On Toric Arrangements

Cohomology and (some) combinatorics

Corrado De Concini joint with Claudio Procesi.

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Partition Functions

Let me start with a simple problem. Take positive integers h_i, $i = 1, \ldots, s$.
We ask, given a (positive) integer n to compute the number of positive integer solutions of the equation

$$n = y_1 h_1 + \cdots + y_s h_s.$$

Equivalently we want to compute the coefficient of x^n in the expansion of

$\prod_i \frac{1}{1 - x^{h_i}}$ or the residue $\frac{1}{2\pi i} \oint x^{-n-1} \prod_i \frac{1}{1 - x^{h_i}} dx$ over a small circle.

The function

$$\prod_i \frac{x^{-n-1}}{1-x^{h_i}}$$

has poles in 0 and in the $m-$th roots of 1, where m is the least common multiple of the numbers h_i and is regular at ∞.
In other words, if $\zeta = e^{2\pi i/m}$ and we write $\prod_i \frac{1}{1-x^{h_i}} = \prod_{i=1}^m \frac{1}{(1-\zeta^i x)^{b_i}}$ with b_i suitable and easily computed integers, the residue theorem gives:

$$\frac{1}{2\pi i}\oint x^{-n-1}\prod_i \frac{1}{1-x^{h_i}}dx = -\sum_{j=1}^m \frac{1}{2\pi i}\oint_{C_j} x^{-n-1}\prod_{i=1}^m \frac{1}{(1-\zeta^i x)^{b_i}}dx \quad (1)$$

where C_j is a small circle around ζ^{-j}.

If we look at the term

$$\frac{1}{2\pi i}\oint_{C_j} x^{-n-1}\prod_{i=1}^{m}\frac{1}{(1-\zeta^i x)^{b_i}}dx$$

a sequence of elementary manipulations show that it equals

$$-(-1)^{b_j}\zeta^{j(n+1-b_j)}\sum_{k+h=b_j-1}(-1)^k\zeta^{jk}\binom{n+k}{k}a_{j,h}.$$

where

$$\sum_{h=0}^{\infty}a_{j,h}w^h \text{ is the power series expansion of } \prod_{i=1,\ i\neq j}^{m}\frac{1}{(1-\zeta^{i-j}-\zeta^i w)^{b_i}}$$

around zero. This formula, summed over all j, answers the question and exhibits the partition function requested as a sum of functions which are polynomials on the cosets modulo m. One calls such a function a periodic polynomial or quasipolynomial.

We can generalize this by taking a system of linear equations

$$\sum_{i=1}^{m} \alpha_i x_i = b$$

with $\alpha_1, \dots \alpha_m, b \in \mathbb{Z}^n$ and we want to compute the number S_b of solutions $(x_1, \dots, x_m)$ of our system such that x_i is a non negative integer. To be sure that this number is finite we assume that there is an integer vector $\ell = (\ell_1, \dots, \ell_n)$ such that

$$\sum_{j=1}^{n} \ell_j \alpha_{i,j} > 0$$

for each $i = 1, \dots m$ and in order to hope to have at least a solution we can also suppose that

$$\sum_{j=1}^{n} \ell_j b_j > 0$$

Another way to see this is to consider the vectors $\alpha_1, \dots \alpha_m$ as giving a map

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n.$$

$$f(a_1, \dots a_m) = a_1\alpha_1 + \cdots + a_m\alpha_m.$$

We can then consider the convex polytope $P_b = f^{-1}(b) \cap \mathbb{R}^{m+}$, $\mathbb{R}^{m+}$ being the positive quadrant. Then our goal is to compute the number of points with integer coordinates inside P_b.

A way to write all the S_b at the same time is to consider $\mathbb{Z}^n$ as the group of characters of the algebraic torus $T = \mathbb{C}^n/\mathbb{Z}^n$ (given $\alpha \in \mathbb{Z}^n$ we denote by e^α the character such that $e^\alpha(t) = e^{2\pi i \sum \alpha_j t_j}$, for $t = (t_1, \dots, t_n) \in T$).

