The sl_2 loop algebra symmetry of the XXZ spin chain: regular Bethe states as highest weight vectors [1]

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Abstract

The Hamiltonian of the XXZ spin chain has the sl_2 loop algebra symmetry if the q parameter is given by a root of unity, $q_0^{2N} = 1$, for an integer N. We show in some sectors that regular Bethe ansatz eigenvectors are highest weight vectors and generate irreducible representations of the sl_2 loop algebra. Moreover, we prove that every finite-dimensional highest weight representation of the sl_2 loop algebra is irreducible. We thus derive the dimensions of the highest weight representation generated by a given regular Bethe state through the Drinfeld polynomial, which is expressed explicitly in terms of the Bethe roots.

1 Introduction

1.1 The sl_2 loop algebra symmetry

The XXZ spin chain is one of the most important exactly solvable quantum systems. The Hamiltonian under the periodic boundary conditions is given by

$$H_{XXZ} = \frac{1}{2}\sum_{j=1}^{L}\left(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z\right) . \tag{1}$$

Here the XXZ anisotropic coupling Δ is related to the q parameter by $\Delta = (q+q^{-1})/2$. Recently it was shown that when q is a root of unity the XXZ Hamiltonian commutes with the generators of the sl_2 loop algebra [4]. Let q_0 be a primitive root of unity satisfying $q_0^{2N} = 1$ for an integer N. We introduce operators $S^{\pm(N)}$ as follows

$$\begin{aligned} S^{\pm(N)} \;\; = & \sum_{1 \le j_1 < \cdots < j_N \le L} q_0^{\frac{N}{2}\sigma^Z} \otimes \cdots \otimes q_0^{\frac{N}{2}\sigma^Z} \otimes \sigma_{j_1}^{\pm} \otimes q_0^{\frac{(N-2)}{2}\sigma^Z} \otimes \cdots \otimes q_0^{\frac{(N-2)}{2}\sigma^Z} \\ & \otimes \sigma_{j_2}^{\pm} \otimes q_0^{\frac{(N-4)}{2}\sigma^Z} \otimes \cdots \otimes \sigma_{j_N}^{\pm} \otimes q_0^{-\frac{N}{2}\sigma^Z} \otimes \cdots \otimes q_0^{-\frac{N}{2}\sigma^Z} . \end{aligned} \tag{2}$$

The operators $S^{\pm(N)}$ are derived from the Nth power of the generators $S^{\pm}$ of the quantum group $U_q(sl_2)$. We also define $T^{(\pm)}$ by the complex conjugates of $S^{\pm(N)}$, i.e. $T^{\pm(N)} = \left(S^{\pm(N)}\right)^*$. The operators, $S^{\pm(N)}$ and $T^{\pm(N)}$, generate the sl_2 loop algebra, $U(L(sl_2))$, in the sector

$$S^Z \equiv 0 \pmod{N} . \tag{3}$$

Here the value of the total spin S^Z is given by an integral multiple of N. It was shown [4] that in the sector (3) the operators commute with the Hamiltonian of the XXZ spin chain:

$$[S^{\pm(N)}, H_{XXZ}] = [T^{\pm(N)}, H_{XXZ}] = 0 . \tag{4}$$

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[1] Poster presentation in *Groups, Homotopy and Configuration Spaces: Conference in honor of Fred Cohen*, July 4-11, 2005, Tokyo, Japan

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1.2 A physical question

For any given (regular) Bethe state $|R\rangle$ with R down spins in the sector (3), we may have the following degenerate eigenvectors

$$S^{-(N)}|R\rangle\,, \quad T^{-(N)}|R\rangle\,, \quad (S^{-(N)})^2|R\rangle\,, \quad T^{-(N)}S^{+(N)}T^{-(N)}|R\rangle\,, \cdots$$

However, it is nontrivial how many of them are linearly independent. The number gives the degree of the spectral degeneracy. We thus want to know the dimensions of the degenerate eigenspace generated by the Bethe state $|R\rangle$.

1.3 Regular solutions of the Bethe ansatz equations

Let us assume that a set of complex numbers, $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$ satisfy the Bethe ansatz equations at a root of unity:

$$\left(\frac{\sinh(\tilde{t}_j + \eta_0)}{\sinh(\tilde{t}_j - \eta_0)} \right)^L = \prod_{k=1;k\neq j}^{M} \frac{\sinh(\tilde{t}_j - \tilde{t}_k + 2\eta_0)}{\sinh(\tilde{t}_j - \tilde{t}_k - 2\eta_0)}\,, \quad \text{for } j = 1, 2, \ldots, R. \tag{5}$$

Here the parameter η is defined by the relation $q = \exp(2\eta)$, and η_0 is given by $q_0 = \exp(2\eta_0)$. If a given set of solutions of the Bethe ansatz equations are finite and distinct, we call them *regular*. A set of regular solutions of the Bethe ansatz equations leads to an eigenvector of the XXZ Hamiltonian. We call it a *regular Bethe state* of the XXZ spin chain or an *regular XXZ Bethe state*, briefly. We formulate the highest weight conjecture as follows: a regular XXZ Bethe state should be a highest weight vector of the sl_2 loop algebra.

