PROJECTIVE UNITARY REPRESENTATIONS OF SMOOTH DELIGNE COHOMOLOGY GROUPS

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Abstract. Generalizing positive energy representations of the loop group of the circle, we construct projective unitary representations of the smooth Deligne cohomology group of a compact oriented Riemannian manifold of dimension $4k+1$. The number of equivalence classes of these representations is shown to be 2 to the $2k$th Betti number of the manifold.

Contents

1. Introduction

The *smooth Deligne cohomology group* ([1]) of a smooth manifold M is the hypercohomology $H^p(M, \mathbb{Z}(q)^\infty_D)$ of the complex of sheaves:

$$\mathbb{Z}(q)^\infty_D : \ \mathbb{Z} \longrightarrow \underline{A}^0 \xrightarrow{d} \underline{A}^1 \xrightarrow{d} \cdots \xrightarrow{d} \underline{A}^{q-1} \longrightarrow 0 \longrightarrow \cdots,$$

where $\mathbb{Z}$ is the constant sheaf, and $\underline{A}^k$ is the sheaf of germs of $\mathbb{R}$-valued differential k-forms on M. In [2], a non-trivial central extension $\tilde{\mathcal{G}}(M)$ of the smooth Deligne cohomology group $\mathcal{G}(M) = H^{2k+1}(M, \mathbb{Z}(2k+1)^\infty_D)$ by $U(1)$ was constructed for a compact oriented smooth $(4k+1)$-dimensional manifold without boundary:

$$1 \longrightarrow U(1) \longrightarrow \tilde{\mathcal{G}}(M) \longrightarrow \mathcal{G}(M) \longrightarrow 1.$$

The group provides a generalization of a central extension of the loop group $LU(1) = C^\infty(S^1, U(1))$: if we take $k = 0$ and $M = S^1$, then we can identify $\mathcal{G}(S^1)$ with $LU(1)$, and $\tilde{\mathcal{G}}(S^1)$ is isomorphic to the central extension $\widehat{L}U(1)/\mathbb{Z}_2$, where $\widehat{L}U(1)$ stands for the *universal (basic) central extension* ([3]) of $LU(1)$.

A motivation for introducing the central extensions $\tilde{\mathcal{G}}(M)$ in [2] is a generalization of Wess-Zumino-Witten model (from a viewpoint of higher gerbes). In a

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construction of the space of conformal blocks in WZW model, *positive energy representations* of central extensions of loop groups are fundamental ingredients. The aim of the work presented here is to explain a construction of representations of $\tilde{\mathcal{G}}(M)$, generalizing positive energy representations of central extensions of the loop group $LU(1)$.

2. Statement of Result

The central extension $\tilde{\mathcal{G}}(M)$ is, by definition, the group defined by a set $\tilde{\mathcal{G}}(M) = \mathcal{G}(M) \times U(1)$ endowed with the group multiplication

$$(f, u) \cdot (g, v) = (f + g, uv \exp 2\pi i S_M(f, g)),$$

where $S_M : \mathcal{G}(M) \times \mathcal{G}(M) \to \mathbb{R}/\mathbb{Z}$ is a 2-cocycle. Hence linear representations of $\tilde{\mathcal{G}}(M)$ correspond to projective representations of $\mathcal{G}(M)$. We remark that the correspondence will be used freely in the sequel. We also remark that $\mathcal{G}(M)$ will be regarded as a topological group in the below. (See the next section for details of the topology.)

Now, our results are as follows:

Theorem 2.1. *Let k be a non-negative integer, M a compact oriented smooth $(4k+1)$-dimensional manifold, and $\mathcal{G}(M) = H^{2k+1}(M, \mathbb{Z}(2k+1)^{\infty}_{D})$ the smooth Deligne cohomology group. We take and fix a Riemannian metric on M. Then, for each homomorphism*

$$\lambda : H^{2k}(M, \mathbb{R})/H^{2k}(M, \mathbb{Z}) \longrightarrow \mathbb{R}/\mathbb{Z},$$

we can construct a projective unitary representation $(\rho_\lambda, \mathcal{H}_\lambda)$ of $\mathcal{G}(M)$ on an infinite dimensional Hilbert space $\mathcal{H}_\lambda$ with its cocycle

$$e^{2\pi i S_M(\cdot,\cdot)} : \mathcal{G}(M) \times \mathcal{G}(M) \longrightarrow U(1).$$

The representation is continuous in the sense that $\rho_\lambda : \mathcal{G}(M) \times \mathcal{H}_\lambda \to \mathcal{H}_\lambda$ is a continuous map.