Then, in a suitable completion of the coordinate ring of T (we can do that because of the assumption about ℓ) we have

$$\prod_{i=1}^{m} \frac{1}{1 - e^{\alpha_i}} = \sum_{b \in \mathbb{Z}^n} S_b e^b.$$

Also in this case one shows (Szenes and Vergne and others) some residue formulae for the S_b which are given as in the simple example given before by computing certain residues of meromorphich n-forms on T with poles on the subtori in T of equation $e^{\alpha_j} - 1$. Computed on certain explicit cycles for the homology of the complement of these subtori.
This approach brings into the picture 3 things:

1. The torus T.
2. The subtori H_{α_j} in T of equation $e^{\alpha_j} - 1$ i.e. the kernels of the characters e^{α_j}.
3. The open set $\mathcal{A} = T - \cup_j H_{\alpha_j}$.

Motivated by these considerations in this talk I will describe some results relative to the geometry of

Toric hyperplane arrangements.

Definition 1 Let T be and algebraic torus of dimension r, with character group Λ. Given a finite subset

$$\Delta = \{(a, \chi) \in \mathbb{C}^* \times \Lambda\},$$

the toric (hyperplane) arrangement associated to Δ is the collection of subvarieties $H_{(a,\chi)} \subset T$ of equation $1 - a\chi = 0$, $(a, \chi) \in \Delta$.

We assume, in this talk, that the χ's for which there is a pair $(a, \chi) \in \Delta$ span $\Lambda \otimes \mathbb{Q}$.

Notice the analogy to the case of (affine) hyperplane arrangements. However there is a at least one "combinatorial" difference.
If we intersects some of the "hyperplanes" we do not necessarily get a connected subvariety.
For example if the intersection of the subvarieties $x = 1$ and $xy^2 = 1$ in a two dimensional torus, consists of the two points $(1, 1)$ and $(1, -1)$.

This brings us to the next definition

Definition 2 Given a subset Δ as above, the intersection poset $\mathcal{P}_\Delta$ of the toric (hyperplane) arrangement associated to Δ is the collection of connected components subvarieties which are intersections of a subset (possibly empty) of the divisors $H_{(a,\chi)} \subset T$ $(a, \chi) \in \Delta$ ordered by reverse inclusion.

There is also a topological difference.
Our toric hyperplanes and their intersections are not linear spaces but rather tori, indeed they are cosets of connected subgroups in T.

So each carries a non trivial topology and has non trivial cohomology groups.

Our goal is to compute the cohomology of the open subset

$$\mathcal{A}_\Delta = T - \cup_{(a,\chi)\in\Delta} H_{(a,\chi)}.$$

Set $A = \mathbb{C}[T] = \mathbb{C}[\Lambda]$. This means that we shall consider A both as the ring of functions on T and as the group algebra of the free abelian group Λ. Set $d = \prod_{(a,\chi)\in\Delta}(1 - a\chi)$.
We start by studying the coordinate ring

$$R = A[\frac{1}{d}]$$

of $\mathcal{A}_\Delta$ as a module over the ring D_T of differential operators on T i.e. the ring generated by A and by the derivations δ_s, $s \in \Lambda^*$ given, for $\chi \in \Lambda$, by

$$\delta_s(\chi) = \langle s, \chi\rangle\chi.$$

We begin by giving a filtration $R_0 \subset R_1 \subset \cdots R_k \cdots$ of R by D_T modules.
It is given as follows:
$R_0 = A$.
R_k is the sum of the subrings in R of the form $A[1/\overline{d}]$ where $\overline{d} = \prod_{(a,\chi)\in\Gamma\subset\Delta,|\Gamma|\leq k}(1 - a\chi)$.

Our first observation is

Theorem. We have

$$R_r = R_{r+k}$$

for each $k > 0$.

