Homotopy invariance of some configuration spaces

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1 Statement of results

Let M be a manifold. The configuration space of k points in M is

$$\mathbb{F}_k(M) = \{(x_1, \dots, x_k) \in M^k \mid x_i \neq x_j\}.$$

We are studying the case where the manifold M is of the particular form $A \times \mathbb{R}$.

1^{st} result : Homotopy construction.

Let A be a manifold. There is a homotopy construction of the configuration space $\mathbb{F}_k(A \times \mathbb{R})$, $k \geq 2$, involving only the natural inclusion $\mathbb{F}_2(A) \hookrightarrow A \times A$.

2^{nd} result : Homotopy invariance.

Let $f : A \xrightarrow{\simeq} B$ be a homotopy equivalence between two manifolds A and B.
Assume that there exists a homotopy equivalence φ_f such that the following square is homotopy commutative :

$$\begin{array}{ccc} \mathbb{F}_2(A) & \hookrightarrow & A^2 \\ \simeq \downarrow \varphi_f & & f \times f \downarrow \simeq \\ \mathbb{F}_2(B) & \hookrightarrow & B^2 \end{array}$$

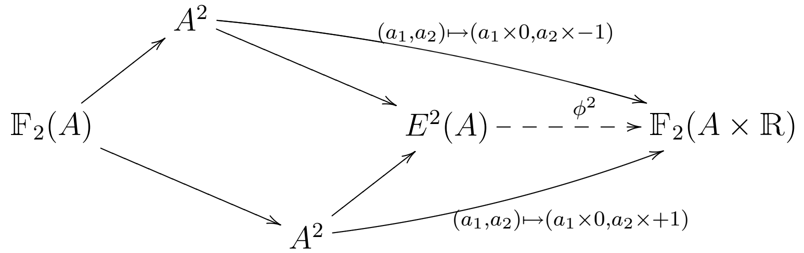
Then, for every $k \geq 2$, the following spaces have the same homotopy type :

- $\mathbb{F}_k(A \times \mathbb{R})$ *and* $\mathbb{F}_k(B \times \mathbb{R})$,
- $\Sigma^{k-2}\mathbb{F}_k(A)$ *and* $\Sigma^{k-2}\mathbb{F}_k(B)$.

For the whole text, we fix two manifolds A and B satisfying the hypothesis above (always true if A and B are closed and 2-connected). The aim of this note is to give an overview of our construction for $k = 2$ and $k = 3$ as well as giving an idea of the underlying proofs. More complete statements and proofs can be found in «Invariance homotopique de certains espaces de configurations», math.AT/0407002.

2 Homotopy construction of $\mathbb{F}_2(A \times \mathbb{R})$ and $\mathbb{F}_3(A \times \mathbb{R})$

Define $E^2(A)$ as the homotopy colimit of the diagram $A^2 \hookleftarrow \mathbb{F}_2(A) \hookrightarrow A^2$. In the diagram below, the outer square is homotopy commutative, hence there is an induced map $\phi^2 : E^2(A) \to \mathbb{F}_2(A \times \mathbb{R})$ commuting up to homotopy with the rest of the diagram.



2.1 Proposition *The map $\phi^2 : E^2(A) \to \mathbb{F}_2(A \times \mathbb{R})$ is a homotopy equivalence.*

Proof. First, consider the commutative diagram of fibrations on the left hand side of the following diagram :

$$\begin{array}{ccccccccc}
A & \longleftarrow & A \setminus q_0 & \longrightarrow & A & \overset{hocolim}{\rightsquigarrow} & A \times \{-1,+1\} \bigcup A \setminus q_0 \times [-1,+1] & \longrightarrow & A \times \mathbb{R} \setminus (q_0, 0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A^2 & \longleftarrow & \mathbb{F}_2(A) & \longrightarrow & A^2 & \overset{hocolim}{\rightsquigarrow} & E^2(A) & \xrightarrow{\phi^2} & \mathbb{F}_2(A \times \mathbb{R}) \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_1} & & \downarrow & & \downarrow{\scriptstyle p_1} \\
A & = & A & = & A & & A & \xrightarrow{\mathbb{1}_A \times \{0\}} & A \times \mathbb{R}
\end{array}$$

