On lower bounds of triple point numbers for 5-colourable 2-knots

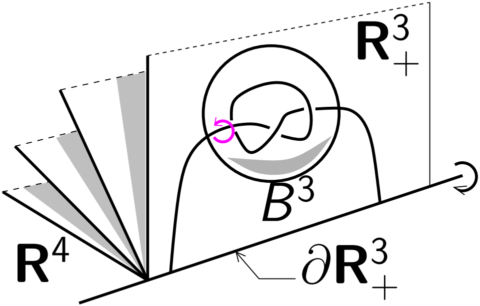
Tsukasa Yashiro (Joint work with Abdul Mohamad)

Sultan Qaboos University

The Fourth East Asian School of Knots and Related Topics
The University of Tokyo
23 January 2008

Zeeman's twist spinning

Let B^3 be a 3–ball in $\mathbf{R}^3_+$ such that $\partial B^3 \cap T(K)$ is the pair of antipodal points of ∂B^3.
An m-**twist-spun knot** obtained from K is defined by rotating the tangle $B^3 \cap T(K)$ about the axis through the antipodal points m times while $\mathbf{R}^3_+$ spins. We denote this 2-knot by $T_m(K)$.



Theorem (Zeeman, 1965)

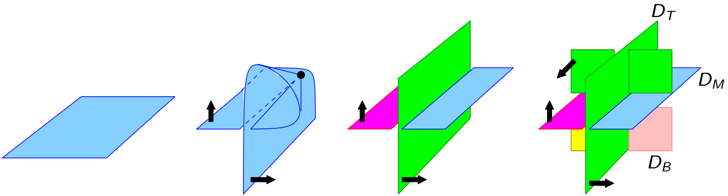
Every m-twist spun knot $T_m(K)$ obtained from K is fibred ($m \geq 1$); the fibre is the one-punctured k-fold branched covering space of S^3 along K.

Corollary (Zeeman, 1965)

For any knot K, 1-twist spun knot obtained from K is trivial.

Surface Diagrams

A **surface-knot** is a connected, oriented, closed surface smoothly embedded in $\mathbf{R}^4$ up to ambient isotopy. If $F \cong S^2$, then F is called a **2-knot**. Let $F \subset \mathbf{R}^4$ be a surface-knot. Let $\pi : \mathbf{R}^4 \to \mathbf{R}^3$; $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$, be the orthogonal projection. A surface diagram of F is a union of the following local diagrams.

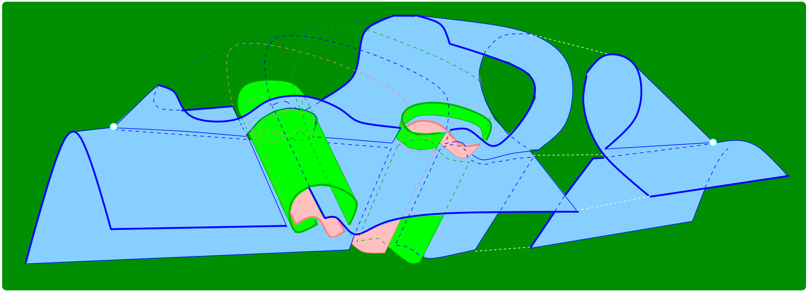


Branch point

Triple point

Twist Spun Trefoil

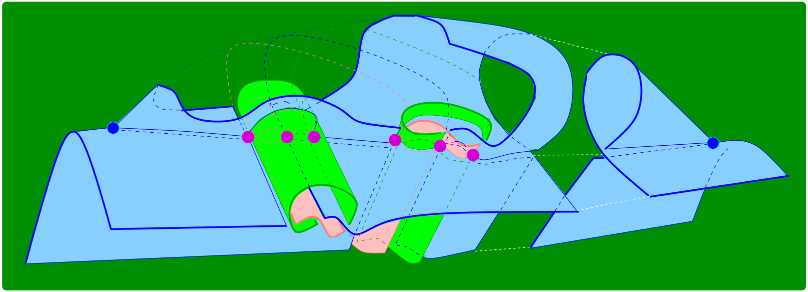
The following is one-twisting part of a surface diagram of the twist spun trefoil.