Theorem 2.2. *The number of the equivalence classes of the projective representations $(\rho_\lambda, \mathcal{H}_\lambda)$ is 2^b, where $b = b_{2k}(M)$ is the $2k$th Betti number of M.*

As an example, we take $k = 0$ and $M = S^1$. In this case, the representation $\mathcal{H}_\lambda$ labeled by $\lambda \in \mathbb{Z}$ gives rise to a positive energy representation of $LU(1)$ of level 2. It is known in [3] that any irreducible positive energy representation of $LU(1)$ of level 2 is isomorphic to either $\mathcal{H}_0$ or $\mathcal{H}_1$.

3. Construction of representations

The key to the construction of $(\rho_\lambda, \mathcal{H}_\lambda)$ is the following proposition:

Proposition 3.1. *Let M be a compact oriented $(4k+1)$-dimensional Riemannian manifold without boundary, and $A^q(M)$ the space of q-forms on M. Then there exists a positive definite inner product $(\ , \ )_V : A^{2k}(M) \times A^{2k}(M) \to \mathbb{R}$ satisfying the following:*

(a) If we topologize $A^{2k}(M)$ by $(\ , \ )_V$, then the topological vector space $A^{2k}(M)$ decomposes into mutually orthogonal subspaces with respect to $(\ , \ )_V$:

$$A^{2k}(M) \cong d(A^{2k-1}(M)) \oplus \mathbb{H}^{2k}(M) \oplus d^*(A^{2k+1}(M)),$$

where $d^* : A^{2k+1}(M) \to A^{2k}(M)$ is defined to be $d^* = - * d *$ by means of the Hodge star operators, and $\mathbb{H}^{2k}(M)$ is the space of Harmonic $2k$-forms on M.

(b) Let V denote the Hilbert space obtained as the completion of $d^*(A^{2k+1}(M))$ with respect to $(\ ,\ )_V$. Then there exists a linear map $J : V \to V$ such that:

(i) $J^2 = -1$;
(ii) $(Jv, Jv')_V = (v, v')_V$ for $v, v' \in V$;
(iii) $(\alpha, J\beta)_V = \int_M \alpha \wedge d\beta$ for $\alpha, \beta \in d^*(A^{2k+1}(M)) \subset V$.

We can see Proposition 3.1 by taking $(\ ,\ )_V$ to be the inner product induced by the Sobolev $H_{\frac{1}{2}}$-norm and $J : V \to V$ to be $J = \tilde{J}/|\tilde{J}|$, where $\tilde{J} : d^*(A^{2k+1}(M)) \to d^*(A^{2k+1}(M))$ is the differential operator $\tilde{J} = *d$.

Note that the smooth Deligne cohomology $\mathcal{G}(M) = H^{2k+1}(M, \mathbb{Z}(2k+1)^\infty_D)$ fits into the exact sequence:

$$0 \longrightarrow A^{2k}(M)/A^{2k}(M)_{\mathbb{Z}} \longrightarrow \mathcal{G}(M) \xrightarrow{\chi} H^{2k+1}(M, \mathbb{Z}) \longrightarrow 0, \tag{1}$$

where $A^{2k}(M)_{\mathbb{Z}}$ consists of closed integral $2k$-forms on M. Since M is assumed to be compact, the exact sequence above splits. Thus, the topology on $A^{2k}(M)$ and the discrete topology on $H^{2k+1}(M, \mathbb{Z})$ makes $\mathcal{G}(M)$ into a Hausdorff topological group such that

$$\mathcal{G}(M) \cong (\mathbb{H}^{2k}(M)/\mathbb{H}^{2k}(M)_{\mathbb{Z}}) \times d^*(A^{2k+1}(M)) \times H^{2k+1}(M, \mathbb{Z}),$$

where $\mathbb{H}^{2k}(M)_{\mathbb{Z}} = \mathbb{H}^{2k}(M) \cap A^{2k}(M)_{\mathbb{Z}}$ is the group of harmonic $2k$-forms with integral periods. We remark that, in the case of $k = 0$ and $M = S^1$, the isomorphism above is a familiar one: $LU(1) \cong U(1) \times \{\phi : S^1 \to \mathbb{R} | \int \phi(t)dt = 1\} \times \mathbb{Z}$.

