On the convergence of the Eilenberg-Moore spectral sequence

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The Eilenberg-Moore spectral sequence

Given a fibration $F \to E \to B$ *and a cohomology theory* K*, can we compute* $K^*(F)$ *from a knowledge of* $K^*(B)$ *(as an algebra) and* $K^*(E)$ *(as a* $K^*(B)$*-algebra)?*

Theorem (Eilenberg, Moore, 1959). *If* B *is simply connected and* $K = H_*(-;\mathbf{F}_p)$ *then there is a convergent spectral sequence*

$$E_2^{**} = \mathrm{Tor}_{**}^{K^*(B)}(K^*(E), K_*) \Longrightarrow K^*(F)$$

• Problems occur when the fibration is not nilpotent, or when K is a nonconnective theory.

Simple connectivity cannot be omitted: take $B = K(\mathbf{Z}/l, 1)$ with $p \nmid l$, and E contractible.
Then $\tilde{K}^*(B) = \tilde{K}^*(E) = 0$, but $\tilde{K}^*(F)$ is nontrivial.

A general construction of the EMSS

The *cobar construction* of a map $E \xrightarrow{p} B$ is the cosimplicial space

$$C^{\mathfrak{n}} = \mathrm{Cobar}^n(E \to B) = E \times B^n$$

with maps

$$E \underset{\mathrm{id}\times *}{\overset{\mathrm{id}\times p}{\rightrightarrows}} E \times B \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix} \cdots$$

The associated tower of total complexes has a very simple shape:

$$\begin{aligned} \mathrm{Tot}^0(C^\bullet) &= E \\ \mathrm{Tot}^i(C^\bullet) &= F \quad \text{for } i \geq 1. \end{aligned}$$

If K is a spectrum then $K \wedge C^\bullet$ is a cosimplicial spectrum with an associated total tower; however, the canonical map

$$\Phi\colon K \wedge \mathrm{Tot}^n(C^\bullet) \to \mathrm{Tot}^n(K \wedge C^\bullet)$$

is almost never an equivalence.

Unlike $\mathrm{Tot}^s(C^\bullet)$, the tower $\mathrm{Tot}^s(K \wedge C^\bullet)$ is not eventually constant.
Thus $K \wedge C^\bullet$ produces an interesting left half-plane spectral sequence, the Bousfield spectral sequence for a cosimplicial spectrum.
If K has Künneth isomorphisms, then its E^2-term is given by

$$E^2_{**} = \mathrm{Cotor}_{K_*(B)}(K_*(E), K_*),$$

and its expected target is $\pi_* \mathrm{Tot}(K \wedge C^\bullet)$.

If $\Phi_* \colon K_n(\mathrm{Tot}^s(C^\bullet)) \to \pi_n \mathrm{Tot}^s(K \wedge C^\bullet)$ is a pro-isomorphism for each n, then

- E^k_{st} becomes eventually constant for every s, t,
- $E^\infty_{s,t-s}$ is nontrivial for only finitely many s,
- (and therefore) the spectral sequence converges to $K_*(F)$ in a strong (pro-constant) sense.

Some non-convergence examples

Even simple connectedness is not enough to ensure convergence if K is nonconnective.

Example. Let $K = K(n)$, $B = K(\mathbf{Z}/p, n+1)$.

Computations of Ravenel-Wilson:

$$\widetilde{K(n)}_*(B) = 0, \text{ but } \widetilde{K(n)}_*(\Omega B) \neq 0$$

Thus the $K(n)$-based EMSS cannot possibly converge for the path-loop fibration on B.

Example. Consider the two path-loop fibrations

$$\begin{array}{ccc} \mathrm{SU}/p & & U/p \\ \downarrow & & \downarrow \\ * & \text{and} & * \\ \downarrow & & \downarrow \\ B\mathrm{SU}/p & & BU/p \end{array}$$

Since $(K/p)_*(B\mathrm{SU}/p) \cong (K/p)_*(BU/p)$, the spectral sequences are isomorphic.
But $(K/p)_*(\mathrm{SU}/p) \ncong (K/p)_*(U/p)$.

Partial convergence results

The convergence for $K = H_*(-;\mathbf{F}_p)$ is quite well-understood for all fibrations:

Theorem (Dwyer). *Let $F \to E \xrightarrow{\pi} B$ be a fibration with F and E connected. The $H_*(-;\mathbf{F}_p)$-based EMSS for a fibration $E \to B$ converges pro-constantly to the homology of the fiber on p-completion towers.*

This means: If

- $\{R_s X\}_s$ denotes the p-completion tower of a space X and
- F_s denotes the fiber of $R_s E \xrightarrow{R_s \pi} R_s B$

then the EMSS for π converges pro-constantly to the total space of the pro-constant tower $H_*(F_s;\mathbf{F}_p)$.

- This generalizes to connective theories.

Two convergence results for $K = K(n)$

Theorem (Tamaki, 1994). *The $K(n)$-based EMSS for the path-loop fibration on $\Omega^{n-1}\Sigma^n X$ converges strongly to $K(n)_*(\Omega^n\Sigma^n X)$ for any space X and any $n \geq 1$.*

• In particular, the path-loop EMSS converges on suspensions (in fact, it collapses at E^2).

Theorem (Jeanneret-Osse, 1999). *The $K(n)$-based cohomological EMSS for $X \to B$ converges whenever $K^*(\Omega B)$ is an exterior algebra on finitely many generators.*

• Useful for classifying spaces of finite loop spaces.

Complete convergence

The notion of pro-constant convergence is too restrictive for the EMSS for nonconnective theories.

