# On the convergence of the Eilenberg-Moore spectral sequence

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#### **The Eilenberg-Moore spectral sequence**

Given a fibration  $F \to E \to B$  and a cohomology theory K, can we compute  $K^*(F)$  from a knowledge of  $K^*(B)$  (as an algebra) and  $K^*(E)$  (as a  $K^*(B)$ -algebra)?

**Theorem (Eilenberg, Moore, 1959).** If *B* is simply connected and  $K = H_*(-;\mathbf{F}_p)$  then there is a convergent spectral sequence

$$E_2^{**} = \operatorname{Tor}_{**}^{K^*(B)}(K^*(E), K_*) \Longrightarrow K^*(F)$$

• Problems occur when the fibration is not nilpotent, or when *K* is a nonconnective theory.

Simple connectivity cannot be omitted: take  $B = K(\mathbb{Z}/l, 1)$ with  $p \nmid l$ , and E contractible. Then  $\tilde{K}^*(B) = \tilde{K}^*(E) = 0$ , but  $\tilde{K}^*(F)$  is nontrivial.

# A general construction of the EMSS

The cobar construction of a map  $E \xrightarrow{p} B$  is the cosimplicial space

$$C^{\mathfrak{n}} = \operatorname{Cobar}^{n}(E \to B) = E \times B^{n}$$

with maps

$$E \stackrel{\operatorname{id} \times p}{\underset{\operatorname{id} \times *}{\rightrightarrows}} E \times B \stackrel{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots$$

The associated tower of total complexes has a very simple shape:

$$Tot^{0}(C^{\bullet}) = E$$
  
$$Tot^{i}(C^{\bullet}) = F \quad \text{for } i \ge 1.$$

If *K* is a spectrum then  $K \wedge C^{\bullet}$  is a cosimplicial spectrum with an associated total tower; however, the canonical map

$$\Phi\colon K\wedge \operatorname{Tot}^n(C^{\bullet})\to \operatorname{Tot}^n(K\wedge C^{\bullet})$$

is almost never an equivalence.

Unlike  $\operatorname{Tot}^{s}(C^{\bullet})$ , the tower  $\operatorname{Tot}^{s}(K \wedge C^{\bullet})$  is not eventually constant.

Thus  $K \wedge C^{\bullet}$  produces an interesting left half-plane spectral sequence, the Bousfield spectral sequence for a cosimplicial spectrum.

If *K* has Künneth isomorphisms, then its  $E^2$ -term is given by

$$E_{**}^2 = \operatorname{Cotor}_{K_*(B)}(K_*(E), K_*),$$

and its expected target is  $\pi_* \operatorname{Tot}(K \wedge C^{\bullet})$ .

If  $\Phi_*$ :  $K_n(\operatorname{Tot}^{s}(C^{\bullet})) \to \pi_n \operatorname{Tot}^{s}(K \wedge C^{\bullet})$  is a pro-isomorphism for each *n*, then

- $E_{st}^k$  becomes eventually constant for every *s*, *t*,
- $E_{s,t-s}^{\infty}$  is nontrivial for only finitely many *s*,
- (and therefore) the spectral sequence converges to K<sub>\*</sub>(F) in a strong (pro-constant) sense.

### Some non-convergence examples

Even simple connectedness is not enough to ensure convergence if K is nonconnective.

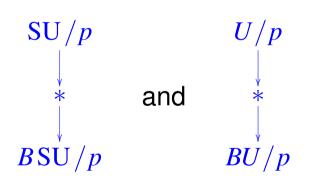
**Example.** Let K = K(n),  $B = K(\mathbb{Z}/p, n+1)$ .

Computations of Ravenel-Wilson:

$$K(n)_*(B) = 0$$
, but  $K(n)_*(\Omega B) \neq 0$ 

Thus the K(n)-based EMSS cannot possibly converge for the path-loop fibration on *B*.