In this diagram there are **six** triple points and **two** branch points. We can reduce the number of triple points into two by isotopy deformations (S. Satoh).

Twist Spun Trefoil

The following is one-twisting part of a surface diagram of the twist spun trefoil.



In this diagram there are **six** triple points and **two** branch points. We can reduce the number of triple points into two by isotopy deformations (S. Satoh).

Triple point numbers

For a surface-knot F, the minimal number of triple points for all possible surface diagrams is called the **triple point number** of F denoted by $t(F)$. A surface diagram D_F of F with $t(F)$ triple points is called a t-**minimal surface diagram**.

Theorem (S. Satoh 2005)

For every 2-knot F with $t(F) \neq 0$,

$$4 \leq t(F).$$

Theorem (S. Satoh and A. Shima 2002, 2004)

Let K be a trefoil knot. Let $T_m(K)$ be m-twist-spinning of K. Then the following holds.

$$t(T_2(K)) = 4, \ t(T_3(K)) = 6.$$

Triple point numbers

For a surface-knot F, the minimal number of triple points for all possible surface diagrams is called the **triple point number** of F denoted by $t(F)$. A surface diagram D_F of F with $t(F)$ triple points is called a t-**minimal surface diagram**.

Theorem (S. Satoh 2005)

For every 2-knot F with $t(F) \neq 0$,

$$4 \leq t(F).$$

Theorem (S. Satoh and A. Shima 2002, 2004)

Let K be a trefoil knot. Let $T_m(K)$ be m-twist-spinning of K. Then the following holds.

$$t(T_2(K)) = 4, \ t(T_3(K)) = 6.$$

Triple point numbers

For a surface-knot F, the minimal number of triple points for all possible surface diagrams is called the **triple point number** of F denoted by $t(F)$. A surface diagram D_F of F with $t(F)$ triple points is called a t-**minimal surface diagram**.

Theorem (S. Satoh 2005)

For every 2-knot F with $t(F) \neq 0$,

$$4 \leq t(F).$$

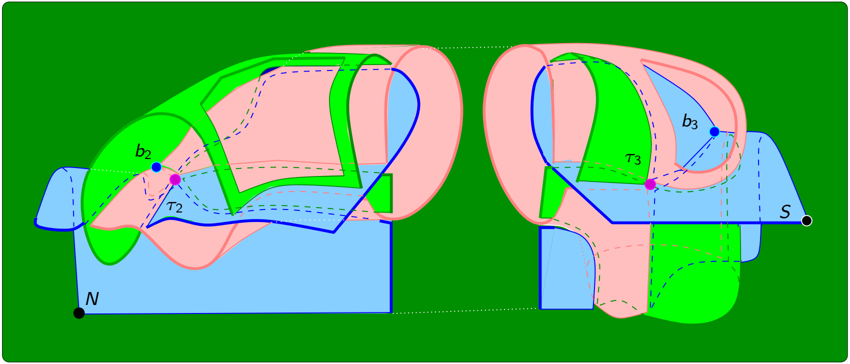
Theorem (S. Satoh and A. Shima 2002, 2004)

Let K be a trefoil knot. Let $T_m(K)$ be m-twist-spinning of K. Then the following holds.

$$t(T_2(K)) = 4, \ t(T_3(K)) = 6.$$

Twist Spun Trefoil (Reduced diagram)

This is a piece of reduced diagram of twist spun trefoil.



This partial diagram has **two** triple points and **two** branch points.

Facts

Theorem (T. Y. 2005)

Let K be the $(2, k)$-torus knot. Then the following holds.

$$t(T_m(K)) \leq m(k-1), (m > 1).$$

Theorem (E. Hatakenaka (2004))

For a 2-twist spun $(2, 5)$-torus knot F, $6 \leq t(F)$.

$$6 \leq t(F) \leq 8.$$

Facts

Theorem (T. Y. 2005)

Let K be the $(2,k)$-torus knot. Then the following holds.

$$t(T_m(K)) \leq m(k-1), (m>1).$$

Theorem (E. Hatakenaka (2004))

For a 2-twist spun $(2,5)$-torus knot F, $6 \leq t(F)$.

$$6 \leq t(F) \leq 8.$$

Facts

Theorem (T. Y. 2005)

Let K be the $(2, k)$-torus knot. Then the following holds.