Proof. By an easy induction if suffices to see that if $\chi_0, \ldots, \chi_s$ are linearly dependent, $\chi_1, \ldots, \chi_s$ are linearly independent and $a_0, \ldots, a_s$ are non zero complex numbers, then the element $1/(1-a_0\chi_0)\cdots(1-a_s\chi_s)$ can be written as a linear combination of fractions whose denominators is a product of at most s of the elements $(1-a_i\chi_i)$. We let
$\Gamma = \{\psi \in \Lambda | \psi^m$ lies in the lattice spanned by $\chi_1, \ldots, \chi_s\}$, $\chi_0 \in \Gamma$.
We choose representatives $\phi_i \in \mathbb{C}[\Gamma]$ of the primitive idempotents e_i of the ring $\mathbb{C}[\Gamma]/(1-a_1\chi_1, \ldots, 1-a_n\chi_n)$. We have $\chi_0 e_i = \beta_i e_i$ with $\beta_i \in \mathbb{C}$. By the definition of the ϕ_i's, write $1 = \sum_i \phi_i + \sum_{j=1}^n b_i(1-a_j\chi_j)$, for some $b_i \in \mathbb{C}[\Gamma]$. So

$$\frac{1}{(1-a_0\psi_0)\cdots(1-a_n\psi_n)} = \frac{\sum_i \phi_i + \sum_{j=1}^n b_i(1-a_j\psi_j)}{(1-a_0\psi_0)\cdots(1-a_n\psi_n)} =$$

$$\sum_i \frac{\phi_i}{(1-a_0\psi_0)\cdots(1-a_n\psi_n)} + \sum_{j=1}^{n} \frac{b_i(1-a_j\psi_j)}{(1-a_0\psi_0)\cdots(1-a_n\psi_n)}$$

Canceling, we only have to look at

$$\frac{\phi_i}{(1-a_0\chi_0)\cdots(1-a_n\chi_n)}.$$

$(1-a_0\chi_0)\phi_i = \sum_{j=1}^{n} c_j(1-a_j\chi_j) + \gamma_i\phi_i$ with $\gamma_i = 1 - a_0\beta_i$. If $\gamma_i = 0$

$$\frac{\phi_i}{(1-a_0\chi_0)\cdots(1-a_n\chi_n)} = \frac{\sum_{j=1}^{n} c_j(1-a_j\chi_j)}{(1-a_0\chi_0)^2(1-a_1\chi_1)\cdots(1-a_n\chi_n)}$$

and we cancel.
If $\gamma_i \neq 0$,

$$\frac{\phi_i}{(1-a_0\chi_0)\cdots(1-a_n\chi_n)} = \gamma_i^{-1}\frac{(1-a_0\chi_0)\phi_i - \sum_{j=1}^{n} c_j(1-a_j\chi_j)}{(1-a_0\chi_0)\cdots(1-a_n\chi_n)},$$

and we cancel again.

Thus now we have the filtration of D_T submodules

$$0 \subset A = R_0 \subset R_1 \subset \cdots \subset R_r = R.$$

The aim is to study the D_T modules R_k/R_{k-1}.
We now make a detour. Suppose $\Gamma \subset \Lambda$ is a sublattice. $\mathbb{C}[\Gamma] \subset \mathbb{C}[\Lambda] = A$. Take a maximal ideal $m \subset \mathbb{C}[\Gamma]$ and consider the space L_Γ of derivations δ_s such that $\delta_s(\chi) = 0$ for $\chi \in \Gamma$. We consider the left ideal J_m in D_T generated by m and L_Γ and define a D_T module

$$N(m) = D_T/J_m$$

Theorem. 1. If Γ is a split direct summand in Λ, then $N(m)$ is a irreducible holonomic D_T module with characteristic variety the conormal bundle to the subvariety defined in T whose ideal is mA.
2. If Γ is not a split direct summand, set

$$\overline{\Gamma} = \{\psi \in \Lambda | \psi^m \in \Gamma \text{ for some } m\}.$$

and let $m_1, \ldots m_s$, $s = |\overline{\Gamma}/\Gamma|$ the maximal ideals in $\mathbb{C}[\overline{\Gamma}]$ over m. Then