Note also that we can use the third property in Proposition 3.1 (b) to prove that the group 2-cocycle $S_M : \mathcal{G}(M) \times \mathcal{G}(M) \to \mathbb{R}/\mathbb{Z}$ is continuous: in fact, on the subgroup $A^{2k}(M)/A^{2k}(M)_{\mathbb{Z}}$, we can express S_M as $S_M(\alpha, \beta) = \int_M \alpha \wedge d\beta \mod \mathbb{Z}$.

Now we outline the construction of $(\rho_\lambda, \mathcal{H}_\lambda)$ in Theorem 2.1, which is a straight generalization of that of positive energy representations of $LU(1)$ in [3].

First, we construct a projective representation of $d^*(A^{2k+1}(M))$. For this aim, let $S : V \times V \to \mathbb{R}$ be the bilinear map $S(v, v') = (v, Jv')$. Then the "symplectic form" S defines a central extension $\tilde{V}$ of V, which is often called the *Heisenberg group*. By virtue of the complex structure J on V, we can apply the standard construction of the representation of the Heisenberg group [3] to our case. Consequently, we obtain a projective representation (ρ, H) of $d^*(A^{2k+1}(M))$.

Next, we construct a projective representation of $A^{2k}(M)/A^{2k}(M)_{\mathbb{Z}}$. The homomorphisms $\lambda : H^{2k}(M, \mathbb{R})/H^{2k}(M, \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ correspond bijectively to the homomorphisms $\lambda : \mathbb{H}^{2k}(M)/\mathbb{H}^{2k}(M)_{\mathbb{Z}} \to \mathbb{R}/\mathbb{Z}$. For each λ, we define a projective representation $(\rho_\lambda, H_\lambda)$ of $A^{2k}(M)/A^{2k}(M)_{\mathbb{Z}}$ by putting $H_\lambda = H$ and $\rho_\lambda(\alpha) = e^{2\pi i \lambda(\eta)}\rho(\nu)$, where we expressed $\alpha \in A^{2k}(M)/A^{2k}(M)_{\mathbb{Z}}$ by $\eta \in \mathbb{H}^{2k}(M)/\mathbb{H}^{2k}(M)_{\mathbb{Z}}$ and $\nu \in d^*(A^{2k+1}(M))$.

Finally, we construct the projective representation $(\rho_\lambda, \mathcal{H}_\lambda)$ of $\mathcal{G}(M)$. Let $\tilde{\mathcal{G}}^0$ be the central extension of $A^{2k}(M)/A^{2k}(M)_{\mathbb{Z}}$ obtained as the restriction of $\tilde{\mathcal{G}}(M)$. The group fits into the exact sequence:

$$1 \longrightarrow \tilde{\mathcal{G}}^0 \longrightarrow \tilde{\mathcal{G}}(M) \longrightarrow H^{2k+1}(M, \mathbb{Z}) \longrightarrow 1.$$

The projective representation $(\rho_\lambda, H_\lambda)$ of $A^{2k}(M)/A^{2k}(M)_{\mathbb{Z}}$ corresponds to a linear representation of $\tilde{\mathcal{G}}^0$. Hence we obtain a linear representation of $\tilde{\mathcal{G}}(M)$ as the induced representation of $(\rho_\lambda, H_\lambda)$, which corresponds to the projective representation $(\rho_\lambda, \mathcal{H}_\lambda)$ of $\mathcal{G}(M)$.

If we take a splitting $\sigma : H^{2k+1}(M, \mathbb{Z}) \to \mathcal{G}(M)$ of the exact sequence (1) suitably, then we can express the representation space $\mathcal{H}_\lambda$ as

$$\mathcal{H}_\lambda = \left\{ \Phi : H^{2k+1}(M, \mathbb{Z}) \to H_\lambda \middle| \sum_\xi \|\Phi(\xi)\|^2 < \infty \right\},$$

and the action of $f \in \mathcal{G}(M)$ on $\Phi \in \mathcal{H}_\lambda$ as

$$(2) \qquad (\rho_\lambda(f)\Phi)(\xi) = e^{2\pi i \{S_M(\sigma(c), \sigma(\xi)+\alpha) - S_M(\sigma(c),\sigma(c))\}} \rho_{\lambda+s(\xi)}(\alpha)\Phi(\xi - c),$$