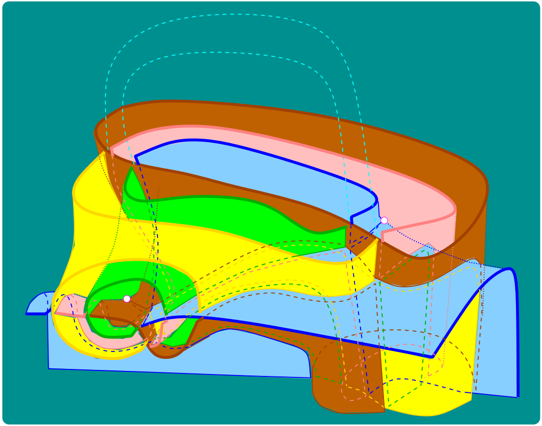
$$t(T_m(K)) \leq m(k-1), (m > 1).$$

Theorem (E. Hatakenaka (2004))

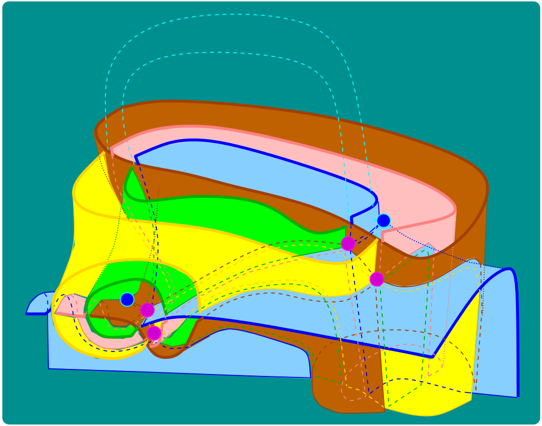
For a 2-twist spun $(2, 5)$-torus knot F, $6 \leq t(F)$.

$$6 \leq t(F) \leq 8.$$

Half-piece of 2-twist Spun $(2,5)$-torus Knot Diagram

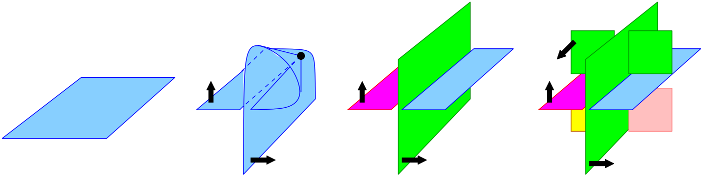


Half-piece of 2-twist Spun $(2, 5)$-torus Knot Diagram



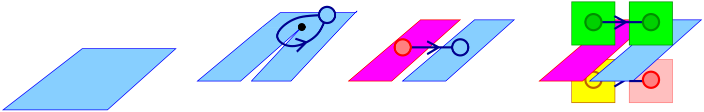
Pre-images of Multiple Points

To investigate triple points in a surface diagram we look at the pre-image of the surface diagram:



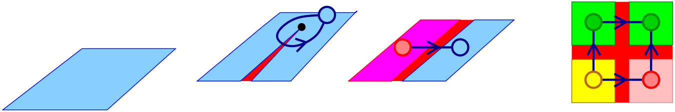
Pre-images of Multiple Points

To investigate triple points in a surface diagram we look at the pre-image of the surface diagram:



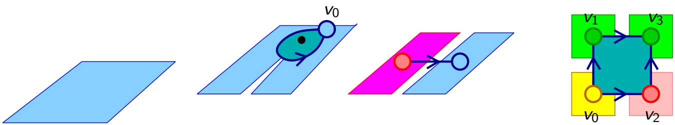
Pre-images of Multiple Points

To investigate triple points in a surface diagram we look at the pre-image of the surface diagram:



Pre-images of Multiple Points

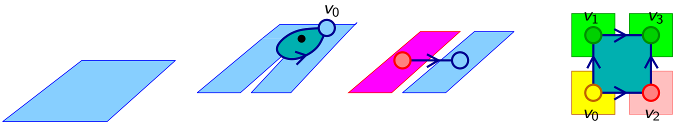
To investigate triple points in a surface diagram we look at the pre-image of the surface diagram:



Cells induced from a branch point and a triple point, are called a **loop disc** and a **rectangle** respectively.

