An algebraic invariant which is associated with a computer topological space in $\mathbf{Z}^n$

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Abstract

One of the new steps into computational algebraic topology is shown. For a computer topological space in $\mathbf{Z}^n$ with one of the general k-adjacency relations, we observe the notions of an n-fold and a regular (k_0, k_1)-covering, respectively. These play significant roles in calculating some algebraic invariants, such as the k- fundamental group and the discrete Deck's transformation group. Finally, the above-mentioned results can be used to study digital images in computer science and is an positive answer of the open question from [2].

1 Notation and terminology

Let $\mathbf{Z}^n$ be the set of points in the Euclidean n-dimensional space with integer coordinates. For $a, b \in \mathbf{Z}$ with $a \lneq b$, the set $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} | a \leq n \leq b\}$ with the 2-adjacency is called a *digital interval* [1]. As a generalization of the commonly used 4- and 8-adjacency of $\mathbf{Z}^2$, and 6-, 18- and 26-adjacency of $\mathbf{Z}^3$, the general adjacency relations in $\mathbf{Z}^n, n \geq 1$, could be obtained as follows[3–5]. $k \in \{3^n - 1 (n \geq 2),\ 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1 (2 \leq r \leq n-1, n \geq 3), 2n (n \geq 1)\}$. In the rest of this paper, we consider a discrete space with k-adjacency (X, k) in a digital picture $(\mathbf{Z}^n, k, 2n, X)$ or $(\mathbf{Z}^n, 2n, 3^n - 1, X)$, where k is one of the general adjacency relations, and $k \neq 2n$ except $n = 1$ owing to the *digital connectivity paradox* [8].

[1] AMS Classification: 57M10; Secondary 57S30, 55R15, 55Q20, 68T10, 68U05
Keywords: computer topological space, (k_0, k_1)-continuity, (k_0, k_1)-homotopy, (k_0, k_1)-homeomorphism, regular (k_0, k_1)-covering, n-fold (k_0, k_1)-covering.

2 Relative (k_0, k_1)-homotopy

The Khalimsky line topology on $\mathbf{Z}$ is induced from the following subbasis $\{[2n-1, 2n+1]_{\mathbf{Z}} | n \in \mathbf{Z}\}$ [2,7]. We now obtain the product topology on $\mathbf{Z}^n$ derived from Khalimsky line topology on $\mathbf{Z}$. Then we call the topology on $\mathbf{Z}^n$ *the product Khalimsky topology* on $\mathbf{Z}^n, n \geq 2$. Further, each space $X \subset \mathbf{Z}^n$ is considered with the relative topological structure $(X, T_X^n) \subset (\mathbf{Z}^n, T^n)$ induced on X by the product Khalimsky topology $(\mathbf{Z}^n, T^n)$, $n \geq 2$. Consequently, we call (X, T_X^n) the *computer topological space*. When considering (X, T_X^n) with a k-adjacency, we use the notation (X, k, T_X^n). For an adjacency relation k, a *simple k-path* in X is the sequence $(x_i)_{i \in [0,m]_{\mathbf{Z}}}$ such that x_i and x_j are k-adjacent if and only if either $j = i+1$ or $i = j+1$. Then the number m is called the length of this path.

Definition 1 *[3] For spaces $(X, k_0, T_X^{n_0})$ and $(Y, k_1, T_Y^{n_1})$, we say that f is (k_0, k_1)-continuous at some point $x_0 \in X$ if, for any $N_{k_1}(f(x_0), \varepsilon) \subset Y$, there is a k_0-neighborhood $N_{k_0}(x_0, \delta) \subset X$ such that $f(N_{k_0}(x_0, \delta)) \subset N_{k_1}(f(x_0), \varepsilon)$, $\varepsilon, \delta \in \mathbf{N}$, where $N_k(x_0, \varepsilon) = \{x \in X | \quad l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}$, and $l_k(x_0, x)$ is the length of a shortest simple k-curve from x_0 to x and $\varepsilon \in \mathbf{N}$. If f is (k_0, k_1)-continuous at any point $x \in X$, then f is called a (k_0, k_1)-continuous map. If $k_0 = k_1$, then we call it a k_0-continuous map.*

Definition 2 *((k_0, k_1)-homotopy rel.A) For spaces $(X, k_0, T_X^{n_0})$ and $(Y, k_1, T_Y^{n_1})$, consider $(A, k_0) \subset (X, k_0)$ with the relative topology on (A, k_0). Let $f, g : X \to Y$ be (k_0, k_1)-continuous functions. Suppose there exist $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that*

- *for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;*

- *for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ defined by*

$F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is $(2, k_1)$-continuous;

- *for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ defined by*

$F_t(x) = F(x, t)$ for all $x \in X$ is (k_0, k_1)-continuous.