1.4 The sl_2 loop algebra via the Drinfeld realization

Finite-dimensional representations of the sl_2 loop algebra, $U(L(sl_2))$, are formulated through the classical analogues of the Drinfeld realization of the quantum sl_2 loop algebra, $U_q(L(sl_2))$. [1, 2] The classical analogues of the Drinfeld generators, $\bar{x}_k^{\pm}$ and $\bar{h}_k$ ($k \in \mathbf{Z}$), satisfy the defining relations in the following:

$$[\bar{h}_j, \bar{x}_k^{\pm}] = \pm 2\bar{x}_{j+k}^{\pm}\,, \quad [\bar{x}_j^{+}, \bar{x}_k^{-}] = \bar{h}_{j+k}\,, \quad \text{for } j, k \in \mathbf{Z}\,. \tag{6}$$

Here $[\bar{h}_j, \bar{h}_k] = 0$ and $[\bar{x}_j^{\pm}, \bar{x}_k^{\pm}] = 0$ for $j, k \in \mathbf{Z}$. Let us now define highest weight vectors. In a representation of $U(L(sl_2))$, a vector Ω is called *a highest weight vector* if Ω is annihilated by generators $\bar{x}_k^{+}$ for all integers k and such that Ω is a simultaneous eigenvector of every generator of the Cartan subalgebra, $\bar{h}_k$ ($k \in \mathbf{Z}$): [1, 2]

$$\begin{aligned} \bar{x}_k^{+}\Omega &= 0\,, \quad \text{for } k \in \mathbf{Z}\,, & (7)\\ \bar{h}_k\Omega &= \bar{d}_k^{+}\Omega\,, \quad \bar{h}_{-k}\Omega = \bar{d}_{-k}^{-}\Omega\,, \quad \text{for } k \in \mathbf{Z}_{\geq 0} & (8) \end{aligned}$$

We call a representation of $U(L(sl_2))$ *highest weight* if it is generated by a highest weight vector. The set of the complex numbers $\bar{d}_k^{\pm}$ given in (8) is called *the highest weight*. It is shown [1] that every finite-dimensional irreducible representation is highest weight. To a finite-dimensional irreducible representation V we associate a unique polynomial through the highest weight $\bar{d}_k^{\pm}$. [1] We call it the Drinfeld polynomial. Here the degree r is given by the weight $\bar{d}_0^{\pm}$. As we shown in [3], the highest weight vector of V is a simultaneous eigenvector of operators $(\bar{x}_0^{+})^k(\bar{x}_1^{-})^k/(k!)^2$ for $k > 0$, and the Drinfeld polynomial of the representation V has another expression as follows

$$P(u) = \sum_{k=0}^{r} \lambda_k (-u)^k\,, \tag{9}$$

where λ_k denote the eigenvalues of operators $(\bar{x}_0^{+})^k(\bar{x}_1^{-})^k/(k!)^2$.

2 Algorithm for evaluating the degenerate multiplicity

2.1 A theorem on the sl_2 loop algebra

We prove in Ref. [3] the following:

Theorem 1 *Every finite-dimensional highest weight representation of the sl_2 loop algebra is irreducible.*

Let Ω be a highest weight vector and V the representation generated by Ω. Suppose that V is finite-dimensional and $\bar{h}_0 \Omega = r\, \Omega$. We define a polynomial $P_\Omega(u)$ through the eigenvalues λ_k such as in eq. (9). We show that the roots of the polynomial $P_\Omega(u)$ are nonzero and finite, and the degree of $P_\Omega(u)$ is given by r. We introduce evaluation parameters a_j by

$$P_\Omega(u) = \prod_{k=1}^{s} (1 - a_k u)^{m_k} \, , \tag{10}$$

where $a_1, a_2, \ldots, a_s$ are distinct, and their multiplicities are given by $m_1, m_2, \ldots, m_s$, respectively. Note that r is given by the sum: $r = m_1 + \cdots + m_s$.

