Homotopy localization with respect to proper classes of maps

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6th July 2005

COE Conference, *Groups, Homotopy, and Configuration Spaces*, Tokyo, 2005

Some foundational references

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Homotopy localization in model categories

A **model category** is a category $\mathcal{M}$ with three distinguished classes of morphisms, called **cofibrations**, **fibrations**, and **weak equivalences**, satisfying the Quillen axioms. We will assume that $\mathcal{M}$ has all small limits and colimits, and that the factorizations stated in the Quillen axioms are functorial.

The homotopy category Ho$\mathcal{M}$ has the same objects as $\mathcal{M}$, and its morphisms from X to Y are homotopy classes of maps from QX to RY, where Q is a cofibrant approximation functor and R is a fibrant approximation functor.

A **homotopy function complex** in a model category $\mathcal{M}$ is any bifunctor taking values in simplicial sets, denoted map$(-,-)$, whose homotopy type is the same as the diagonal of the bisimplicial set $\mathcal{M}(X^*, Y_*)$, where $X^* \to X$ is a cosimplicial resolution of X and $Y \to Y_*$ is a simplicial resolution of Y. Homotopy function complexes exist in any model category. If the given model category is **simplicial**, then Map$(Q-, R-)$ is a homotopy function complex, where Map$(-,-)$ is given by the simplicial enrichment, Q is a cofibrant approximation functor, and R is a fibrant approximation functor.

If $\mathcal{M}$ is any model category and $\mathcal{S}$ is a class of maps in $\mathcal{M}$, an object Y is called **$\mathcal{S}$-local** if it is fibrant and, for each $f\colon A \to B$ in $\mathcal{S}$, the induced map of simplicial sets

$$\mathrm{map}(f, Y)\colon \mathrm{map}(B, Y) \longrightarrow \mathrm{map}(A, Y)$$

is a weak equivalence, where $\mathrm{map}(-,-)$ is any homotopy function complex. We also refer to this condition by saying that Y is **simplicially orthogonal** to all the maps in $\mathcal{S}$.

A **homotopy localization** in $\mathcal{M}$ is a functor $L\colon \mathcal{M} \to \mathcal{M}$ preserving weak equivalences, taking fibrant values, and equipped with a natural transformation $\eta\colon \mathrm{Id} \to L$ which is idempotent in the homotopy category $\mathrm{Ho}\mathcal{M}$, that is, for every object X, the maps $L\eta_X$ and η_{LX} from LX to LLX coincide in $\mathrm{Ho}\mathcal{M}$ and are isomorphisms.

A homotopy localization L is called an **$\mathcal{S}$-localization**, where $\mathcal{S}$ is a class of maps, if the image of L in $\mathrm{Ho}\mathcal{M}$ is precisely the closure under isomorphisms of the class of $\mathcal{S}$-local objects.

Existence of f-localizations

The existence of an f-localization functor L_f of simplicial sets for every single map f was proved by Bousfield and Dror Farjoun. Essentially the same construction works in any left-proper model category $\mathcal{M}$ that is cofibrantly generated and whose underlying category is locally presentable. Such model categories are called **combinatorial**.

A model category is **left proper** if any pushout of a weak equivalence along a cofibration is a weak equivalence. It is **right proper** if any pullback of a weak equivalence along a fibration is a weak equivalence.

A model category is **cofibrantly generated** if there are sets of maps I and J with small domains where I detects trivial fibrations and J detects fibrations.

A category $\mathcal{C}$ is **locally presentable** if it is cocomplete and there is a regular cardinal λ and a set $\mathcal{X}$ of λ-presentable objects such that every object of $\mathcal{C}$ is a λ-directed colimit of objects from $\mathcal{X}$. This concept was originally introduced by Gabriel and Ulmer in 1971. An object X of a category $\mathcal{C}$ is called λ-**presentable**, where λ is a regular cardinal, if the functor $\mathcal{C}(X, -)$ preserves λ-directed colimits.

In fact, if $\mathcal{M}$ is left proper and combinatorial, then, for every map f, there is a new model category structure $\mathcal{M}_f$ with the same objects, where the weak equivalences are the maps h such that $L_f(h)$ is a weak equivalence in $\mathcal{M}$.

Some important special cases

Localization at sets of primes: f is the coproduct of the power maps $q\colon S^1 \to S^1$ where q belongs to the complement of the given set of primes.

Homological localizations: f is the coproduct of a sufficiently large set of equivalences for a given homology theory. The same method does *not* work for *cohomology* theories. The existence of cohomological localizations of simplicial sets or spectra is still an open problem.

Quillen's plus-construction: f sends the coproduct of a sufficiently large set of acyclic simplicial sets to a point. A much smaller f was described by [Berrick–C, 1999].

The unstable motivic category: $\mathcal{M}$ is the model category of simplicial presheaves over the Grothendieck site of smooth schemes over a field k with the Nisnevich topology, and f is the trivial map from the affine line $\mathbf{A}^1$ to the point Spec(k).