**Example.** Consider the two path-loop fibrations



Since  $(K/p)_*(BSU/p) \cong (K/p)_*(BU/p)$ , the spectral sequences are isomorphic.

But  $(K/p)_*(SU/p) \ncong (K/p)_*(U/p)$ .

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## **Partial convergence results**

The convergence for  $K = H_*(-; \mathbf{F}_p)$  is quite well-understood for all fibrations:

**Theorem (Dwyer).** Let  $F \to E \xrightarrow{\pi} B$  be a fibration with *F* and *E* connected. The  $H_*(-;\mathbf{F}_p)$ -based EMSS for a fibration  $E \to B$  converges pro-constantly to the homology of the fiber on *p*-completion towers.

This means: If

- {*R<sub>s</sub>X*}<sub>s</sub> denotes the *p*-completion tower of a space
  *X* and
- $F_s$  denotes the fiber of  $R_s E \xrightarrow{R_s \pi} R_s B$

then the EMSS for  $\pi$  converges pro-constantly to the total space of the pro-constant tower  $H_*(F_s; \mathbf{F}_p)$ .

• This generalizes to connective theories.

#### Two convergence results for K = K(n)

**Theorem (Tamaki, 1994).** The K(n)-based EMSS for the path-loop fibration on  $\Omega^{n-1}\Sigma^n X$  converges strongly to  $K(n)_*(\Omega^n\Sigma^n X)$  for any space X and any  $n \ge 1$ .

• In particular, the path-loop EMSS converges on suspensions (in fact, it collapses at  $E^2$ ).

**Theorem (Jeanneret-Osse, 1999).** The K(n)-based cohomological EMSS for  $X \rightarrow B$  converges whenever  $K^*(\Omega B)$  is an exterior algebra on finitely many generators.

• Useful for classifying spaces of finite loop spaces.

# **Complete convergence**

The notion of pro-constant convergence is too restrictive for the EMSS for nonconnective theories.

**Definition.** For  $C^{\bullet} = \operatorname{Cobar}^{\bullet}(E \to B)$ , let

- $T^s = \operatorname{Tot}^s(K \wedge C^{\bullet});$
- $F^{s}K_{*}(F) = \ker\left(K_{*}(F) \xrightarrow{\Phi} \pi_{*}T^{s}\right)$

The EMSS is called

- weakly convergent to  $K_*F$  if  $\frac{F^sK_*F}{F^{s+1}K_*F} \rightarrow E_{\infty}^{s,*}$  is iso;
- *strongly convergent* if in addition,

 $\lim^{\varepsilon} F^{s} K_{*}(F)$  for  $\varepsilon = 0, 1;$ 

• completely convergent if also  $\lim^{1} \pi_{*}T^{s} = 0$ .

In a completely convergent spectral sequence, infinitely many differentials at (s,t) and infinite filtrations are possible, but in a controlled way.

### New convergence results

From now on, let  $K_*$  be a graded field (e.g. K = K(n)).

Call a fibration *EM-convergent* if its *K*-based EMSS is completely convergent to  $K_*(F)$ .

**Theorem 1.** Let *B* be a space such that the path-loop fibration on *B* is EM-convergent. Then so is any fibration  $E \rightarrow B$ .

Hodgkin proved a cohomological version of this result under the assumption that  $K^*(B)$  has finite global dimension.

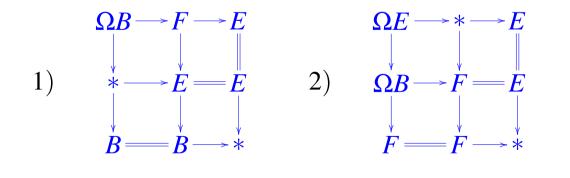
**Theorem 2.** Let  $X \rightarrow B_i$  (i = 1, 2) be two EM-convergent fibrations with fibers  $F_i$ . Then

 $F_1 \rightarrow B_2$  is EM-convergent iff  $F_2 \rightarrow B_1$  is EM-convergent.

