On lower bounds of triple point numbers for 5-colourable 2-knots

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Let $B^3$ be a 3–ball in $\mathbb{R}^3_+$ such that $\partial B^3 \cap T(K)$ is the pair of antipodal points of $\partial B^3$.

An \textit{m-twist-spun knot} obtained from $K$ is defined by rotating the tangle $B^3 \cap T(K)$ about the axis through the antipodal points $m$ times while $\mathbb{R}^3_+$ spins. We denote this 2-knot by $T_m(K)$. 
Theorem (Zeeman, 1965)

Every $m$-twist spun knot $T_m(K)$ obtained from $K$ is fibred ($m \geq 1$); the fibre is the one-punctured $k$-fold branched covering space of $S^3$ along $K$.

Corollary (Zeeman, 1965)

For any knot $K$, 1-twist spun knot obtained from $K$ is trivial.
A **surface-knot** is a connected, oriented, closed surface smoothly embedded in $\mathbb{R}^4$ up to ambient isotopy. If $F \cong S^2$, then $F$ is called a **2-knot**. Let $F \subset \mathbb{R}^4$ be a surface-knot. Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$; 
$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$, be the orthogonal projection. A surface diagram of $F$ is a union of the following local diagrams.
Twist Spun Trefoil

The following is one-twisting part of a surface diagram of the twist spun trefoil.

In this diagram there are six triple points and two branch points. We can reduce the number of triple points into two by isotopy deformations (S. Satoh).
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Triple point numbers

For a surface-knot $F$, the minimal number of triple points for all possible surface diagrams is called the **triple point number** of $F$ denoted by $t(F)$. A surface diagram $D_F$ of $F$ with $t(F)$ triple points is called a **$t$-minimal surface diagram**.

**Theorem (S. Satoh 2005)**

For every 2-knot $F$ with $t(F) \neq 0$,

$$4 \leq t(F).$$

**Theorem (S. Satoh and A. Shima 2002, 2004)**

Let $K$ be a trefoil knot. Let $T_m(K)$ be $m$-twist-spinning of $K$. Then the following holds.

$$t(T_2(K)) = 4, \quad t(T_3(K)) = 6.$$
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$$t(T_2(K)) = 4, \quad t(T_3(K)) = 6.$$
This is a piece of reduced diagram of twist spun trefoil.

This partial diagram has two triple points and two branch points.
Facts

**Theorem (T. Y. 2005)**

Let $K$ be the $(2, k)$-torus knot. Then the following holds.

$$t(T_m(K)) \leq m(k - 1), (m > 1).$$

**Theorem (E. Hatakenaka (2004))**

For a 2-twist spun $(2, 5)$-torus knot $F$, $6 \leq t(F)$.

$$6 \leq t(F) \leq 8.$$
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Cells induced from a branch point and a triple point, are called a **loop disc** and a **rectangle** respectively.
To investigate triple points in a surface diagram we look at the pre-image of the surface diagram:

We denote the rectangle in the figure by \((v_0; v_0v_1, v_0v_2; v_3)\).
The closure of the pre-image of double curves in $D_F$ is a union of two families of arcs called the **double decker set**. The union of blue arcs is the **upper decker set** and the union of red arcs is the **lower decker set**.
Pre-images of Multiple Points

We denote the lower decker set by $S_b$. $F \setminus S_b = \{R_0, \ldots, R_n\}$. Let $N(S_b)$ be a small neighbourhood of $S_b$ in $F$. $F \setminus N(S_b) = \{V_0, \ldots, V_n\}$; $V_i \subset R_i$ ($i = 0, \ldots, n$). The quotient map $q : F \rightarrow F/\sim$ is defined by $q(V_i) = v_i$, ($i = 0, \ldots, n$). The quotient space has a cell-complex structure. We will denote the cell-complex by $K_{DF}$.

A subcomplex of $K_{DF}$ induced from a simple closed curve in $S_b$ is called a bubble.
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A subcomplex of $K_{DF}$ induced from a simple closed curve in $S_b$ is called a **bubble**.
A **parcel** of $K_{DF}$ is a subcomplex $K$ without free edges of $K_{DF}$ such that for any finite set $S$ of vertices of $K_{DF}$, $|K| \setminus |S|$ is connected. Let $F$ be a surface-knot and let $D_F$ be a $t$-minimal surface diagram. Then there is a cell-complex $K_{DF}$ such that $K_{DF}$ can be decomposed into parcels $K_1, \ldots, K_n$ such that

$$K_{DF} = K_1 + \cdots + K_n,$$

$$= R_{DF} + B_{DF}.$$

where $R_{DF}$ is the union of rectangles and $B_{DF}$ be the union of bubbles.
We add a weight on an edge as follows.