Then we say that F is a (k_0, k_1)-homotopy between f and g, and f and g are (k_0, k_1)-homotopic in Y and we use the notation $f \simeq_{(k_0, k_1)} g$ [1].

- *If, further, for all $t \in [0, m]_{\mathbf{Z}}$ and every $x_0 \in A$, $F(x_0, t) = x_0$, i.e., the induced map F_t on A is fixed, then we say that the homotopy is a (k_0, k_1)-homotopy rel.A.*

Specially, if $A = \{x_0\}$, then the (k_0, k_1)-homotopy rel.A is called a pointed

(k_0, k_1)-homotopy [1].

If $X = [0, m_X]_{\mathbf{Z}}$, for all $t \in [0, m]_{\mathbf{Z}}$, we have $F(0,t) = F(0,0)$ and $F(m_X, t) = F(m_X, 0)$, then we say that F holds the endpoints fixed.

In order to understand the pointed k-homotopy in the current paper in relation with the k-fundamental group, we need to use the notion of the *trivial extension*[7]:

Let $F_1^k(X, x_0)$ be the set $\{f | f$ is a k-loop based at $x_0\}$. For members $f : [0, m_f]_{\mathbf{Z}} \to X$, $g : [0, m_g]_{\mathbf{Z}} \to X$ of $F_1^k(X, x_0)$, we have a map $f * g : [0, m_f + m_g]_{\mathbf{Z}} \to X$ [7] defined by

$$f * g(t) = \begin{cases} f(t) & \text{if} \quad 0 \le t \le m_f; \\ g(t - m_f) & \text{if} \quad m_f \le t \le m_f + m_g. \end{cases}$$

Consequently, the k-fundamental group is induced from the based k-homotopy *rel.*$\{x_0\}$ in Definition 2 which is the computer topological version of the pointed digital (k_0, k_1)-homotopy in [1,4–6]. The k-homotopy class of a pointed loop f as defined in [1] is denoted by $[f]$. We have $g \in [f]$ if and only if there is a homotopy, holding the endpoint fixed, between *trivial extensions* F, G of f, g, respectively, where a trivial extension F of f is a map that, roughly speaking, follows the same path as f with pauses for rest. Then, if $f_1, f_2, g_1, g_2 \in F_1^k(X, p)$, $f_1 \in [f_2]$, and $g_1 \in [g_2]$, then $f_1 * g_1 \in [f_2 * g_2]$, *i.e.*, $[f_1 * g_1] = [f_2 * g_2]$ [1,4–7]. Therefore

$$\pi_1^k(X, x_0) := \{[f] | f \in F_1^k(X, x_0)\}$$

is a group with the operation $[f] \cdot [g] = [f * g]$, is called the *k-fundamental group* of (X, x_0).

3 N-fold regular (k_0, k_1)-covering

In the following, we shall assume that each space (X, k) is k-connected unless otherwise stated. For spaces $(X, k_0, T_X^{n_0})$ and $(Y, k_1, T_Y^{n_1})$, a map $h : X \to Y$ is called a (k_0, k_1)-*homeomorphism* if h is (k_0, k_1)-continuous and bijective and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous.

Definition 3 *For spaces $(E, k_0, T_E^{n_0})$ and $(B, k_1, T_B^{n_1})$, let $p : E \to B$ be a (k_0, k_1)-continuous surjection. Suppose, for any $b \in B$, there exists $\varepsilon \in \mathbf{N}$ such that*

(C1) for some index set M,
$p^{-1}(N_{k_1}(b,\varepsilon)) = \cup_{\alpha\in M} E_\alpha$*, where* $E_\alpha = N_{k_0}(e_\alpha,\varepsilon), e_\alpha \in p^{-1}(b)$*;*
(C2) if $\alpha,\beta \in M$ *and* $\alpha \neq \beta$*, then* $E_\alpha \cap E_\beta = \phi$*;*
(C3) the restriction map p on E_α, denoted by $p|_{E_\alpha} : E_\alpha \to N_{k_1}(b,\varepsilon)$*, is a* (k_0,k_1)*-homeomorphism for all* $\alpha \in M$*.*

Then the map p is called a (k_0,k_1)-covering map and (E,p,B) is said to be a (k_0,k_1)-covering. Furthermore, according to the cardinality of the index set M, $\sharp M = n$, we say that the (k_0,k_1)-covering is an n-fold (k_0,k_1)-covering.

In this paper, let $SC_k^{n,l} = (w_i)_{i\in[0,l-1]_{\mathbf{Z}}}$ in $\mathbf{Z}^n$ such that w_i and w_j are adjacent if and only if $j = i \pm 1 (mod\, l)$ and further, there is an $N_k(w_i,1) \subset \mathbf{Z}^n$ for any point $w_i \in SC_k^{n,l}$ which is called a simple closed k-curve with distinct l elements.