$$N(m) \simeq \oplus_{i=1}^s N(m_i).$$

What do this modules have to do with our previous considerations? Easy. Take a basis $\psi_1, \ldots \psi_t$ of Γ. There are non zero complex numbers $a_1, \ldots, a_t$ with

$$m = (1 - a_1\psi_1, \ldots, 1 - a_t\psi_t).$$

Consider the ring

$$A[\frac{1}{\prod_{i=1}^t (1 - a_i\psi_i)}]$$

as a D_T module. Then

Theorem. The D_T module

$$\frac{A[\frac{1}{\prod_{i=1}^t(1-a_i\psi_i)}]}{\sum_{j=1}^t A[\frac{1}{\prod_{i\neq j}(1-a_i\psi_i)}]}$$

is isomorphic to

$$N(m).$$

The class of 1 in $N(m)$ maps to the class of $1/\prod_{i=1}^t(1 - a_i\psi_i)$.

We can try to use this to describe our modules R_k/R_{k-1}. This is indeed the case.
Fix an element W in the intersection poset $\mathcal{P}_\Delta$ of codimension k. Put

$$\Delta_W = \{(a, \chi) \in \Delta | a\chi = 1 \text{ on } W\}$$

Project Δ_W to Λ and remark that we get an injection. Set Γ_W equal to the sublattice generated by this image and as before

$$\overline{\Gamma}_W = \{\psi \in \Lambda | \psi^m \in \Gamma_W \text{ for some } m\}.$$

There is a unique maximal ideal $m_W \in \mathbb{C}[\overline{\Gamma}_W]$ such that

1. $m'_W = m_W \cap \mathbb{C}[\Gamma_W]$ is the maximal ideal generated by the elements $1 - a\chi$ with $(a, \chi) \in \Delta_W$.

2. $m_W A$ is the ideal of W.

By the previous construction we get an irreducible D_T module associated to W

$$N(m_W) := N(W).$$

These modules will be the irreducible components of our R_k/R_{k-1}.

We need to describe the multiplicities.

Fix a total ordering on Δ_W.

Definition 3. A non broken circuit in Δ_W is a finite subset $(a_1, \chi_1) < \cdots < (a_k, \chi_k) \in \Delta_W$ such that

1. $\chi_1, \dots, \chi_k$ are linearly independent.
2. If $(a, \chi) \in \Delta_W$, $(a, \chi) \le (a_j, \chi_j)$ and $\chi, \chi_j, \dots, \chi_k$ are linearly dependent, then $(a, \chi) = (a_j, \chi_j)$.

To such a non broken circuit $(a_1, \chi_1), \dots, (a_k, \chi_k)$ we associate an element in R_k as follows We take the idempotent $\varepsilon \in \mathbb{C}[\overline{\Gamma}_W]/m'_W$ killed by m_W and a representative $\phi \in \mathbb{C}[\overline{\Gamma}_W]$. We than take

$$\frac{\phi}{(1 - a_1\chi_1) \cdots (1 - a_k\chi_k)} \in R_k.$$

and remark that its class modulo R_{k-1} is independent from the choice of ϕ.

The classes of these elements are linearly independent and span a vector subspace $V(W)$ which is independent from the choice of the total ordering on Δ_W.

This is the space "of multiplicities".

We are almost ready to finish our description of R_k/R_{k-1} as a D_T module. Let us consider the Moëbius function $\mu(W_1, W_2)$ for the intersection poset $\mathcal{P}_W$. Set $\mu(W) = \mu(T, W)$.

Theorem. 1. The D_T module R_k/R_{k-1} is semisimple and it is canonically isomorphic to the module

$$\bigoplus_{W \in \mathcal{P}_\Delta,\ c(W)=k} M(W) \otimes V(W).$$

2. The dimension of $V(W)$ equals $(-1)^{c(W)}\mu(W)$. where $c(W)$ is the codimension of W.

Cohomology

The next question is how to use this to compute cohomology. The way is through the algebraic de Rham complex. By a result of Grothendieck we know that $H^*(\mathcal{A}_\Delta, \mathbb{C})$ is the cohomology of the complex