The edge is denoted by $v_i v_j$ and the weight is $v_k$. We write $w(v_i v_j) = v_k$. 

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A quandle $X$ is a non-empty set with a binary operation $(a, b) \mapsto a \ast b$ such that

1. For any $a \in X$, $a \ast a = a$,
2. For any $a, b \in X$, there is a unique $c \in X$ such that $c \ast b = a$.
3. For any $a, b, c \in X$, $(a \ast b) \ast c = (a \ast c) \ast (b \ast c)$.

Let $\mathcal{V}$ and $\mathcal{E}$ be the set of vertices and edges in $K_{DF}$. A colouring of $K_{DF}$ is a map

$$\text{Col}: \mathcal{V} \cup \mathcal{E} \rightarrow X$$

defined by $\text{Col}(v) = w(v)$ for $v \in \mathcal{V}$ and $\text{Col}(e) = w(e)$ for $e \in \mathcal{E}$. 
Let $C_n(X)$ ($n \geq 1$) be a free abelian group generated by $n$-tuples $(x_1, \ldots, x_n) \in X^n$. Let $C_n^D(X)$ be a sub group of $C_n(X)$ generated by $(x_1, \ldots, x_n)$ such that $x_i = x_j$ for some $1 \leq i, j, \leq n$ and $(|i - j| = 1)$. We denote the quotient group $C_n(X)/C_n^D(X)$ by $C_n^Q(X)$.
We define a chain group $C_2(K_{DF})$ of $K_{DF}$. A homomorphism $\text{Col}_\#: C_2(K_{DF}) \to C_3^Q(X)$ is induced from the colouring of $D_F$.

**Theorem (T.Y.)**

Let $F$ be a 2-knot. Suppose that $F$ is coloured by a finite quandle $X$. Then the following holds.

$$4\mu(F) \leq t(F). \quad (1)$$
If $K_{DF}$ has the maximal rank of $H_2(|R_{DF}|; \mathbb{Z})$ for all $t$-minimal surface diagrams, $K_{DF}$ is said to be maximal. Suppose that $K_{DF}$ is maximal. We view $K_i \subset R_{DF}, (i = 1, \ldots, r \leq n)$ as chains in $C_2(K_{DF})$.

$$\mu(F) = \min_{D_F} \# \{K_i \subset R_{DF} | \text{Col}_{\#}(K_i) \neq 0, K_{DF} \text{ is maximal} \}. \quad (1)$$

Theorem (T.Y.)

Let $F$ be a 2-knot. Suppose that $F$ is coloured by a finite quandle $X$. Then the following holds.

$$4\mu(F) \leq t(F). \quad (2)$$
If $K_{DF}$ has the maximal rank of $H_2(|R_{DF}|; \mathbb{Z})$ for all $t$-minimal surface diagrams, $K_{DF}$ is said to be **maximal**. Suppose that $K_{DF}$ is maximal. We view $K_i \subset R_{DF}, (i = 1, \ldots, r \leq n)$ as chains in $C_2(K_{DF})$.

\[ \mu(F) = \min_{DF} \# \{ K_i \subset R_{DF} \mid \text{Col}_i(K_i) \neq 0, K_{DF} \text{ is maximal} \}. \quad (1) \]

**Theorem (T.Y.)**

Let $F$ be a 2-knot. Suppose that $F$ is coloured by a finite quandle $X$. Then the following holds.

\[ 4\mu(F) \leq t(F). \quad (2) \]
The dihedral quandle \((X, \ast)\) of order \(p > 0\) \((p\ is\ prime)\) is a quandle \(X = \{0, \ldots, p - 1\}\) with the binary operation \((i, j) \mapsto 2j - i \mod p\).

**Theorem (M-Y)**

Let \(F\) be a 2-knot and let \(D_F\) be a \(t\)-minimal surface diagram. Suppose that \(F\) is non-trivially coloured by \(R_5\) and that \(\mu(F) = 1\). Suppose that \(B_{D_F} = \emptyset\) and there are no cancelling pair of triple points also \(D_F\) represents a non-trivial quandle homology class in \(H_3^Q(X)\). Then

\[7 \leq t(F).\]  

(3)
The dihedral quandle \((X, \ast)\) of order \(p > 0\) (\(p\) is prime) is a quandle \(X = \{0, \ldots, p - 1\}\) with the binary operation 
\((i, j) \mapsto 2j - i \pmod{p}\).