Definition. For $C^\bullet = \mathrm{Cobar}^\bullet(E \to B)$, let

- $T^s = \mathrm{Tot}^s(K \wedge C^\bullet)$;
- $F^s K_*(F) = \ker\left(K_*(F) \xrightarrow{\Phi} \pi_* T^s\right)$

The EMSS is called

- *weakly convergent* to K_*F if $\frac{F^s K_* F}{F^{s+1} K_* F} \to E_\infty^{s,*}$ is iso;
- *strongly convergent* if in addition,

$$\lim{}^{\varepsilon} F^s K_*(F) \quad \text{for} \quad \varepsilon = 0,\ 1;$$

- *completely convergent* if also $\lim^1 \pi_* T^s = 0$.

In a completely convergent spectral sequence, infinitely many differentials at (s,t) and infinite filtrations are possible, but in a controlled way.

New convergence results

From now on, let K_* be a graded field (e.g. $K = K(n)$).

Call a fibration *EM-convergent* if its K-based EMSS is completely convergent to $K_*(F)$.

Theorem 1. *Let B be a space such that the path-loop fibration on B is EM-convergent. Then so is any fibration $E \longrightarrow B$.*

Hodgkin proved a cohomological version of this result under the assumption that $K^*(B)$ has finite global dimension.

Theorem 2. *Let $X \to B_i$ $(i = 1, 2)$ be two EM-convergent fibrations with fibers F_i. Then*

$F_1 \to B_2$ is EM-convergent iff $F_2 \to B_1$ is EM-convergent.

The fiber, in both cases, is the fiber of $X \to B_1 \times B_2$.

Corollary. *Let $F \to E \to B$ be a fibration such that the path-loop fibrations on F and on B are EM-convergent. Then so is the path-loop fibration (and hence any fibration) on E.*

Proof.

$$1)\quad \begin{array}{ccccc} \Omega B & \longrightarrow & F & \longrightarrow & E \\ \downarrow & & \downarrow & & \| \\ * & \longrightarrow & E & = & E \\ \downarrow & & \downarrow & & \downarrow \\ B & = & B & \longrightarrow & * \end{array} \qquad 2)\quad \begin{array}{ccccc} \Omega E & \longrightarrow & * & \longrightarrow & E \\ \downarrow & & \downarrow & & \| \\ \Omega B & \longrightarrow & F & = & E \\ \downarrow & & \downarrow & & \downarrow \\ F & = & F & \longrightarrow & * \end{array}$$

□

Theorem 3. *Any fibration over a zeroth space of a K-module spectrum is EM-convergent.*

Idea of the proof:

By Theorem 1, it is enough to consider the path-loop fibration.

The spectral sequence splits as an (infinite) tensor product of path-loop spectral sequences of $\underline{K}_k$, the kth space of the spectrum K.

$K_*(\underline{K}_*)$ is a Hopf ring, computed by Wilson (1984):

As a $\mathbf{Z}/(2p^n-2)$-bigraded ring,

$$K(n)_*\underline{K(n)}_* \cong \bigotimes_{j_0<p^n-1} \bigwedge \left(a^I b^J e_1\right) \otimes \bigotimes_{i_0=0 \text{ or } j_0<p^n-1} P_{t_1(I)+1}\left(a^I b^J\right)$$

where $k \in \mathbf{Z}$, $I = (i_0, \dots, i_{n-1})$ with $i_k = 0$ or 1, $j_k < p^n$.

From this knowledge, the EMSS can be computed explicitly. For $K = K(n)$, it collapses at $E_{**}^{p^{n+1}}$ and is completely convergent.

A generalization of Dwyer's convergence

Theorems 2 and 3 allow an inductive argument to show:

Corollary. *Let $F \to E \to B$ be an arbitrary fibration. Let K be a field spectrum. Then the K_*-based EMSS converges completely to K_* of the fiber on K-completion towers.*

The K-completion tower (Bendersky-Thompson) is the analog of the p-completion tower: it is associated to the cosimplicial space

$$K^\bullet(X) = \Omega^\infty(K \wedge \Omega^\infty(K \wedge \ldots \Omega^\infty(K \wedge X)\cdots)).$$

(We may want to use S-algebras to make the cosimplicial identities hold strictly.)

Steps of the proof

• We may assume that E is contractible.

• By Theorem 3, all spaces in the cosimplicial resolution $K^\bullet(B)$ are EM-convergent.

• By induction and Theorem 2, the spaces $\mathrm{Tot}^t K^\bullet(B)$ are also EM-convergent for all t.

• The cosimplicial spectrum

$$\{K \wedge \mathrm{Cobar}^s(K^t(B))\}_t \simeq \mathrm{const}(K \wedge \mathrm{Cobar}^s(B))$$

is contractible for all s by the contraction

$$K \wedge K(X) = K \wedge \Sigma^\infty \Omega^\infty (K \wedge X) \to K \wedge K \wedge X \to K \wedge X.$$

Thus, passing to total spaces with respect to s,

$$\mathrm{Tot}(K \wedge \mathrm{Cobar}^\bullet(B)) \simeq \{K \wedge F_t\}_t.$$

A $\lim^1$-consideration finishes the proof about convergence.

A final question

The previous corollary gives a satisfactory answer about Eilenberg-Moore convergence, yet it cannot state the target without the technical tool of completion towers.

It makes sense to ask when the target group

$$\pi_* \operatorname{holim}_t K \wedge F_t$$

coincides with the K-homology of the localized fibration. I do not know any counterexamples and in fact believe in the

Conjecture. *Any fibration of K-local spaces is EM-convergent. In particular, the EMSS for a fibration $E \longrightarrow B$ converges to the homology of the fiber of the localized fibration.*