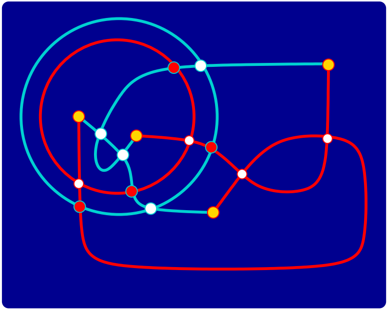
Pre-images of Multiple Points

To investigate triple points in a surface diagram we look at the pre-image of the surface diagram:



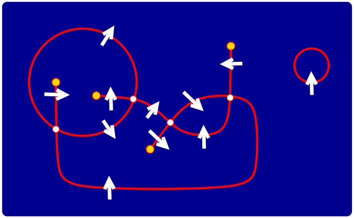
We denote the rectangle in the figure by $(v_0; v_0v_1, v_0v_2; v_3)$.

Pre-images of Multiple Points



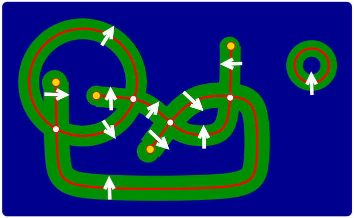
The closure of the pre-image of double curves in D_F is a union of two families of arcs called the **double decker set**. The union of blue arcs is the **upper decker set** and the union of red arcs is the **lower decker set**.

Pre-images of Multiple Points



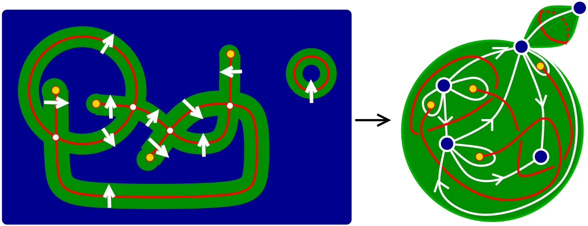
We denote the lower decker set by S_b. $F \setminus S_b = \{R_0, \ldots, R_n\}$. Let $N(S_b)$ be a small neighbourhood of S_b in F.
$F \setminus N(S_b) = \{V_0, \ldots, V_n\}$; $V_i \subset R_i$ $(i = 0, \ldots, n)$.
The quotient map $q : F \to F/_\sim$ is defined by $q(V_i) = v_i$, $(i = 0, \ldots, n)$. The quotient space has a cell-complex structure. We will denote the cell-complex by K_{D_F}.
A subcomplex of K_{D_F} induced from a simple closed curve in S_b is called a **bubble**.

Pre-images of Multiple Points



We denote the lower decker set by S_b. $F \setminus S_b = \{R_0, \ldots, R_n\}$. Let $N(S_b)$ be a small neighbourhood of S_b in F.
$F \setminus N(S_b) = \{V_0, \ldots, V_n\}$; $V_i \subset R_i$ $(i = 0, \ldots, n)$.
The quotient map $q : F \to F/_\sim$ is defined by $q(V_i) = v_i$, $(i = 0, \ldots, n)$. The quotient space has a cell-complex structure.
We will denote the cell-complex by K_{D_F}.
A subcomplex of K_{D_F} induced from a simple closed curve in S_b is called a **bubble**.

