# On the geometry of certain slices of character varieties of knots

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The Fourth East Asian School of Knots and Related Topics

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- 1 Introduction
  - Motivation

## 2 Preliminaries

- SU(2)-Representations & characters
- Result concerning binary dihedral

### 3 Result & Example

- Statement
- Idea of the construction
- Example

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Motivation		

*K* ⊂ S<sup>3</sup> : a knot,
 *E<sub>K</sub>* = S<sup>3</sup> \ *N*(*K*) : the knot exterior, (π<sub>1</sub>(*E<sub>K</sub>*) is called the knot group of *K*)

A binary dihedral representation: For a Wirtinger presentation,

$$\pi_1(E_K) = \langle x_1, \dots, x_k | r_1, \dots, r_{k-1} \rangle$$
  
$$\pi_1(E_K) \to \operatorname{SU}(2)$$
  
$$x_i \mapsto \begin{pmatrix} 0 & e^{\theta_i \sqrt{-1}} \\ -e^{-\theta_i \sqrt{-1}} & 0 \end{pmatrix}$$

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### Remark

Binary dihedral representations appear in various areas of 3-dimensional topology, concerning representations.

We focus on these special representations and the features. Our purpose is to describe the following things:

- the invariance of binary dihedral representations in the character variety.
- what kind of representations is related to the branched cover of S<sup>3</sup>.

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### SU(2)-Representations & characters

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Result & Example

SU(2)-Representations & characters

Definitions of representations and characters

Definition (the SU(2)-representation space)

$${\cal R}({\cal E}_{{\cal K}})=\{
ho:\pi_1({\cal E}_{{\cal K}}) o {
m SU}(2)=\left(egin{array}{c} {a\ -b\ \overline{a}} \end{array}
ight) \quad {
m homomorphism}\}$$

where  $a, b \in \mathbb{C}$  such that  $|a|^2 + |b|^2 = 1$ .

### Definition (the SU(2)-character variety)

$$X(E_{\mathcal{K}}) = \left\{ \begin{array}{ccc} \chi_{\rho} : \pi_{1}(E_{\mathcal{K}}) & \to & \mathbb{R} \\ \gamma & \mapsto & \operatorname{tr} \rho(\gamma) \end{array} \middle| \rho \in \mathcal{R}(E_{\mathcal{K}}) \right\}$$

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SU(2)-Representations & characters

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SU(2)-Representations & characters

### Fact

- Both of R(E<sub>K</sub>) and X(E<sub>K</sub>) have the structure of algebraic varieties.
- The following identification exists:

 $X(E_{\mathcal{K}}) = \mathcal{R}(E_{\mathcal{K}})/conj$  $ho \underset{conj}{\sim} 
ho' \Leftrightarrow \exists A \in \mathrm{SU}(2), \ 
ho' = A 
ho A^{-1}$ 

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SU(2)-Representations & characters

### Example of SU(2)-representation

$$K = \sum_{i=1}^{N}$$

### Figure: The trefoil knot

$$\pi_1(E_{\mathcal{K}}) = \langle x, y \,|\, y^{-1}xy = xyx^{-1} \rangle$$

$$\rho_0(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_0(y) = \begin{pmatrix} 0 & \xi \\ -\xi^{-1} & 0 \end{pmatrix}, \quad \xi = e^{2\pi\sqrt{-1/3}}.$$

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SU(2)-Representations & characters

### Example of SU(2)-representation

$$K = \bigoplus_{y \in \mathcal{Y}_x} = \bigcup_{y \in \mathcal{Y}_x}$$

Figure: The trefoil knot

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(A binary dihedral representation)

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#### Result concerning binary dihedral

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#### Result concerning binary dihedral

### We want to show the following things.

- {binary dihedral rep.} forms certain fixed point set in the character variety.
- {binary dihedral rep.} is related to
   {abelian reps for the two-fold branched cover of S<sup>3</sup>}.

We keep to prepare some notions.

(a subset in  $X(E_K)$ , concerning binary dihedral representations)

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Introduction	

Result concerning binary dihedral

### Definition (Trace function)

 $\mu$ : the meridian of K,

$$I_{\mu}: X(E_{\mathcal{K}}) \to \mathbb{R}$$
$$\chi_{\rho} \mapsto \chi_{\rho}(\mu) = \operatorname{tr} \rho(\mu) (= 2 \cos \theta)$$

$$S_c(K) := I_\mu^{-1}(c) \subset X(E_K)$$

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### **Definition** (Slice)

For  $c \in [-2, 2]$ ,

$$\mathsf{S}_{c}(\mathsf{K}):=\mathit{I}_{\mu}^{-1}(c)\subset\mathsf{X}(\mathsf{E}_{\mathsf{K}})$$

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### We focus on $S_0(K)$ .

