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# The homotopy type of the complement of a coordinate subspace arrangement


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Problem

## COORDINATE SUBSPACE ARRANGEMENT

is a finite set $\mathcal{CA} = \{L_1, \dots, L_r\} \subset \mathbb{C}^n$ of coordinate subspaces, that is,

$$L_\omega = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_{i_1} = \dots = z_{i_k} = 0\},$$

where $\omega = \{i_1, \dots, i_k\} \subset [n]$ and its complement $\mathsf{U}(\mathcal{CA})$ is defined as

$$\mathsf{U}(\mathcal{CA}) := \mathbb{C}^n \setminus \bigcup_{i=1}^{r} L_i.$$

**GOAL**: The homotopy type of $\mathsf{U}(\mathcal{CA})$.

Toric topology-main definitions and constructions

## SIMPLICIAL COMPLEXES

$V = \{v_1, \dots, v_n\} = [n]$ set of vertices

$K := \{\sigma_1, \dots, \sigma_s : \sigma_i \subset V\}$ $(\varnothing \in K)$ – abstract simplicial complex closed under formation of subsets

$\sigma \in K$ – simplex $\quad \dim(K) = d$ if $\sharp\sigma \leq d + 1$ for all $\sigma \in K$

## STANLEY-REISNER FACE RING

$R$ – commutative ring with unit;

$\deg(v_i) = 2$ – topological grading

$R[V] = R[v_1, \dots, v_n]$ graded polynomial algebra on $V$ over $R$

Given $\sigma \subset [n]$, set

$$v^\sigma := \prod_{i \in \sigma} v_i, \quad v^\sigma = v_{i_1} \dots v_{i_r} \quad \text{for } \sigma = \{i_1, \dots, i_r\}.$$

The Stanley-Reisner algebra (or face ring) of $K$ is

$$R[K] := R[v_1, \dots, v_n]/(v^\sigma : \sigma \notin K).$$

`''Topological models" for the algebraic objects`

## DAVIS–JANUSZKIEWICZ SPACE $\mathrm{DJ}(K)$

–topological realisation of the Stanley–Reisner ring $R[K]$, that is,

$$H^*(\mathrm{DJ}(K); R) = R[K] \quad (\text{for } R = \mathbb{Z} \text{ or } R = \mathbb{Z}/2).$$

Davis–Januszkiewicz $\quad \mathrm{DJ}(K) = ET^n \times_{T^n} \mathcal{Z}_K$

Buchstaber–Panov through a simple colimit of nice blocks

Assume $R = \mathbb{Z}$. Denote $\mathbb{C}P^\infty = BS^1$, thus $BT^n = (\mathbb{C}P^\infty)^n$

For $\omega \subset [n]$, define

$$BT^\omega := \{(x_1, \ldots, x_n) \in BT^n : x_i = * \text{ if } i \notin \omega\}.$$

For $K$ on $[n]$, the Davis-Januszkiewicz space of $K$ is given by

$$\mathrm{DJ}(K) := \bigcup_{\sigma \in K} BT^\sigma \subset BT^n.$$

## MOMENT–ANGLE COMPLEX $\mathcal{Z}_K$

Torus $T^n \subset (D^2)^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| \le 1, \forall i\}$

For arbitrary $\sigma \subset [n]$, define

$B_\sigma := \{(z_1, \ldots, z_n) \in (D^2)^n : |z_i| = 1 \quad i \notin \sigma\}.$

$$B_\sigma \cong (D^2)^{|\sigma|} \times T^{n-|\sigma|}$$

For $K$ on $[n]$, define the moment–angle complex $\mathcal{Z}_K$ by

$$\mathcal{Z}_K := \bigcup_{\sigma \in K} B_\sigma \subset (D^2)^n.$$

$B_\sigma$ invariant under the action of $T^n \quad \rightsquigarrow \quad T^n$ acts on $\mathcal{Z}_K$

**Proposition.** *The moment–angle complex $\mathcal{Z}_K$ is the homotopy fibre of the inclusion*

$$i\colon \mathrm{DJ}(K) \longrightarrow BT^n.$$

**Proposition.** $H^*_{T^n}(\mathcal{Z}_K) = \mathbb{Z}[K]$

## Arrangements and their complements

For $K$ on set $[n]$, define the complex coordinate subspace arrangement as

$$\mathcal{CA}(K) := \{L_\sigma : \sigma \notin K\}$$

and its complement in $\mathbb{C}^n$ by

$$\mathrm{U}(K) := \mathbb{C}^n \setminus \bigcup_{\sigma\notin K} L_\sigma.$$

If $L \subset K$ is a subcomplex, then $\mathrm{U}(L) \subset \mathrm{U}(K)$.