Definition 4 *A (k_0,k_1)-covering (E,p,B) is called a radius n-(k_0,k_1)-covering if there is an $\varepsilon \geq n$ in Definition 3.*

Example 3.1 *(1) While the map* $p_1 : \mathbf{Z} \to SC_k^{n,l} = (c_t)_{t\in[0,l-1]_{\mathbf{Z}}}$ *with* $p_1(t) = c_{t(mod\, l)}$ *is a radius 2-$(2,l)$-covering map, $l \geq 6$, the map* $p_2 : \mathbf{Z} \to SC_{3^n-1}^{n,4} = (d_t)_{t\in[0,3]_{\mathbf{Z}}}$ *with* $p_2(t) = d_{t(mod\, 4)}$ *is not a radius 2-$(2,8)$-covering map.*

Lemma 3.2 *Given spaces* $((E,e_0),k_0,T^{n_0}_{(E,e_0)})$, $((B,b_0),k_1,T^{n_1}_{(B,b_0)})$*, let* $p : E \to B$ *be a (k_0,k_1)-covering map such that $p(e_0) = b_0$. Then any k_1-path* $f : [0,m]_{\mathbf{Z}} \to B$ *beginning at some point b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at some point e_0.*

Lemma 3.3 *For spaces* $((E,e_0),k_0,T^{n_0}_{(E,e_0)})$*, and* $((B,b_0),k_1,T^{n_1}_{(B,b_0)})$*, let* $p : (E,e_0) \to (B,b_0)$ *be a pointed radius 2-(k_0,k_1)-covering map. For k_0-paths* $g_0 : [0,m_0]_{\mathbf{Z}} \to (E,e_0)$ *and* $g_1 : [0,m_1]_{\mathbf{Z}} \to (E,e_0)$ *that begin at the point* $e_0 = g_0(0) = g_1(0)$*, if $p\circ g_0$ and $p\circ g_1$ are k_1-homotopic rel.*$\{p(e_0), p(g_0(m_0)) = p(g_1(m_1))\}$ *in B, then g_0 and g_1 are k_0-homotopic rel.*$\{e_0, g_0(m_0) = g_1(m_1)\}$ *in E.*

Definition 5 *For spaces* $((E,e_0),k_0,T^{n_0}_{(E,e_0)})$ *and* $((B,b_0),k_1,T^{n_1}_{(B,b_0)})$*, let* $p : (E,e_0) \to (B,b_0)$ *be a pointed (k_0,k_1)-covering map. If* $p_*\pi_1^{k_0}(E,e_0)$ *is a normal subgroup of* $\pi_1^{k_1}(B,b_0)$*, then we say that $((E,e_0),p,(B,b_0))$ is a regular (k_0,k_1)-covering.*

Definition 6 *For the pointed (k_0,k_1)-covering $((E,e_0),p,(B,b_0))$, we say that a k_0-homeomorphism $h : (E,e_0) \to (E,e_0)$ is a covering transformation such that $p = p\circ h$.*

Then the set of all covering transformations for the pointed (k_0,k_1)-covering $((E,e_0),p,(B,b_0))$ forms a group under the operation of composition, called an *automorphism group of* (E,e_0) *over* (B,b_0), and denoted by $Aut((E,e_0)|(B,b_0))$

which is a group with the operation of composition $\circ$.

Theorem 3.4 *Let* $((E, e_0), p, (B, b_0))$ *be a regular radius* 2*-*(k_0, k_1)*-covering. Then* $Aut((E, e_0)|(B, b_0))$ *is isomorphic to* $\pi_1^{k_1}(B, b_0)/p_*\pi_1^{k_0}(E, e_0)$.

Corollary 3.5 $\pi_1^{3^n-1}(SC_{3^n-1}^{n,l})$ *is isomorphic to* $l\mathbf{Z}$ *if* $l \geq 6$.

Theorem 3.6 *For an* $m|q$*-fold* (k_0, k_1)*-covering* $((SC_{k_0}^{n_0,ml}, w_0), p, (SC_{k_1}^{n_1,ql}, v_0))$ *such that* $SC_{k_1}^{n_1,ql}$ *is not* k_1*-contractible,* $Aut((SC_{k_0}^{n_0,ml}, w_0)|(SC_{k_1}^{n_1,ql}, v_0))$ *is isomorphic to a finite group* $\mathbf{Z}_{m|q}$.

Theorem 3.7 *Let* (E, e_0) *be* k_0*-connected and* (B, b_0) *be* k_1*-connected. If* $((E, k_0), p, (B, b_0))$ *is a pointed regular radius* 2*-*(k_0, k_1)*-covering such that* (E, e_0) *is simply* k_0*-connected, then* $Aut((E, e_0)|(B, b_0))$ *is isomorphic to* $\pi_1^{k_1}((B, b_0))$.

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