Large-cardinal axioms

There is a hierarchy of set-theoretical statements that cannot be proved using the ordinary ZFC axioms (Zermelo–Fraenkel axioms with the axiom of choice), yet they are believed to be consistent with ZFC. Their negations are consistent with ZFC. For example, such axioms include the existence of strongly inaccessible cardinals or the existence of measurable cardinals.

The following statements are in the large-cardinal hierarchy:

Vopěnka's Principle (VP): *The category of ordinals cannot be fully embedded into the category of graphs (binary relations).*

WVP: *The opposite of the category of ordinals cannot be fully embedded into the category of graphs.*

It is known that VP implies WVP, and that WVP is equivalent to the following statement: *Every full subcategory of a locally presentable category closed under limits is reflective.*

A full subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is called **reflective** if the embedding $\mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint $L: \mathcal{C} \to \mathcal{D}$, which is called a **reflection** or a **localization**. A full subcategory $\mathcal{A}$ is called **coreflective** if the embedding $\mathcal{A} \hookrightarrow \mathcal{C}$ has a right adjoint $\mathcal{C} \to \mathcal{A}$.

VP is equivalent to the statement that every full subcategory of a locally presentable category closed under colimits is coreflective.

VP implies that every full subcategory of a locally presentable category closed under limits is the orthogonal complement of a single morphism.

Theorem [C–Scevenels–Smith, 2005] *Assume that VP holds. Let $\mathcal{S}$ be any (possibly proper) class of maps in the category of simplicial sets. Then $\mathcal{S}$-localization exists and is equivalent to L_f for some single map f. Moreover, if L is any homotopy localization on simplicial sets, then $L \simeq L_f$ for some f.*

Therefore, VP implies the existence of cohomological localizations.

An example of a homotopy localization on simplicial sets that is not equivalent to L_f for any map f was displayed under the assumption that measurable cardinals do not exist (a statement that is consistent with ZFC, but incompatible with VP).

Open problem: Find an example of a class of maps of simplicial sets $\mathcal{S}$ such that $\mathcal{S}$-localization can be proved *not* to exist, under the negation of some large-cardinal principle.

From spaces to spectra, and beyond

The above theorem was extended by [C–Chorny, 2005] to any left-proper combinatorial model category $\mathcal{M}$. This applies to the model category of symmetric spectra over simplicial sets.

Theorem [C–Gutiérrez–Rosický] *Assuming VP, every descending chain of simplicial orthogonality classes*

$$\ldots \subseteq \mathcal{D}_\alpha \subseteq \mathcal{D}_{\alpha-1} \subseteq \ldots \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_0$$

indexed by the ordinals in a combinatorial model category stabilizes.

We say that $\mathcal{D}_\alpha$ is a **simplicial orthogonality class** if there is a class of maps $\mathcal{S}_\alpha$ such that $\mathcal{D}_\alpha$ is precisely the class of all objects X such that

$$\mathrm{map}(f, X)\colon \mathrm{map}(B, X) \to \mathrm{map}(A, X)$$

is a weak equivalence for every $f\colon A \to B$ in $\mathcal{S}_\alpha$.

Theorem [C–Gutiérrez–Rosický] *Let $\mathcal{M}$ be a stable combinatorial model category. Assuming VP, every localizing subcategory of* $\mathrm{Ho}\mathcal{M}$ *is coreflective and every colocalizing subcategory of* $\mathrm{Ho}\mathcal{M}$ *is reflective.*

This answers a question posed by [Hovey–Palmieri–Strickland, 1997].

A pointed model category $\mathcal{M}$ is **stable** if the suspension operator is invertible in $\mathrm{Ho}\mathcal{M}$. Hence, $\mathrm{Ho}\mathcal{M}$ is a triangulated category.

If $\mathcal{T}$ is a triangulated category, a subcategory is **localizing** if it is triangulated and closed under retracts and coproducts. A subcategory is **colocalizing** if it is triangulated and closed under retracts and products.

Open problem: Find an example of a localizing subcategory in the homotopy category of a stable combinatorial model category that fails to be coreflective, under the negation of some large-cardinal principle.

Two counterexamples

Example [Neeman, 2001] In the triangulated category $K(\mathbb{Z})$ of chain complexes of abelian groups and homotopy classes of chain maps, there are localizations that are not equivalent to L_f for any map f. In fact, the derived category $D(\mathbb{Z})$ embeds into $K(\mathbb{Z})$ and the kernel of the reflection $K(\mathbb{Z}) \to D(\mathbb{Z})$ is the class of acyclic chain complexes, which is not generated by any set. The category $K(\mathbb{Z})$ is the homotopy category of a model category that fails to be cofibrantly generated.

Example [C–Neeman] There is a triangulated category $\mathcal{T}$ (namely, chain complexes of small modules over the free ring on all the ordinals, with homotopy classes of chain maps), in which there is a localizing subcategory which is not coreflective. Indeed, the class $\mathcal{A}$ of acyclic chain complexes is closed under triangles, retracts and coproducts, yet it is not the class of chain complexes annihilated by any localization functor defined on $\mathcal{T}$ (i.e., $\mathcal{A}$-nullification does not exist). This triangulated category $\mathcal{T}$ is also the homotopy category of a model category that fails to be cofibrantly generated.