The fiber, in both cases, is the fiber of  $X \rightarrow B_1 \times B_2$ .

**Corollary.** Let  $F \rightarrow E \rightarrow B$  be a fibration such that the path-loop fibrations on F and on B are EM-convergent. Then so is the path-loop fibration (and hence any fibration) on E.

Proof.



**Theorem 3.** Any fibration over a zeroth space of a *K*-module spectrum is EM-convergent.

Idea of the proof:

By Theorem 1, it is enough to consider the path-loop fibration.

The spectral sequence splits as an (infinite) tensor product of path-loop spectral sequences of  $\underline{K}_k$ , the *k*th space of the spectrum *K*.

 $K_*(\underline{K}_*)$  is a Hopf ring, computed by Wilson (1984):

As a  $\mathbb{Z}/(2p^n - 2)$ -bigraded ring,  $K(n)_* \underbrace{K(n)}_{j_0 < p^n - 1} \cong \bigotimes_{\substack{i_0 = 0 \text{ or } j_0 < p^n - 1}} \wedge (a^I b^J e_1) \otimes \bigotimes_{i_0 = 0 \text{ or } j_0 < p^n - 1}} P_{t_1(I) + 1} (a^I b^J)$ where  $k \in \mathbb{Z}$ ,  $I = (i_0, \dots, i_{n-1})$  with  $i_k = 0$  or  $1, j_k < p^n$ .

From this knowledge, the EMSS can be computed explicitly. For K = K(n), it collapses at  $E_{**}^{p^{n+1}}$  and is completely convergent.

# A generalization of Dwyer's convergence

Theorems 2 and 3 allow an inductive argument to show:

**Corollary.** Let  $F \rightarrow E \rightarrow B$  be an arbitrary fibration. Let *K* be a field spectrum. Then the  $K_*$ -based EMSS converges completely to  $K_*$  of the fiber on *K*-completion towers.

The *K*-completion tower (Bendersky-Thompson) is the analog of the p-completion tower: it is associated to the cosimplicial space

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K^{\bullet}(X) = \Omega^{\infty}(K \wedge \Omega^{\infty}(K \wedge \ldots \Omega^{\infty}(K \wedge X) \cdots)).
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(We may want to use S-algebras to make the cosimplicial identities hold strictly.)

# **Steps of the proof**

- We may assume that E is contractible.
- By Theorem 3, all spaces in the cosimplicial resolution  $K^{\bullet}(B)$  are EM-convergent.
- By induction and Theorem 2, the spaces  $\operatorname{Tot}^{t} K^{\bullet}(B)$  are also EM-convergent for all *t*.
- The cosimplicial spectrum

 $\{K \wedge \operatorname{Cobar}^{s}(K^{t}(B))\}_{t} \simeq \operatorname{const}(K \wedge \operatorname{Cobar}^{s}(B))$ 

is contractible for all s by the contraction

 $K \wedge K(X) = K \wedge \Sigma^{\infty} \Omega^{\infty}(K \wedge X) \to K \wedge K \wedge X \to K \wedge X.$ 

Thus, passing to total spaces with respect to s,

 $\operatorname{Tot}(K \wedge \operatorname{Cobar}^{\bullet}(B)) \simeq \{K \wedge F_t\}_t.$ 

A lim<sup>1</sup>-consideration finishes the proof about convergence.

# A final question

The previous corollary gives a satisfactory answer about Eilenberg-Moore convergence, yet it cannot state the target without the technical tool of completion towers.

It makes sense to ask when the target group

#### $\pi_* \operatorname{holim}_t K \wedge F_t$

coincides with the K-homology of the localized fibration. I do not know any counterexamples and in fact believe in the

**Conjecture.** Any fibration of *K*-local spaces is *EM*-convergent. In particular, the EMSS for a fibration  $E \rightarrow B$  converges to the homology of the fiber of the localized fibration.