The spaces appearing in the top line (in red) are the fibers above a point $q_0 \in A$. Since the base spaces of those fibrations are identical, a result of Puppe asserts that the homotopy fiber of the induced map $E^2(A) \to A$ is the homotopy colimit of the various fibers. Hence, the right hand side of the diagram is a morphism of fibrations in which two of the three horizontal maps are homotopy equivalence : $\phi^2 : E^2(A) \to \mathbb{F}_2(A \times \mathbb{R})$ is therefore a homotopy equivalence. $\square$

In a similar way, we define a space $E^3_1(A)$ and a map $\phi^3_1 : E^3_1(A) \to \mathbb{F}_3(A \times \mathbb{R})$ by taking the homotopy colimit of the following diagram :

$$\begin{array}{ccccccccc}
A^3 & \hookleftarrow & A \times \mathbb{F}_2(A) & \hookrightarrow & A^3 & \hookleftarrow & \mathbb{F}_2(A) \rtimes A & \hookrightarrow & A^3 \overset{hocolim}{\rightsquigarrow} E^3_1(A) \\
& \searrow{\scriptstyle a_1 \times 0, a_2 \times -1, a_3 \times -2} & & & \downarrow{\scriptstyle a_1 \times 0, a_2 \times -1, a_3 \times -1/2} & & & \swarrow{\scriptstyle a_1 \times 0, a_2 \times -1, a_3 \times +2} & \downarrow{\scriptstyle \phi^3_1} \\
& & & & \mathbb{F}_3(A \times \mathbb{R}) & & & & \mathbb{F}_3(A \times \mathbb{R})
\end{array}$$

where $\mathbb{F}_2(A) \rtimes A = \{(a_1, a_2, a_3) \in A^3 \mid a_3 \neq a_1\} \cong \mathbb{F}_2(A) \times A$.

2.2 Proposition *The following diagram is a homotopy pullback square :*

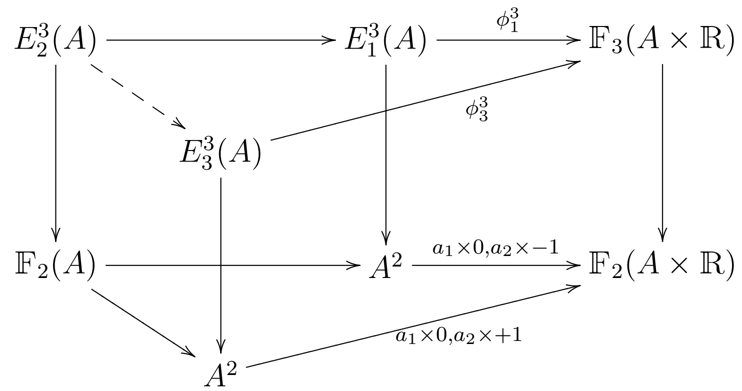
$$\begin{array}{ccc}
E^3_1(A) & \xrightarrow{\phi^3_1} & \mathbb{F}_3(A \times \mathbb{R}) \\
\downarrow & h.p.b & \downarrow \\
A^2 & \xrightarrow[a_1 \times 0, a_2 \times -1]{} & \mathbb{F}_2(A \times \mathbb{R})
\end{array}$$

Proof. We argue along the lines of the proof of 2.1. Consider the following homotopy commutative diagram where each column is a fibration sequence and the spaces on the top line are the homotopy fibers above $(a_1, a_2) \in A^2$.

$$\begin{array}{ccccccccccccc}
A & \longleftarrow & A \setminus q_2 & \longrightarrow & A & \longleftarrow & A \setminus q_1 & \longrightarrow & A & \overset{hocolim}{\rightsquigarrow} & F & \xrightarrow{\simeq} & A \times \mathbb{R} \setminus \{q_2 \times -1, q_1 \times 0\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A^3 & \longleftarrow & A \times \mathbb{F}_2(A) & \longrightarrow & A^3 & \longleftarrow & \mathbb{F}_2(A) \rtimes A & \longrightarrow & A^3 & \overset{hocol}{\rightsquigarrow} & E^3_1(A) & \xrightarrow{\phi^3_1} & \mathbb{F}_3(A \times \mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & h.p.b & \downarrow \\
A^2 & = & A^2 & = & A^2 & = & A^2 & = & A^2 & & A^2 & \xrightarrow[a_1 \times 0, a_2 \times -1]{} & \mathbb{F}_2(A \times \mathbb{R})
\end{array}$$