**Proposition.** *The assignment*

$$K \mapsto \mathrm{U}(K)$$

*defines a one–to–one order preserving correspondence*

$$\left\{ \begin{array}{c} \textit{simplicial} \\ \textit{complexes on } [n] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{complements of} \\ \textit{coordinate subspace} \\ \textit{arrangements in } \mathbb{C}^n \end{array} \right\}.$$

## CONNECTION BETWEEN $\mathcal{Z}_K$ AND $\mathrm{U}(K)$

**Theorem** (Buchstaber–Panov). *There is an equivariant deformation retraction*

$$\mathrm{U}(K) \xrightarrow{\simeq} \mathcal{Z}_K.$$

## COHOMOLOGY OF $\mathrm{U}(K)$

**Theorem** (Buchstaber–Panov). *The following isomorphism of graded algebra holds*

$$H^*(\mathrm{U}(K); k) \cong \mathrm{Tor}_{k[v_1,\ldots,v_n]}(k[K], k) \cong H\big[\Lambda[u_1, \ldots, u_n] \otimes k[K], d\big].$$

`hints from ALGEBRA and COMBINATORICS`

**Definition.** The Stanley-Reisner ring $k[K]$ is Golod if all Massey products in $\mathrm{Tor}_{k[v_1,\ldots,v_n]}(k[K], k)$ vanish.

**Definition.** A simplicial complex $K$ is shifted if there is an ordering $\sigma \in K,\ v' < v \Rightarrow (\sigma - v) \cup v' \in K$.

**Proposition.** *If $K$ is shifted, then its face ring $k[K]$ is Golod.*

## THE MAIN THEOREM (G., Theriault)

Let $K$ be a shifted complex. Then $\mathcal{Z}_K$ is a wedge of spheres.

Back to COMBINATORICS

PROBLEM: Determine the homotopy type of the complement of arbitrary codimension coordinate subspace arrangements.

STRATEGY:

1⟩ determine the simplicial complex $K$ which corresponds to a codimenison–$i$ coordinate subspace arrangement, $\mathsf{U}(K)$;

2⟩ associate to the determined simplicial complex $K$ its Davis–Januszkiewicz space, i.e, $\mathsf{DJ}(K)$;

3⟩ looking at the fibration

$$\mathcal{Z}_K \longrightarrow \mathsf{DJ}(K) \longrightarrow BT^n,$$

describe the homotopy type of $\mathcal{Z}_K$.

**1⟩** Look at an $i+2$–codimension coordinate subspace in $\mathbb{C}^n$, that is,

$$L_\omega = \big\{(z_1,\dots,z_n) \in \mathbb{C}^n : z_{j_1} = \dots = z_{j_{i+2}} = 0\big\}, \omega = \{j_1,\dots,j_{i+2}\}.$$

Then $K = \mathsf{sk}^i(\Delta^{n-1})$.

Hence, $\quad \mathbb{C}^n \backslash \mathcal{CA}^{i+2} = \mathsf{U}\big(\mathsf{sk}^i(\Delta^{n-1})\big)$.

**2⟩** A colimit model of the Davis-Januszkiewicz space for $K$ is given by

$$\mathsf{DJ}(K) := \bigcup_{\sigma\in K} BT^\sigma \subset BT^n, \quad \sharp\text{vertices in } K.$$

Then we have

$$\mathsf{DJ}(K) = T^n_{n-1-i}$$

$$= \big\{(z_1,\dots,z_n) : \text{at least } n-1-i \text{ coordinates are } *\big\} \subset (\mathbb{C}P^\infty)^n.$$

$\mathbf{3}\rangle$ Determine the homotopy fibre $\mathcal{Z}_K$ of the fibration sequence

$$(\mathcal{Z}_K)^n_k \longrightarrow T^n_k \longrightarrow (\mathbb{C}P^\infty)^n \text{ for } 1 \leq k \leq n-1.$$

Let $X_1, \ldots, X_n$ be path-connected spaces.
There is a filtration of $X_1 \times \ldots \times X_n$ given by

$$T^n_n \longrightarrow T^n_{n-1} \longrightarrow \cdots \longrightarrow T^n_0$$

were $T^n_k = \{(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n : \text{ at least } k \text{ of } x_i\text{'s are } * \}$.

**Theorem** (Porter; G., Theriault)**.** *For* $n \geq 1$*, and* $k$ *such that* $1 \leq k \leq n-1$*, the homotopy fibre* $F^n_k$ *of the inclusion* $i\colon T^n_k \longrightarrow X_1 \times \ldots \times X_n$ *decomposes as*

$$F^n_k \simeq \bigvee_{j+n-k+1}^{n} \left( \bigvee_{1 \leq i_1 < \ldots < i_j \leq n} \binom{j-1}{n-k} \Sigma^{n-k} \Omega X_{i_1} \wedge \ldots \wedge \Omega X_{i_j} \right).$$

$\square$

Take for $X_1 = \ldots = X_n = \mathbb{C}P^\infty$. Then we have the inclusion

$$i\colon T^n_k(\mathbb{C}P^\infty) \longrightarrow (\mathbb{C}P^\infty)^n.$$

It follows that

$$F^n_k \simeq \bigvee_{j=n-k+1}^{n} \left( \binom{n}{j} \binom{j-1}{n-k} \Sigma^{n-k} \underbrace{\Omega\mathbb{C}P^\infty \wedge \ldots \wedge \Omega\mathbb{C}P^\infty}_{j} \right)$$

$$\simeq \bigvee_{j=n-k+1}^{n} \binom{n}{j} \binom{j-1}{n-k} S^{n+j-k}.$$

$\boxed{\texttt{Family}}$

$$\mathcal{F}_t = \{K - \text{simplicial complex} | \Sigma^t \mathcal{Z}_K \text{a wedge of spheres}\}$$

Notice that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_t \subset \ldots \subset \mathcal{F}_\infty$.