Theorem 1 shows that V is irreducible and that $P_\Omega(u)$ coincides with the Drinfeld polynomial of V. We thus obtain the Drinfeld polynomial for a given highest weight vector through the highest weight via (9). Furthermore, we have the following:

Proposition 2 *Let V be such a finite-dimensional highest weight representation that has evaluation parameters a_j with multiplicities m_j for $j = 1, 2, \ldots, s$. Then, V is isomorphic to the following tensor product of evaluation representations: $V_{m_1}(a_1) \otimes V_{m_2}(a_2) \otimes \cdots \otimes V_{m_s}(a_s)$. The dimensions of V are given by the product $(m_1 + 1)(m_2 + 1) \cdots (m_s + 1)$.*

2.2 Regular Bethe states as highest weight vectors

For the XXZ spin chain at roots of unity, Fabricius and McCoy made important observations on the highest weight conjecture [5, 6, 7]. Motivated by them, we show in Ref. [3] the following:

Theorem 3 *Every regular Bethe state $|R\rangle$ in the sector $S^Z \equiv 0$ (mod N) at q_0 is a highest weight vector of the sl_2 loop algebra.*

By the method of the algebraic Bethe ansatz, we derive the following relations:

$$\begin{aligned} S^{+(N)} |R\rangle = T^{+(N)} |R\rangle &= 0 \, , \\ \left(S^{+(N)}\right)^k \left(T^{-(N)}\right)^k /(k!)^2 |R\rangle &= \mathcal{Z}_k^+ |R\rangle \quad \text{for } k \in \mathbf{Z}_{\geq 0} \, , \\ \left(T^{+(N)}\right)^k \left(S^{-(N)}\right)^k /(k!)^2 |R\rangle &= \mathcal{Z}_k^- |R\rangle \quad \text{for } k \in \mathbf{Z}_{\geq 0} \, . \end{aligned} \tag{11}$$

Here, the operators $S^{\pm(N)}$, $T^{+(N)}$, $T^{-(N)}$ and $2S^Z/N$ satisfy the same defining relations of the sl_2 loop algebra as generators $\bar{x}_0^{\pm}$, $\bar{x}_{-1}^{+}$, $\bar{x}_1^{-}$ and $\bar{h}_0$, respectively, and hence the relations (11) correspond to (7) and (8).

Eigenvalues $\mathcal{Z}_k^{\pm}$ are explicitly evaluated as

$$\mathcal{Z}_k^+ = (-1)^{kN} \tilde{\chi}_{kN}^+ \, , \quad \mathcal{Z}_k^- = (-1)^{kN} \tilde{\chi}_{kN}^- \, .$$

Here the $\tilde{\chi}_m^{\pm}$ have been defined by the coefficients of the following expansion with respect to small x:

$$\frac{\phi(x)}{\tilde{F}^{\pm}(x q_0) \tilde{F}^{\pm}(x q_0^{-1})} = \sum_{j=0}^{\infty} \tilde{\chi}_j^{\pm} \, x^j \tag{12}$$

where $\phi(x) = (1 - x)^L$ and $\tilde{F}^{\pm}(x) = \prod_{j=1}^{R} (1 - x \exp(\pm 2 \tilde{t}_j))$.

2.3 Drinfeld polynomials of regular Bethe states and the degeneracy

The Drinfeld polynomial for a regular Bethe state $|R\rangle$ in the sector $S^Z \equiv 0 \pmod N$ is evaluated by putting $\lambda_k = (-1)^{kN} \tilde{\chi}_{kN}^{+}$ in eq. (9). The coefficients $\tilde{\chi}_{kN}^{\pm}$ are explicitly given by [3]

$$\tilde{\chi}_{kN}^{\pm} = \sum_{n=0}^{\min(L,kN)} (-1)^n \begin{pmatrix} L \\ n \end{pmatrix} \sum_{n_1+\cdots+n_R=kN-n} e^{\pm \sum_{j=1}^{R} 2n_j \tilde{t}_j} \prod_{j=1}^{R} [n_j+1]_{q_0} \qquad (13)$$

Here $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ and the sum is taken over all nonnegative integers $n_1, n_2, \ldots, n_R$ satisfying $n_1 + \cdots + n_R = kN - n$. We thus obtain the Drinfeld polynomial for the regular Bethe state.

Corollary 4 *Let $|R\rangle$ be a regular Bethe state such as in theorem (3). If the Drinfeld polynomial of the representation V generated by $|R\rangle$ gives evaluation parameters a_j with multiplicities m_j for $j = 1, 2, \ldots, s$, then we have* $\dim V = (m_1 + 1)(m_2 + 1) \cdots (m_s + 1)$.

In particular, when $m_j = 1$ for $j = 1, 2, \ldots, s$, we have $r = s$ and the dimensions are given by the rth power of 2, 2^r, where $r = (L - 2R)/N$.

Note: Theorem 3 generalizes the $su(2)$ symmetry of the XXX spin chain [10].

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