$$\Omega(\mathcal{A}_\Delta) = R \otimes \wedge(\frac{dx_1}{x_1}, \dots, \frac{dx_r}{x_r})$$

where $x_1, \dots x_r$ is a basis for Λ and $\wedge(\frac{dx_1}{x_1}, \dots, \frac{dx_r}{x_r})$ is the exterior algebra of invariant differential forms.
The differential is given by

$$d(f\frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}}) = \sum_{i=1}^{r} \delta_i(f)\frac{dx_i}{x_i} \wedge \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}}$$

$$\delta_i = x_i\frac{\partial}{\partial x_i}.$$

The definition of the differential implies that setting $\Omega_k = R_k \otimes \wedge(\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r})$, Ω_k is a subcomplex and we get a filtration of $\Omega(\mathcal{A}_\Delta)$

$$0 \subset \Omega_0 \subset \cdots \subset \Omega_k \subset \cdots \subset \Omega_r = \Omega(\mathcal{A}_\Delta)$$

by subcomplexes. In particular exact sequences of complexes

$$0 \to \Omega_{k-1} \to \Omega_k \to \Omega_k/\Omega_{k-1} = R_k/R_{k-1} \otimes \wedge(\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}) \to 0$$

and a direct sum of complexes

$$\Omega_k/\Omega_{k-1} = \oplus_{W \in \mathcal{P}_\Delta,\ c(W)=k} M(W) \otimes V(W) \otimes \wedge(\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}).$$

All this is for free. A little bit of extra work is needed to show that

The sequence of complexes

$$0 \to \Omega_{k-1} \to \Omega_k \to \Omega_k/\Omega_{k-1} = R_k/R_{k-1} \otimes \wedge(\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}) \to 0$$

splits (non canonically).

Putting everything together we have shown that our cohomology groups can be written as a sum of local contributions one for each element W of the intersection poset $\mathcal{P}_\Delta$ and this local contribution is the cohomology of the complex

$$M(W) \otimes V(W) \otimes \wedge(\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}).$$

For this we have

Proposition. The cohomology of

$$M(W) \otimes V(W) \otimes \wedge(\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}).$$

is isomorphic as a graded vector space to

$$H^*(W, \mathbb{C}) \otimes V(W)[c(W)].$$

i.e. the h-cohomology group is isomorphic to $H^{h-c(W)}(W) \otimes V(W)$.

We should remark that in this way one also obtains bases for the cohomology given by some "explicit" differential forms. Also we should remark that topologically, W is a torus of dimension $r - c(w)$, so we know that its cohomology is an exterior algebra on $r - c(w)$ generators. Summarizing

Theorem. $H^h(\mathcal{A}_\Delta)$ is (non canonically) isomorphic to

$$\oplus_{W \in \mathcal{P}_\Delta} H^{h-c(W)}(W) \otimes V(W).$$

If we put together the formula in terms of the Moëbius function for the dimension of $V(W)$, the fact that W is a torus and the above theorem, we get a combinatorial formula for the Poincàre polynomial of $\mathcal{A}_\Delta$ which resembles the one for linear hyperplane arrangements. This formula appears in the work of Looijenga.

$$\sum_h dim\ H^h(\mathcal{A}_\Delta)t^h = \sum_{W\in\mathcal{P}_\Delta} \mu(W)(1+t)^{r-c(W))}(-t)^{c(W)}.$$

In particular the Euler characteristic equals

$$\chi(\mathcal{A}_\Delta) = \sum_{W\in\mathcal{P}_\Delta, W \text{ a point}} (-1)^r \mu(W).$$

Multiplicative Structure

The cohomology is a ring so one would like to know the multiplicative structure of $H^*(\mathcal{A}_\Delta)$. At the moment we know this under an assumption.

Definition 4. A finite subset $\Theta \subset \Lambda$ is called unimodular if any proper subset in Θ spans a split direct summand of Λ.
$\Delta \subset \mathbb{C}^* \times \Lambda$ is unimodular it its projection to Λ is unimodular.