**Theorem (M-Y)**

Let \(F\) be a 2-knot and let \(D_F\) be a t-minimal surface diagram. Suppose that \(F\) is non-trivially coloured by \(R_5\) and that \(\mu(F) = 1\). Suppose that \(B_{D_F} = \emptyset\) and there are no cancelling pair of triple points also \(D_F\) represents a non-trivial quandle homology class in \(H_3^Q(X)\). Then

\[7 \leq t(F).\]
Finding Cycles

Edge Condition

Suppose that $K_{DF} = R_{DF}$. For a non-degenerate triple $(p, q, r)$ of a 3-cycle, $(q, r)$ represents a non-degenerate edge $e$.

$(0, 1, 2)$ induces the edge $(1, 2)$. The edge $(1, 2)$ must exist in $K_{DF}$. 

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Finding Cycles

Cycles with length $n < 7$

There is no cycles with length $n < 7$ satisfying conditions of Theorem [M-Y].

To find cycles with the length $n$.

1. Generate a list of non-degenerate $R_5$-3-simplexes.
2. Generate a list of boundaries of quandle 3-simplexes obtained in (1).
3. Produce a 2-chain from $n$ of boundaries from the list obtained in (2).
4. Check whether or not the 2-chain is zero and it satisfies the edge-condition.
Triangle condition

Let $X = R_5$. Let $\sigma = (a, b, c) \in X^3$ be a non-degenerate quandle 3-simplex. $T_\sigma$ is a triangle if $2b - a - c \equiv 0$ or $a \equiv c \pmod{5}$.

A rectangle $T_\sigma = (v_0; v_0 v_1, v_0 v_2; v_3)$ induced from a triple $(a, b, c) \in X^3$ is a rectangle with $\text{Col}(v_0) = a$, $\text{Col}(v_0 v_1) = b$, and $\text{Col}(v_0 v_2) = c$. $\text{Col}(v_1) = a * b$, $\text{Col}(v_1 v_3) = c$, $\text{Col}(v_2) = a * c$, $\text{Col}(v_2 v_3) = b * c$, $\text{Col}(v_3) = (a * b) * c$. 

$\sigma = (a, b, c) \mapsto T_\sigma = b$ or $T'_\sigma = (a * b) * c$. 

n triple points ($n \leq 5$)
Let $X = R_5$. Let $\sigma = (a, b, c) \in X^3$ be a non-degenerate quandle 3-simplex. $T_\sigma$ is a triangle if $2b - a - c \equiv 0$ or $a \equiv c \pmod{5}$.

A rectangle $T_\sigma = (\nu_0; \nu_0 \nu_1, \nu_0 \nu_2; \nu_3)$ induced from a triple $(a, b, c) \in X^3$ is a rectangle with $\text{Col}(\nu_0) = a$, $\text{Col}(\nu_0 \nu_1) = b$, and $\text{Col}(\nu_0 \nu_2) = c$. $\text{Col}(\nu_1) = a \ast b$, $\text{Col}(\nu_1 \nu_3) = c$, $\text{Col}(\nu_2) = a \ast c$, $\text{Col}(\nu_2 \nu_3) = b \ast c$, $\text{Col}(\nu_3) = (a \ast b) \ast c$.
$n$ triple points ($n \leq 5$)

Enumerating cycles of four quandle 3-simplexes, our programs give 60 cycles. 12 of them are shown in the table below.

