geq p} H^{\otimes m}$, two-sided ideals, $(p \geq 1)$,

$\Theta_n := \{\theta : F_n \to 1 + \widehat{T}_1$; group homomorphism, $\theta(\gamma) \equiv 1 + [\gamma] \bmod \widehat{T}_2, (\forall \gamma)\}$,
the space of ($\mathbb{R}$-valued) **Magnus expansions** of F_n.
$\theta \in \Theta_n$, a **Magnus expansion** in a generalized sense.

Two kinds of group actions on the space Θ_n.

(i) *The automorphism group* $\mathrm{Aut}(F_n)$ *acts on* Θ_n *by*

$$\varphi \cdot \theta := |\varphi| \circ \theta \circ \varphi^{-1}, \quad (\varphi \in \mathrm{Aut}(F_n),\ \theta \in \Theta_n). \tag{1.1}$$

Here $|\varphi|$ *is the induced map on* $H = H_1(F_n; \mathbb{R})$, *which acts on the tensor algebra* $\widehat{T}$ *in a natural way.*

(ii) *The (projective limit of) Lie group(s)*

$$\mathrm{IA}(\widehat{T}) := \{U : \widehat{T} \to \widehat{T};\ \mathbb{R}\text{-algebra automorphism}, \\ U(\widehat{T}_p) = \widehat{T}_p, (\forall p \geq 1),\ U = 1_H \text{ on } \widehat{T}_1/\widehat{T}_2 = H.\}$$

acts on Θ_n *by* $U \cdot \theta := U \circ \theta$, $(U \in \mathrm{IA}(\widehat{T}),\ \theta \in \Theta_n)$. *We have a natural bijection of sets*

$$\mathrm{IA}(\widehat{T}) \cong \prod_{m=1}^{\infty} \mathrm{Hom}(H, H^{\otimes m+1}) \cong \prod_{m=1}^{\infty} H^* \otimes H^{\otimes m+1}, \quad U \mapsto U|_H. \tag{1.2}$$

Moreover **this action is free and transitive!**

These two kinds of group actions induce **the Johnson map** associated to a fixed $\theta \in \Theta_n$

$$\tau^\theta : \mathrm{Aut}(F_n) \to \mathrm{IA}(\widehat{T}) = \prod_{m=1}^{\infty} H^* \otimes H^{\otimes m+1}, \quad \varphi \mapsto \tau^\theta(\varphi) = (\tau^\theta_m(\varphi))$$

by

$$\tau^\theta(\varphi) \circ \theta = \varphi \cdot \theta\ (= |\varphi| \circ \theta \circ \varphi^{-1}), \quad (\varphi \in \mathrm{Aut}(F_n)). \tag{1.3}$$

The restriction of **the p-th Johnson map** $\tau^\theta_p : \mathrm{Aut}(F_n) \to H^* \otimes H^{\otimes p+1}$ to some subgroup $\mathcal{M}(p) \subset \mathcal{M}_{g,1} \subset \mathrm{Aut}(F_{2g})$ coincides with the classical p-th Johnson homomorphism of the mapping class groups.

Since the bijection (1.2) is **not** a homomorphism, the Johnson maps τ^θ_p's are **not** homomorphisms. They satisfy an infinite sequence of cochain relations including

$$\tau^\theta_1(\varphi\psi) = \tau^\theta_1(\varphi) + |\varphi|\tau^\theta_1(\psi) \tag{1.4}$$

$$\tau^\theta_2(\varphi\psi) = \tau^\theta_2(\varphi) + (\tau^\theta_1(\varphi) \otimes 1 + 1 \otimes \tau^\theta_1(\varphi))|\varphi|\tau^\theta_1(\psi) + |\varphi|\tau^\theta_2(\psi) \tag{1.5}$$

for any φ and $\psi \in \mathrm{Aut}(F_n)$.