In this case one can completely determine the multiplicative structure of $H^*(\mathcal{A}_\Delta)$. Indeed one has the following "formality" statement which gives explicit generators for $H^*(\mathcal{A}_\Delta)$. A similar statement which I am not going to write give the analogous of the Orlik Solomon relations as well.

Theorem. Consider the subalgebra B of the algebra $\Omega(\mathcal{A}_\Delta)$ of differential forms generated by the forms

$$d\log x_i \quad d\log(1 - a\chi)$$

$\{x_i\}$ a basis of Λ, $(a, \chi) \in \Delta$. Then

1. Every form in B is closed, so that we get a morphism of graded algebras
$$\tau : B \to H^*(\mathcal{A}_\Delta, \mathbb{C}).$$

2. The homomorphism τ is an isomorphism.

To finish some problems.

1. Does $H^*(\mathcal{A}_\Delta, \mathbb{Z})$ have torsion? I would guess not.

2. Is $H^*(\mathcal{A}_\Delta)$ formal for a general Δ? Also in this case I would guess not.

3. Does the above theorem hold over $\mathbb{Z}$? (it should be said that the relations have integer coefficients)? I would say yes.

Some examples

Among the most studied hyperplane arrangements there are the so called reflection arrangements. i.e. one takes a finite group W generated by reflections and takes the hyperplanes H_s fixed by a reflection $s \in W$.

Here is a toric analogue.

Take a semisimple algebraic group (for example $Sl(n)$). Take a maximal torus in $T \subset G$ (for example the torus of diagonal matrices of determinant 1). Consider the root system $R \subset X(T) = \Lambda$. We can the consider the corresponding Toric arrangement and try to study $\mathcal{A}_R$.
In this case we also have the action of the Weyl group W (in our case it is the symmetric group S_n), so we might try to study $H^*(\mathcal{A}_R)$ as e representation of W. This case has been studied before (Lehrer, Loojienga) and a number of things are known.

Here is a list of known and not known facts.

1. The fundamental group. Consider the affine Weyl group $\tilde{W}$. $\tilde{W}$ is the semidirect product of W and of the sublattice $Q \subset \Lambda$ spanned by R, so we have a homomorphism $\pi : \tilde{W} \to W$. Take the Artin group $\mathcal{B}_{\tilde{W}}$ associated to $\tilde{W}$. We have a homomorphism $p : \mathcal{B}_{\tilde{W}} \to \tilde{W}$.
Then

Theorem. If G is simply connected, $\pi_1(\mathcal{A}_R) = \ker \pi p$.

This is similar to the case of reflection arrangements where the fundamental group is the pure Artin group.
However it is not known whether $\mathcal{A}_R$ is a K$(\pi, 1)$ space (in the reflection arrangements case this is a celebrated result of Deligne).

2. Euler characteristic. By experimenting with the formula for the Euler characteristic given before one finds

Theorem. If G is simply connected, as a virtual character,

$$\chi_W(\mathcal{A}_R) = (-1)^r Reg$$

Reg being the regular character of W. In particular we have

$$\chi(\mathcal{A}_R) = (-1)^r |W|.$$

This last statement was not proved using the method explained here. Instead one uses an explicit complex of $\mathbb{Z}[W]$ modules which has been introduced by Salvetti and which computes the homology of $\mathcal{A}_R$ as a $\mathbb{Z}[W]$-module.

It is not clear how to use our methods to compute Euler characteristic. In fact it is not easy to enumerate points in the intersection poset $\mathcal{P}_\Delta$.

Trying to do this in the case of root systems one discovers another curious combinatorial fact.
Consider an extended Dynkin diagram D. For any node a of D the diagrams D_a obtained by removing a is of finite type and has a finite Weyl group W_a. W_a has its reflection representation V_a.
One knows that the polynomial functions on V_a form a polynomial ring generated by homogeneous polynomial of respective degrees $d_1^{(a)}, \dots, d_r^{(a)}$. One has

$$\sum_{a \in D} \frac{(d_1^{(a)} - 1) \cdots (d_r^{(a)} - 1)}{d_1^{(a)} \cdots d_r^{(a)}} = 1$$