where $c = \chi(f) \in H^{2k+1}(M, \mathbb{Z})$ and $\alpha = f - \sigma(\chi(f)) \in A^{2k}(M)/A^{2k}(M)_\mathbb{Z}$. The homomorphism $s : H^{2k+1}(M, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{H}^{2k}(M)/\mathbb{H}^{2k}(M)_\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ is given by defining $s(\xi) : \mathbb{H}^{2k}(M)/\mathbb{H}^{2k}(M)_\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to be $s(\xi)(\eta) = 2\int_M \eta \wedge \xi_\mathbb{R} \mod \mathbb{Z}$, where $\xi_\mathbb{R} \in A^{2k+1}(M)_\mathbb{Z}$ is a de Rham representative of the real image of $\xi \in H^{2k+1}(M, \mathbb{Z})$.

4. Classification of representations

Under an isomorphism $\mathrm{Hom}(\mathbb{H}^{2k}(M)/\mathbb{H}^{2k}(M)_\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}^b$, we can identify the image of s with $(2\mathbb{Z})^b \subset \mathbb{Z}^b$ by the Poincaré duality. Hence the classification in Theorem 2.2 follows from the next proposition:

Proposition 4.1. *The projective unitary representations $(\rho_\lambda, \mathcal{H}_\lambda)$ and $(\rho_{\lambda'}, \mathcal{H}_{\lambda'})$ are equivalent if and only if there is $\xi \in H^{2k+1}(M, \mathbb{Z})$ such that $\lambda' = \lambda + s(\xi)$.*

The proof of the "if " part is as follows: we fix a decomposition of $H^{2k+1}(M, \mathbb{Z})$ into the free part F^{2k+1} and the torsion part T^{2k+1}. For $(\rho_\lambda, \mathcal{H}_\lambda)$ and $\xi \in F^{2k+1}$, the representation $(\rho'_\lambda, \mathcal{H}_\lambda)$ given by $\rho'_\lambda = \rho_\lambda(\sigma(\xi))^{-1} \circ \rho_\lambda \circ \rho_\lambda(\sigma(\xi))$ is apparently isomorphic to $(\rho_\lambda, \mathcal{H}_\lambda)$. Using the expression (2), we can show $\rho'_\lambda = \rho_{\lambda+s(\xi)}$.

The proof of the "only if" part is as follows: the expression (2) implies that the representation of $\tilde{\mathcal{G}}^0$ obtained as the restriction of $(\rho_\lambda, \mathcal{H}_\lambda)$ is the Hilbert space direct sum $\widehat{\oplus}_\xi H_{\lambda+s(\xi)}$, where ξ runs through the elements in $H^{2k+1}(M, \mathbb{Z})$. Because $\mathcal{H}_{\lambda'}$ is induced from the representation $H_{\lambda'}$ of $\tilde{\mathcal{G}}^0$, the "Frobenius reciprocity" yields an isomorphism of (abstract) linear spaces:

$$\mathrm{Hom}_{\tilde{\mathcal{G}}(M)}(\mathcal{H}_\lambda, \mathcal{H}_{\lambda'}) \longrightarrow \mathrm{Hom}_{\tilde{\mathcal{G}}^0}(\widehat{\oplus}_\xi H_{\lambda+s(\xi)}, H_{\lambda'}).$$

The restriction maps also yield an injective linear map

$$\mathrm{Hom}_{\tilde{\mathcal{G}}^0}(\widehat{\oplus}_\xi H_{\lambda+s(\xi)}, H_{\lambda'}) \longrightarrow \prod_\xi \mathrm{Hom}_{\tilde{\mathcal{G}}^0}(H_{\lambda+s(\xi)}, H_{\lambda'}).$$

Now we can show that: if $\lambda' \neq \lambda + s(\xi)$, then $\mathrm{Hom}_{\tilde{\mathcal{G}}^0}(H_{\lambda+s(\xi)}, H_{\lambda'}) = 0$, so that $\mathrm{Hom}_{\tilde{\mathcal{G}}(M)}(\mathcal{H}_\lambda, \mathcal{H}_{\lambda'}) = 0$. This completes the proof.

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