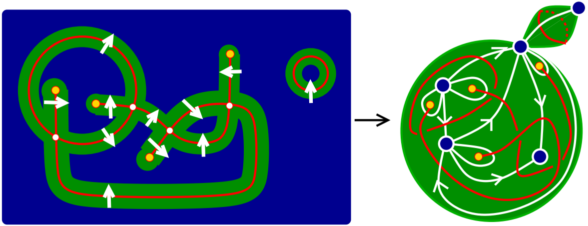
Pre-images of Multiple Points



The quotient map $q : F \to F/_\sim$ is defined by $q(V_i) = v_i$, $(i = 0, \ldots, n)$. The quotient space has a cell-complex structure. We will denote the cell-complex by K_{D_F}.

A subcomplex of K_{D_F} induced from a simple closed curve in S_b is called a **bubble**.

Pre-images of Multiple Points



The quotient map $q : F \to F/_{\sim}$ is defined by $q(V_i) = v_i$, $(i = 0, \ldots, n)$. The quotient space has a cell-complex structure. We will denote the cell-complex by K_{D_F}.

A subcomplex of K_{D_F} induced from a simple closed curve in S_b is called a **bubble**.

Parcels

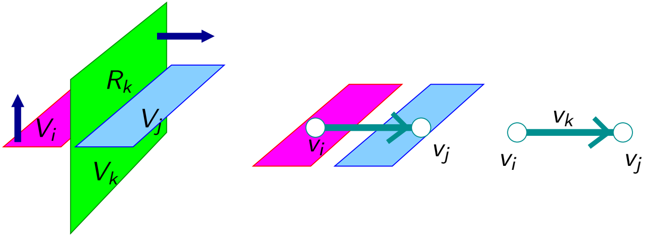
A **parcel** of K_{D_F} is a subcomplex K without free edges of K_{D_F} such that for any finite set S of vertices of K_{D_F}, $|K| \setminus |S|$ is connected. Let F be a surface-knot and let D_F be a t-minimal surface diagram. Then there is a cell-complex K_{D_F} such that K_{D_F} can be decomposed into parcels $K_1, \ldots, K_n$ such that

$$\begin{aligned} K_{D_F} &= K_1 + \cdots + K_n, \\ &= R_{D_F} + B_{D_F}. \end{aligned}$$

where R_{D_F} is the union of rectangles and B_{D_F} be the union of bubbles.

Weights

We add a weight on an edge as follows.



The edge is denoted by $v_i v_j$ and the weight is v_k. We write $w(v_i v_j) = v_k$.

Quandle Colourings

A **quandle** X is a non-empty set with a binary operation $(a, b) \mapsto a * b$ such that

1. For any $a \in X$, $a * a = a$,
2. For any $a, b \in X$, there is a unique $c \in X$ such that $c * b = a$.
3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

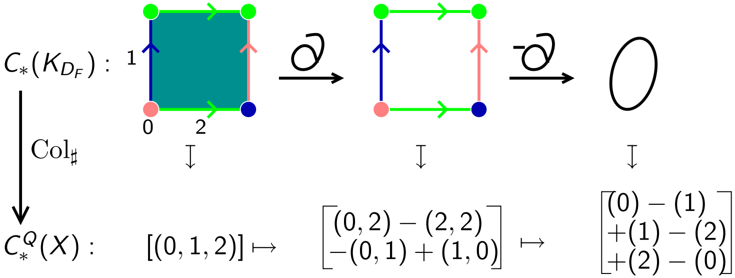
Let $\mathcal{V}$ and $\mathcal{E}$ be the set of vertices and edges in K_{D_F}. A colouring of K_{D_F} is a map

$$\mathrm{Col} : \mathcal{V} \cup \mathcal{E} \to X$$

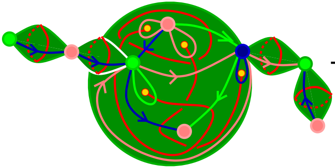
defined by $\mathrm{Col}(v) = w(v)$ for $v \in \mathcal{V}$ and $\mathrm{Col}(e) = w(e)$ for $e \in \mathcal{E}$.