Again, the space F is the homotopy colimit of the diagram with five spaces appearing in red. Explicitly the homotopy fiber F is $(A \setminus q_2) \times [-2, -\frac{1}{2}] \bigcup A \times \{-2, -\frac{1}{2}, 2\} \bigcup (A \setminus q_1) \times [-\frac{1}{2}, 2]$. The result follows since the restriction of ϕ_1^3 given by $\phi_1^3 : F \to A \times \mathbb{R} \setminus \{q_2 \times -1, q_1 \times 0\}$ is a homotopy equivalence. $\square$

Also, we define a space $E_2^3(A)$ as the homotopy pullback of the map $E_1^3(A) \to A^2$ along the natural inclusion $\mathbb{F}_2(A) \hookrightarrow A^2$. Finally, there is a space $E_3^3(A)$, constructed in a similar way to $E_1^3(A)$, such that the front square of the following diagram is a homotopy pullback :



In this diagram, each vertical square with full arrows is a homotopy pullback. From the universal property of a homotopy pullback, there exists a map $E_2^3(A) \to E_3^3(A)$ which commutes up to homotopy with the diagram and such that the completed square is also a homotopy pullback. As seen before, the bottom face of this diagram is a homotopy pushout. The cube lemma asserts that the top face is also a homotopy pushout, i.e. we have the following proposition :

2.3 Proposition *The homotopy colimit of $E_1^3(A) \leftarrow E_2^3(A) \to E_3^3$, denoted by $E^3(A)$, has the same homotopy type as $\mathbb{F}_3(A \times \mathbb{R})$.*

3 Homotopy invariance of $\mathbb{F}_2(A \times \mathbb{R})$ and $\mathbb{F}_3(A \times \mathbb{R})$

The assumption on the manifolds A and B implies that the following diagram is homotopy commutative and that the vertical arrows are homotopy equivalences. Hence, using homotopy colimit, we have a homotopy equivalence between $\mathbb{F}_2(A \times \mathbb{R}) \simeq E^2(A)$ and $\mathbb{F}_2(B \times \mathbb{R}) \simeq E^2(B)$.

$$\begin{array}{ccccccc}
A^2 & \longleftarrow & \mathbb{F}_2(A) & \longrightarrow & A^2 & \overset{hocolim}{\rightsquigarrow} & E^2(A) \\
{\scriptstyle f\times f}\downarrow\simeq & & {\scriptstyle \varphi_f}\downarrow\simeq & & \simeq\downarrow{\scriptstyle f\times f} & & \downarrow\simeq \\
B^2 & \longleftarrow & \mathbb{F}_2(B) & \longrightarrow & B^2 & \overset{hocolim}{\rightsquigarrow} & E^2(B)
\end{array}$$

In a similar way, using the diagram below, the spaces $E_1^3(A)$ and $E_1^3(B)$ have the same homotopy type.

$$\begin{array}{ccccccccccc}
A^3 & \longleftarrow & A \times \mathbb{F}_2(A) & \longrightarrow & A^3 & \longleftarrow & \mathbb{F}_2(A) \rtimes A & \longrightarrow & A^3 & \overset{hocolim}{\rightsquigarrow} & E_1^3(A) \\
{\scriptstyle f^3}\downarrow\simeq & & {\scriptstyle f\times\varphi_f}\downarrow\simeq & & \simeq\downarrow{\scriptstyle f^3} & & {\scriptstyle \varphi_f\rtimes f}\downarrow\simeq & & \simeq\downarrow{\scriptstyle f^3} & & \downarrow\simeq \\
B^3 & \longleftarrow & B \times \mathbb{F}_2(B) & \longrightarrow & B^3 & \longleftarrow & \mathbb{F}_2(B) \rtimes B & \longrightarrow & B^3 & \overset{hocolim}{\rightsquigarrow} & E_1^3(B)
\end{array}$$