For the free and transitive action of $\mathrm{IA}(\widehat{T})$ on Θ_n we have the Maurer-Cartan form $\eta = (\eta_p) \in \Omega^1(\Theta_n)\widehat{\otimes}\mathrm{LieIA}(\widehat{T}) = \prod_{p=1}^{\infty} \Omega^1(\Theta_n) \otimes H^* \otimes H^{\otimes(p+1)}$. Using Chen's iterated integrals we have an integral presentation of the total Johnson map

$$\tau^\theta(\varphi)^{-1} = 1 + \sum\nolimits_{m=1}^{\infty} \int_\theta^{|\varphi|\circ\theta\circ\varphi^{-1}} \overbrace{\eta\eta\cdots\eta}^{m} \tag{1.6}$$

for any $\varphi \in \mathrm{Aut}(F_n)$. Especially the closed 1-form η_1 represents the cohomology class $[\tau_1^\theta] \in H^1(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes 2})$. From the Maurer-Cartan formula $d\eta = \eta \wedge \eta$ we obtain

$$d\eta_p = \sum_{s=1}^{p-1} \underbrace{(\eta_s \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \eta_s)}_{p-s+1} \circ \eta_{p-s}, \tag{1.7}$$

which suggests us a close relation between the Johnson maps and the Stasheff associahedrons K_p (J.D. Stasheff, Trans. Amer. Math. Soc., **108**(1963) 275–292).

Consider the double cochain complex

$$C^{*,*} := C^*(K_{p+1}; \Omega^*(\Theta_n; H^* \otimes H^{\otimes(p+1)})^{\mathrm{Aut}(F_n)}),$$

that is, the cellular cochain complex of K_{p+1} with values in the de Rham complex of Θ_n with twisted coefficients in $\mathrm{Aut}(F_n)$-module $H^* \otimes H^{\otimes(p+1)}$. The formula (1.7) means the Maurer-Cartan forms η_p's induce a p-cocycle $Y_p \in Z^p(C^{*,*})$, whose cohomology class

$$[Y_p] \in H^p(C^{*,*}) \cong H^p(\Omega^*(\Theta_n; H^* \otimes H^{\otimes(p+1)})^{\mathrm{Aut}(F_n)})$$

induces the $(0, p+2)$-twisted Morita-Mumford class on the moduli space $\mathbb{M}_{g,1}$.

§2. Applications to Cohomology of $\mathrm{Aut}(F_n)$.

The formula (1.4) means τ_1^θ is a 1-cocycle of $\mathrm{Aut}(F_n)$ with values in $H^* \otimes H^{\otimes 2}$. If we restrict it to the mapping class group $\mathcal{M}_{g,1} \subset \mathrm{Aut}(F_{2g})$ we have

Theorem 2.1. $\tau_1^\theta|_{\mathcal{M}_{g,1}} = \frac{1}{6} m_{0,3} \in H^1(\mathcal{M}_{g,1}; H^{\otimes 3})$.

Here it should be remarked H^* and its dual H^* are $\mathcal{M}_{g,1}$-isomorphic to each other by the intersection form for the closed surface. $m_{i,j} \in H^{2i+j-2}(\mathcal{M}_{g,1}; \Lambda^j H)$ is the (i, j) twisted Morita-Mumford class (K-., Invent. math., **131** (1998), 137–149). As was shown in S. Morita and K-., Math. Research Lett. **3** (1996), 629–641, all the algebraic combinations of the twisted Morita-Mumford classes using by the intersection form are just the polynomials of all the Morita-Mumford classes. Consequently the 1-cocycle τ_1^θ, or equivalently, the closed 1-form η_1 yields all the Morita-Mumford classes.

D. Johnson, Topology **24** (1985), 127–144, proved the first Johnson homomorphism induces an isomorphism of the abelianization or the Torelli group onto the space $\Lambda^3 H_{\mathbb{Z}}$ (or $\Lambda^3 H_{\mathbb{Z}}/H_{\mathbb{Z}}$) up to 2-torsions. A similar result holds for the group $\mathrm{IA}_n := \mathrm{Ker}(\mathrm{Aut}(F_n) \to \mathrm{GL}(H))$.

Theorem 2.2. *The first Johnson map* τ_1^θ *induces an isomorphism* $\tau_1 : {IA_n}^{\mathrm{abel}} \overset{\cong}{\to} {H_\mathbb{Z}}^* \otimes \Lambda^2 H_\mathbb{Z}$.