Quandle Chain Complex

Let $C_n(X)$ $(n \geq 1)$ be a free abelian group generated by n-tuples $(x_1, \dots, x_n) \in X^n$. Let $C_n^D(X)$ be a sub group of $C_n(X)$ generated by $(x_1, \dots, x_n)$ such that $x_i = x_j$ for some $1 \leq i, j, \leq n$ and $(|i-j| = 1)$. We denote the quotient group $C_n(X)/C_n^D(X)$ by $C_n^Q(X)$.



We define a chain group $C_2(K_{D_F})$ of K_{D_F}. A homomorphism $\mathrm{Col}_\sharp : C_2(K_{D_F}) \to C_3^Q(X)$ is induced from the colouring of D_F.



$\xrightarrow{\mathrm{Col}_\sharp} C_3^Q(X)$

Theorem (T.Y.)

Let F be a 2-knot. Suppose that F is coloured by a finite quandle X. Then the following holds.

$$4\mu(F) \leq t(F). \tag{1}$$

If K_{D_F} has the maximal rank of $H_2(|R_{D_F}|;\mathbb{Z})$ for all t-minimal surface diagrams, K_{D_F} is said to be **maximal**. Suppose that K_{D_F} is maximal. We view $K_i \subset R_{D_F}, (i = 1, \dots, r \leq n)$ as chains in $C_2(K_{D_F})$.

$$\mu(F) = \min_{D_F} \#\{K_i \subset R_{D_F} | \ \mathrm{Col}_\sharp(K_i) \neq 0, K_{D_F} \text{ is maximal}\}. \quad (1)$$

Theorem (T.Y.)

Let F be a 2-knot. Suppose that F is coloured by a finite quandle X. Then the following holds.

$$4\mu(F) \leq t(F). \quad (2)$$

If K_{D_F} has the maximal rank of $H_2(|R_{D_F}|;\mathbb{Z})$ for all t-minimal surface diagrams, K_{D_F} is said to be **maximal**. Suppose that K_{D_F} is maximal. We view $K_i \subset R_{D_F}, (i = 1, \ldots, r \leq n)$ as chains in $C_2(K_{D_F})$.

$$\mu(F) = \min_{D_F} \#\{K_i \subset R_{D_F} | \ \mathrm{Col}_\sharp(K_i) \neq 0, K_{D_F} \text{ is maximal}\}. \quad (1)$$

Theorem (T.Y.)

Let F be a 2-knot. Suppose that F is coloured by a finite quandle X. Then the following holds.

$$4\mu(F) \leq t(F). \quad (2)$$

The dihedral quandle $(X, *)$ of order $p > 0$ (p is prime) is a quandle $X = \{0, \ldots, p-1\}$ with the binary operation $(i, j) \mapsto 2j - i \pmod{p}$.

Theorem (M-Y)

Let F be a 2-knot and let D_F be a t-minimal surface diagram. Suppose that F is non-trivially coloured by R_5 and that $\mu(F) = 1$. Suppose that $B_{D_F} = \emptyset$ and there are no cancelling pair of triple points also D_F represents a non-trivial quandle homology class in $H_3^Q(X)$. Then

$$7 \leq \mathrm{t}(F). \tag{3}$$

The dihedral quandle $(X, *)$ of order $p > 0$ (p is prime) is a quandle $X = \{0, \ldots, p-1\}$ with the binary operation $(i, j) \mapsto 2j - i \ (\mathrm{mod} p)$.

Theorem (M-Y)

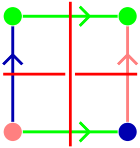
Let F be a 2-knot and let D_F be a t-minimal surface diagram. Suppose that F is non-trivially coloured by R_5 and that $\mu(F) = 1$. Suppose that $B_{D_F} = \emptyset$ and there are no cancelling pair of triple points also D_F represents a non-trivial quandle homology class in $H_3^Q(X)$. Then

$$7 \leq \mathrm{t}(F). \qquad (3)$$

Finding Cycles

Edge Condition

Suppose that $K_{D_F} = R_{D_F}$. For a non-degenerate triple (p, q, r) of a 3-cycle, (q, r) represents a non-degenerate edge e.