The same kind of homotopy equivalence can be found between $E_3^3(A)$ and $E_3^3(B)$. Using a technical lemma[1], we show that there exists a homotopy equivalence $E_2^3(A) \to E_2^3(B)$ making the following diagram commutative up to homotopy :

$$\begin{array}{ccccccc}
E_1^3(A) & \longleftarrow & E_2^3(A) & \longrightarrow & E_3^3(A) & \overset{hocolim}{\rightsquigarrow} & E^3(A) \\
\downarrow \simeq & & \downarrow \simeq & & \simeq \downarrow & & \downarrow \simeq \\
E_1^3(B) & \longleftarrow & E_2^3(B) & \longrightarrow & E_3^3(B) & \overset{hocolim}{\rightsquigarrow} & E^3(B)
\end{array}$$

Hence there is a homotopy equivalence between $\mathbb{F}_3(A \times \mathbb{R}) \simeq E^3(A)$ and $\mathbb{F}_3(B \times \mathbb{R}) \simeq E^3(B)$.

4 Homotopy invariance of $\Sigma\mathbb{F}_3(A)$

In the previous part, we have seen that the two spaces $E_2^3(A)$ and $E_2^3(B)$ have the same homotopy type. We are going to show that, up to one suspension, those spaces contain informations about $\mathbb{F}_3(A)$ and $\mathbb{F}_3(B)$. First, define the space $G_2^3(A)$ as the homotopy colimit of the diagram $\mathbb{F}_2(A) \times A \hookleftarrow \mathbb{F}_3(A) \hookrightarrow \mathbb{F}_2(A) \times A$. Now, from the definition of $E_2^3(A)$, we know that the following square is a homotopy pullback :

$$\begin{array}{ccc}
E_2^3(A) & \longrightarrow & \mathbb{F}_3(A \times \mathbb{R}) \\
\downarrow & h.p.b & \downarrow \\
\mathbb{F}_2(A) & \longrightarrow & \mathbb{F}_2(A \times \mathbb{R})
\end{array}$$

As before, we show that the space $G_2^3(A)$ can be also described as the homotopy pullback of the same maps :

$$\begin{array}{ccccccccc}
A & \longleftarrow & A \setminus \{q_0, q_1\} & \longrightarrow & A & \overset{hocolim}{\rightsquigarrow} & F & \overset{\simeq}{\longrightarrow} & A \times \mathbb{R} \setminus \{q_0 \times 0, q_1 \times 0\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{F}_2(A) \times A & \longleftarrow & \mathbb{F}_3(A) & \longrightarrow & \mathbb{F}_2(A) \times A & \overset{hocol}{\rightsquigarrow} & G_2^3(A) & \longrightarrow & \mathbb{F}_3(A \times \mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & h.p.b & \downarrow \\
\mathbb{F}_2(A) & = & \mathbb{F}_2(A) & = & \mathbb{F}_2(A) & & \mathbb{F}_2(A) & \xrightarrow{1_{\mathbb{F}_2(A)} \times \{0\}} & \mathbb{F}_2(A \times \mathbb{R})
\end{array}$$

where $F = A \times \{-1, +1\} \bigcup A \setminus \{q_0, q_1\} \times [-1, +1]$. Hence, the two spaces $E_2^3(A)$ and $G_2^3(A)$ have the same homotopy type and there is a map $\mathbb{F}_2(A) \times A \coprod \mathbb{F}_2(A) \times A \to E_2^3(A)$ which cofiber is $\Sigma\mathbb{F}_3(A)_+$.

With some more work, we can show that the left square of the following diagram is commutative up to homotopy :

$$\begin{array}{ccccc}
\mathbb{F}_2(A) \times A \coprod \mathbb{F}_2(A) \times A & \longrightarrow & E_2^3(A) & \longrightarrow & \Sigma\mathbb{F}_3(A)_+ \\
\varphi_f \times f \coprod \varphi_f \times f \downarrow \simeq & \circlearrowright_H & \downarrow \simeq & & \downarrow \simeq \\
\mathbb{F}_2(B) \times B \coprod \mathbb{F}_2(B) \times B & \longrightarrow & E_2^3(B) & \longrightarrow & \Sigma\mathbb{F}_3(B)_+
\end{array}$$

Then, we can deduce that the configuration spaces $\mathbb{F}_3(A)$ and $\mathbb{F}_3(B)$ have the same homotopy type up to one suspension.

[1]cf. Proposition A.8 in the appendix of the paper