Using the cohomology class $[\tau_1^\theta] \in H^1(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes 2})$ we obtain

Theorem 2.3. *Suppose* $1-n$ *is invertible in a commutative ring* R. *Then we have a natural decomposition of the cohomology group*

$$H^*(\mathrm{Aut}(F_n); M) = H^*(\mathrm{Out}(F_n); M) \oplus H^{*-1}(\mathrm{Out}(F_n); {H_R}^* \otimes M)$$

for any $R[\mathrm{Out}(F_n)]$*-module* M. *Especially,* $\pi^* : H^*(\mathrm{Out}(F_n); M) \to H^*(\mathrm{Aut}(F_n); M)$ *is an injection.*

For details, see K.-, preprint, arXiv:math.GT/0505497.

§3. Harmonic Magnus Expansions

For a (equivalent class of) triple $[C, P_0, v] \in \mathbb{M}_{g,1}$ we define the fundamental group $\pi_1(C, P_0, v)$ by all the homotopy classes of the loops ℓ satisfying the conditions $\ell(]0,1[) \subset C - \{P_0\}$, $\ell(0) = \ell(1) = P_0$, and $\frac{d\ell}{dt}(0) = -\frac{d\ell}{dt}(1) = v$. It has a natural group structure isomorphic to the free group F_{2g}.

The map taking the harmonic 1-form representing its own cohomology class $H = H^1(C; \mathbb{R}) \to \Omega^1(C)$ can be regarded as a H-valued 1-form $\omega_{(1)} \in \Omega^1(C) \otimes H$. We have $\int_C \omega_{(1)} \wedge \omega_{(1)} = I \in H^{\otimes 2}$, the intersection form. We denote by $\delta_0 : C^\infty(C) \to \mathbb{R}$, $f \mapsto f(P_0)$, the delta 2-current on C at P_0. Then we have a $\widehat{T}$-valued 1-current $\omega = \sum_{p \geq 1} \omega_{(p)}$, $\omega_{(p)} \in A^1(C) \otimes H^{\otimes p}$, satisfying the (modified) integrability condition

$$d\omega = \omega \wedge \omega - I \cdot \delta_0, \tag{3.1}$$

$\omega_{(p)} = \omega_{(1)}$ for $p = 1$, and the normalization condition $\int_C \omega_{(p)} \wedge *\varphi = 0$ for any closed 1-form φ and each $p \geq 2$. Moreover, using Chen's iterated integrals, we can define a Magnus Expansion

$$\theta = \theta^{(C,P_0,v)} : \pi_1(C, P_0, v) \to 1 + \widehat{T}_1, \quad [\ell] \mapsto \sum_{m=0}^{\infty} \int_\ell \overbrace{\omega\omega \cdots \omega}^{m}.$$

The Magnus expansions $\theta^{(C,P_0,v)}$ for all the triples (C, P_0, v) define a canonical real analytic map

$$\theta : \mathcal{T}_{g,1} \to \Theta_{2g},$$

which we call **the harmonic Magnus expansion on the universal family of Riemann surfaces**. The pullbacks of the Maurer-Cartan forms η_p's give the canonical differential forms representing the Morita-Mumford classes and their higher relations. Our main result in this poster is

Theorem 3.1. *For any* $[C, P_0, v] \in \mathbb{M}_{g,1}$ *we have*

$$(\theta^*\eta)_{[C,P_0,v]} = \Re(2N(\omega'\omega' - 2{\omega_{(1)}}'{\omega_{(1)}}')) \in T^*_{[C,P_0,v]}\mathbb{M}_{g,1} \otimes \widehat{T}_3.$$

Here $N : \widehat{T}_1 \to \widehat{T}_1$ *is defined by* $N|_{H^{\otimes m}} := \sum_{k=0}^{m-1} \begin{pmatrix} 1 & 2 & \cdots & m-1 & m \\ 2 & 3 & \cdots & m & 1 \end{pmatrix}^k$, *and the meromorphic quadratic differential* $N(\omega'\omega')$ *is regarded as a* $(1,0)$*-cotangent vector at* $[C, P_0, v] \in \mathbb{M}_{g,1}$ *in a natural way.*