$(0, 1, 2)$

$(1, 2)$

$(0, 1, 2)$ induces the edge $(1, 2)$. The edge $(1, 2)$ must exists in K_{D_F}.

Finding Cycles

Cycles with length $n < 7$

There is no cycles with length $n < 7$ satisfying conditions of Theorem [M-Y].

To find cycles with the length n.

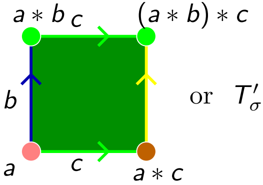
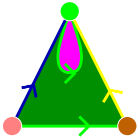
(1) Generate a list of non-degenerate R_5-3-simplexes.
(2) Generate a list of boundaries of quandle 3-simplexes obtained in (1).
(3) Produce a 2-chain from n of boundaries from the list obtained in (2).
(4) Check whether or not the 2-chain is zero and it satisfies the edge-condition.

n triple points ($n \leq 5$)

Triangle condition

Let $X = R_5$. Let $\sigma = (a, b, c) \in X^3$ be a non-degenerate quandle 3-simplex. T_σ is a triangle if $2b - a - c \equiv 0$ or $a \equiv c \pmod{5}$,

A rectangle $T_\sigma = (v_0; v_0v_1, v_0v_2; v_3)$ induced from a triple $(a, b, c) \in X^3$ is a rectangle with $\mathrm{Col}(v_0) = a$, $\mathrm{Col}(v_0v_1) = b$, and $\mathrm{Col}(v_0v_2) = c$. $\mathrm{Col}(v_1) = a * b$, $\mathrm{Col}(v_1v_3) = c$, $\mathrm{Col}(v_2) = a * c$, $\mathrm{Col}(v_2v_3) = b * c$, $\mathrm{Col}(v_3) = (a * b) * c$.

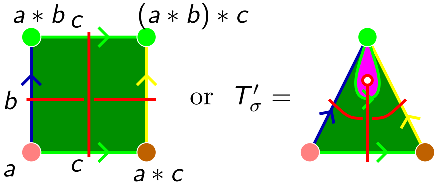
$\sigma = (a, b, c) \mapsto T_\sigma =$  or $T'_\sigma =$ 

n triple points ($n \leq 5$)

Triangle condition

Let $X = R_5$. Let $\sigma = (a, b, c) \in X^3$ be a non-degenerate quandle 3-simplex. T_σ is a triangle if $2b - a - c \equiv 0$ or $a \equiv c \pmod 5$,

A rectangle $T_\sigma = (v_0; v_0v_1, v_0v_2; v_3)$ induced from a triple $(a, b, c) \in X^3$ is a rectangle with $\mathrm{Col}(v_0) = a$, $\mathrm{Col}(v_0v_1) = b$, and $\mathrm{Col}(v_0v_2) = c$. $\mathrm{Col}(v_1) = a * b$, $\mathrm{Col}(v_1v_3) = c$, $\mathrm{Col}(v_2) = a * c$, $\mathrm{Col}(v_2v_3) = b * c$, $\mathrm{Col}(v_3) = (a * b) * c$.

$\sigma = (a, b, c) \mapsto T_\sigma =$  or $T'_\sigma =$

n triple points ($n \leq 5$)

Enumerating cycles of four quandle 3-simplexes, our programs give 60 cycles. 12 of them are shown in the table below.