<table>
<thead>
<tr>
<th></th>
<th>Cycles with length 4</th>
<th>2b-a-c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1(0, 1, 3)-1(0, 2, 3) 1(1, 0, 3)-1(1, 4, 3)</td>
<td>4, 1, 1, 4</td>
</tr>
<tr>
<td>2</td>
<td>1(0, 1, 3) 1(1, 0, 3) 1(2, 1, 3) 1(4, 0, 3)</td>
<td>4, 1, 2, 3</td>
</tr>
<tr>
<td>3</td>
<td>1(0, 1, 3) 1(1, 0, 3) 1(2, 4, 3) 1(4, 2, 3)</td>
<td>4, 1, 3, 2</td>
</tr>
<tr>
<td>4</td>
<td>-1(0, 1, 3) 1(0, 2, 3)-1(1, 0, 3) 1(1, 4, 3)</td>
<td>4, 1, 1, 4</td>
</tr>
<tr>
<td>5</td>
<td>-1(0, 1, 3)-1(1, 0, 3)-1(2, 1, 3)-1(4, 0, 3)</td>
<td>4, 1, 2, 3</td>
</tr>
<tr>
<td>6</td>
<td>-1(0, 1, 3)-1(1, 0, 3)-1(2, 4, 3)-1(4, 2, 3)</td>
<td>4, 1, 3, 2</td>
</tr>
<tr>
<td>7</td>
<td>1(0, 1, 4)-1(0, 3, 4)-1(3, 0, 4) 1(3, 2, 4)</td>
<td>3, 2, 3, 2</td>
</tr>
<tr>
<td>8</td>
<td>1(0, 1, 4) 1(1, 3, 4) 1(2, 0, 4) 1(3, 2, 4)</td>
<td>3, 2, 4, 2</td>
</tr>
<tr>
<td>9</td>
<td>-1(0, 1, 4) 1(0, 3, 4) 1(3, 0, 4)-1(3, 2, 4)</td>
<td>3, 2, 3, 2</td>
</tr>
<tr>
<td>10</td>
<td>-1(0, 1, 4)-1(1, 3, 4)-1(2, 0, 4)-1(3, 2, 4)</td>
<td>3, 2, 4, 2</td>
</tr>
<tr>
<td>11</td>
<td>1(0, 2, 1) 1(2, 0, 1) 1(3, 0, 1) 1(4, 2, 1)</td>
<td>3, 2, 1, 4</td>
</tr>
<tr>
<td>12</td>
<td>-1(0, 2, 1)-1(2, 0, 1)-1(3, 0, 1)-1(4, 2, 1)</td>
<td>3, 2, 1, 4</td>
</tr>
</tbody>
</table>
Twist Spun knots
Triple Point Numbers
Cell-Complexes for Surface-knots

Results

$n$ triple points ($n \leq 5$)

<table>
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<tr>
<td>2</td>
<td>1(0, 1, 3) 1(1, 0, 3) 1(2, 1, 3) 1(4, 0, 3)</td>
<td>4, 1, 2, 3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>:</td>
</tr>
</tbody>
</table>

The third column shows sequences of non-zero numbers $2b - a - c \pmod{5}$. Thus any simplex $\sigma$ in each cycle is non-degenerate. This implies that $K_{DF}$ contains no branch points. This contradicts the $t$-minimality of $D_F$. 

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Six triple points

Enumerating cycles with length six, our programs give essentially 36 3-chains.

<table>
<thead>
<tr>
<th>Type A</th>
<th>$\partial(4\text{-chain})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 $(0, 1, 0)-(0, 2, 0)-(0, 3, 0)$ $+(0, 4, 0)-(1, 2, 0)-(4, 3, 0)$</td>
<td>$= \partial(-(0, 1, 2, 0)-(0, 4, 3, 0))$</td>
</tr>
<tr>
<td>2 $(0, 1, 0)-(0, 2, 0)-(0, 3, 0)$ $+(0, 4, 0)-(1, 4, 0)-(4, 1, 0)$</td>
<td>$= \partial((0, 3, 4, 0)+(0, 2, 1, 0))$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>32 $-(0, 1, 2)-(1, 3, 2)-(2, 0, 2)$ $-(2, 4, 2)-(3, 1, 2)-(4, 3, 2)$</td>
<td>$= \partial(-(0, 3, 1, 2)-(0, 4, 3, 2)$ $-(1, 2, 3, 2)-(4, 1, 3, 2)$ $-(4, 0, 1, 2)-(3, 2, 1, 2))$</td>
</tr>
</tbody>
</table>
Let $K_{DF}$ be a cell-complex for a coloured $D_F$. $K'_{DF}$ is defined by replacing $T_\sigma \in R_{DF}$ with $T'_\sigma$, where $\sigma = (a, b, c) \in \mathbb{X}^3$ is the colour triple of the rectangle.

**Lemma**

Let $K'_{DF}$ be the deformed cell-complex from $K_{DF}$ by the above deformation. Then

$$\beta_1(|K_{DF}|) \geq \beta_1(|K'_{DF}|),$$

(4)

where $\beta_1$ is the first Betti number.
Let $c$ be a cycle of Type B. Then $K_c$ is obtained. When we construct $|K_c|$, $|K_c|$ is homeomorphic to a torus. This implies that $\beta_1(|K_c|) = 2$. This contradicts Lemma 1.

<table>
<thead>
<tr>
<th>Type $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
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