We have shown if $K$ -shifted, then $K \in \mathcal{F}_0 \quad (sk^i(\Delta^{n-1}) \in \mathcal{F}_0)$

Want: make simplicial complexes out of our building blocks

## WHAT CAN HOMOTOPY SEE?

### DISJOINT UNION OF SIMPLICIAL COMPLEXES

Let $K_1 \in \mathcal{F}_t$ and $K_2 \in \mathcal{F}_s$. Then $K_1 \coprod K_2 \in \mathcal{F}_m$, $m = \max\{t, s\}$.

$\mathcal{Z}_K \simeq (\prod_i S^1 * \prod_j S^1) \vee (\mathcal{Z}_{K_1} \rtimes \prod_i S^1) \vee (\prod_j S^1 \ltimes \mathcal{Z}_{K_2})$

### GLUING ALONG A COMMON FACE

Let $K = K_1 \bigcup_\sigma K_2$. If $K_1, K_2 \in \mathcal{F}_0$, then $K \in \mathcal{F}_0$.

$\mathcal{Z}_K \simeq (\prod S^1 * \prod S^1) \vee (\mathcal{Z}_{K_1} \rtimes \prod S^1) \vee (\mathcal{Z}_{K_2} \rtimes \prod S^1)$

### JOIN OF SIMPLICIAL COMPLEXES

$K_1, K_2$ simplicial complexes on sets $S_1$ and $S_2$, belonging to $\mathcal{F}_t$ and $\mathcal{F}_s$.

The *join* $K_1 * K_2 := \{\sigma \subset S_1 \cup S_2 : \sigma = \sigma_1 \cup \sigma_2, \sigma_1 \in K_1, \sigma_2 \in K_1\}$

Notice $k[K_1 * K_2] = k[K_1] \otimes k[K_2]$

Therefore for the join of $K_1$ and $K_2$ we get a product fibration

$$DJ(K_1) \times DJ(K_2) \longrightarrow BT^{m_1} \times BT^{m_2}$$

hence $\mathcal{Z}_{K_1 * K_2} \simeq \mathcal{Z}_{K_1} \times \mathcal{Z}_{K_2}$ and $K_1 * K_2 \in \mathcal{F}_m,\ m = \max\{t, s\} + 1$.

## Our contribution to ALGEBRA

Let $A$ be a polynomial ring on $n$ variables $k[x_1, \ldots, x_n]$ over a field $k$ and $S = A/I$, where $I$ is a homogeneous ideal, i.e, $S = k[K]$ for some simplicial complex $K$.

**PROBLEM**: The nature of $\mathrm{Tor}_S(k, k)$.

The Poincaré series

$$P(S) = \sum_{i=0}^{\infty} b_i t^i \quad \text{where} \quad b_i = \dim_k \mathrm{Tor}_i^S(k, k)$$

**PROBLEM**: The rationality of $P(S)$.

**Theorem.** *(Golod) There exist non-negative integers $n, c_1, \ldots, c_n$ such that*

$$P(S) \leq \frac{(1+t)^n}{1 - \sum_{i=1}^{n} c_i t^{i+1}}.$$

**Theorem.** *(G.,Theriault) There is a topological proof of Golod's inequality.*

**Theorem.** *(Buchstaber-Panov-Ray)*

$$\mathrm{Tor}^*_{k[K]}(k,k) \cong H^*(\Omega DJ(K);k).$$

Looking at the split fibration

$$\Omega\mathcal{Z}_K \longrightarrow \Omega\mathrm{DJ}(K) \longrightarrow T^n$$

$$\mathrm{Tor}^*_S(k,k) \cong H^*(\Omega\mathrm{DJ}(K)) = H^*(T^n) \otimes H^*(\Omega\mathcal{Z}_K)$$

Using the bar resolution,

$$P(H^*(\Omega\mathcal{Z}_K)) \leq P(T(\Sigma^{-1}H^*(\mathcal{Z}_K))).$$

Therefore

$$P(R) = (1+t)^n P(H^*(\Omega\mathcal{Z}_K)) \leq (1+t)^n P(T(\Sigma^{-1}H^*(Z_K)))$$

$$= \frac{(1+t)^n}{1 - P(\Sigma^{-1}H^*(\mathcal{Z}_K))}.$$

Equality is obtained when $H^*(\mathcal{Z}_K)$ is Golod.

**Corollary.** *When $K \in \mathcal{F}_0$, then $P(k[K])$ is rational.*