	Cycles with length 4	2b-a-c
1	1(0, 1, 3)-1(0, 2, 3) 1(1, 0, 3)-1(1, 4, 3)	4, 1, 1, 4
2	1(0, 1, 3) 1(1, 0, 3) 1(2, 1, 3) 1(4, 0, 3)	4, 1, 2, 3
3	1(0, 1, 3) 1(1, 0, 3) 1(2, 4, 3) 1(4, 2, 3)	4, 1, 3, 2
4	-1(0, 1, 3) 1(0, 2, 3)-1(1, 0, 3) 1(1, 4, 3)	4, 1, 1, 4
5	-1(0, 1, 3)-1(1, 0, 3)-1(2, 1, 3)-1(4, 0, 3)	4, 1, 2, 3
6	-1(0, 1, 3)-1(1, 0, 3)-1(2, 4, 3)-1(4, 2, 3)	4, 1, 3, 2
7	1(0, 1, 4)-1(0, 3, 4)-1(3, 0, 4) 1(3, 2, 4)	3, 2, 3, 2
8	1(0, 1, 4) 1(1, 3, 4) 1(2, 0, 4) 1(3, 2, 4)	3, 2, 4, 2
9	-1(0, 1, 4) 1(0, 3, 4) 1(3, 0, 4)-1(3, 2, 4)	3, 2, 3, 2
10	-1(0, 1, 4)-1(1, 3, 4)-1(2, 0, 4)-1(3, 2, 4)	3, 2, 4, 2
11	1(0, 2, 1) 1(2, 0, 1) 1(3, 0, 1) 1(4, 2, 1)	3, 2, 1, 4
12	-1(0, 2, 1)-1(2, 0, 1)-1(3, 0, 1)-1(4, 2, 1)	3, 2, 1, 4

n triple points ($n \leq 5$)

	Cycles with length 4	2b-a-c
1	1(0, 1, 3)-1(0, 2, 3) 1(1,0, 3)-1(1, 4, 3)	4, 1, 1, 4
2	1(0, 1, 3) 1(1, 0, 3) 1(2, 1, 3) 1(4, 0, 3)	4, 1, 2, 3
⋮	⋮	⋮

The third column shows sequences of non-zero numbers $2b - a - c \pmod 5$. Thus any simplex σ in each cycle is non-degenerate. This implies that K_{D_F} contains no branch points. This contradicts the t-minimality of D_F.

Six triple points

Enumerating cycles with length six, our programs give essentially 36 3-chains.

	Type A	∂**(4-chain)**
1	(0, 1, 0)-(0, 2, 0)-(0, 3, 0) +(0, 4, 0) -(1, 2, 0)-(4, 3, 0)	$=\partial$(-(0, 1, 2, 0)-(0, 4, 3, 0))
2	(0, 1, 0)-(0, 2, 0)-(0, 3, 0) +(0, 4, 0)-(1, 4, 0)-(4, 1, 0)	$=\partial$((0, 3, 4, 0)+(0, 2, 1, 0))
⋮	⋮	⋮
32	-(0, 1, 2)-(1, 3, 2)-(2, 0, 2) -(2, 4, 2)-(3, 1, 2)-(4, 3, 2)	$=\partial$(-(0, 3, 1, 2)-(0, 4, 3, 2) -(1, 2, 3, 2) -(4, 1, 3, 2) -(4, 0, 1, 2)-(3, 2, 1, 2))

Let K_{D_F} be a cell-complex for a coloured D_F. K'_{D_F} is defined by replacing $T_\sigma \in R_{D_F}$ with T'_σ, where $\sigma = (a, b, c) \in X^3$ is the colour triple of the rectangle.

Lemma

Let K'_{D_F} be the deformed cell-complex from K_{D_F} by the above deformation. Then

$$\beta_1(|K_{D_F}|) \geq \beta_1(|K'_{D_F}|), \tag{4}$$

where β_1 is the first Betti number.

Type B

Let c be a cycle of Type B. Then K_c is obtained. When we construct $|K_c|$, $|K_c|$ is homeomorphic to a torus. This implies that $\beta_1(|K_c|) = 2$. This contradicts Lemma 1.

	Type B
1	(0, 1, 0) 1(0, 4, 3)+(1, 0, 3)+(1, 2, 0)-(2, 0, 4)-(3, 0, 4)
2	-(0, 1, 0)-(0, 4, 3)-(1, 0, 3)-(1, 2, 0)+ (2, 0, 4)+(3, 0, 4)
3	(0, 1, 3)-(1, 3, 0)-(1, 4, 1)+(2, 1, 3)-(4, 1, 0)-(4, 2, 1)
4	-(0, 1, 3)+ (1, 3, 0)+(1, 4, 1)-(2, 1, 3)+(4, 1, 0)+(4, 2, 1)