Remark

 $S_0(K) \supset \{\chi_\rho \mid \rho : \text{binary dihedral}\}$ 

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Example of  $X(E_K)$ 

By E. Klassen,

$$K = \bigotimes$$

### Figure: $X(E_{\kappa})$

abelian: the subset  $\{\chi_{\rho} | \rho(\pi_1(E_{\mathcal{K}})) \subset SU(2), abelian\},$ non-abelian: the subset  $\{\chi_{\rho} | \rho(\pi_1(E_{\mathcal{K}})) \subset SU(2), non-abelian\}.$ 

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$$K = \bigcap_{I_{\mu} = 2} \underbrace{\bigcap_{abelian}}_{I_{\mu} = -2}$$
  
Figure:  $X(E_{\kappa})$ 

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#### Statement

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#### Statement

### $C_2$ :two-fold cover of $E_K$



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Statement

# $C_2$ :two-fold cover of $E_K$



 $\Sigma_2$ :two–fold branched cover of S<sup>3</sup> along K



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#### Statement

### Lemma

$$\exists \iota : X(E_{\mathcal{K}}) \to X(E_{\mathcal{K}})$$
 involution  
*i.e.*,  $\iota^2 = id$ .

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# Example of $\iota$



### Figure: Involution *i*

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# Example of $\iota$



### Figure: Involution *i*

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#### Statement

### Theorem

$$\exists \Phi: S_0(E_K) \to X(\Sigma_2)$$

and  $\Phi$  :  $S_0(E_K) \rightarrow \text{Im } \Phi$  two–fold branched covering such that  $\iota$  acts as the covering transformation. Moreover the branched set is given as follows:

 $S_0(E_{\mathcal{K}})^{\iota} = \{\chi_{\rho} \mid \rho : \text{binary dihedral}\}$  $\cup \{\chi_{\rho} \mid \rho : \text{abelian}, \rho(\mu) = \begin{pmatrix} \sqrt{-1} & 0\\ 0 & -\sqrt{-1} \end{pmatrix}\}$  $\Phi(S_0(E_{\mathcal{K}})^{\iota}) = \{\chi_{\rho'} \mid \rho' \in R(\Sigma_2), \text{abelian}\}$ 

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#### Idea of the construction

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  - Idea of the construction
  - Example

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Idea of the construction

### The construction of the map $\Phi$

 $\pi_1(C_2) \xrightarrow{\pi} \pi_1(\Sigma_2)$ 

### Figure: Maps among the character varieties

# Intersection $X(E_K)$ with $X(\Sigma_2)$ in $X(C_2)$ gives a correspondence.

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Figure: Maps among the character varieties

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Figure: Idea of Φ

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Figure: Idea of Φ

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#### Example

# Outline

- 1 Introduction Motivation
- 2 Preliminaries
  - SU(2)-Representations & characters
  - Result concerning binary dihedral
- 3 Result & Example
  - Statement
  - Idea of the construction
  - Example

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#### Example

Example of  $\Phi$ 

$$K = \bigoplus_{y \in Y_x} = \bigoplus_{y \in Y_x}$$

$$\pi_1(E_{\mathcal{K}}) = \langle x, y \, | \, y^{-1} x y = x y x^{-1} \rangle$$

$$\rho_0(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_0(y) = \begin{pmatrix} 0 & \xi \\ -\xi^{-1} & 0 \end{pmatrix}$$

where 
$$\xi = e^{2\pi \sqrt{-1}/3}$$

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A binary dihedral representation  $\rho_0$  is given by

$$\rho_0(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_0(\mathbf{y}) = \begin{pmatrix} 0 & \xi \\ -\xi^{-1} & 0 \end{pmatrix}$$

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#### Example

$$S_0(\mathcal{K}) = \{\chi_{\rho_0}\}$$
$$\cup \{\chi_{\rho_{ab}} \mid \rho_{ab}(x) = \begin{pmatrix} \sqrt{-1} & 0\\ 0 & -\sqrt{-1} \end{pmatrix}, \rho_{ab}(y) = \begin{pmatrix} \sqrt{-1} & 0\\ 0 & -\sqrt{-1} \end{pmatrix}\}$$
$$\Sigma_2 = L(3, 1): \text{Lens space,}$$

$$\pi_1(\Sigma_2) = \langle \gamma \, | \, \gamma^3 = 1 \rangle,$$

$$X(\Sigma_2) = \{\chi_{\rho'} \mid \rho'(\gamma) = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}\} \cup \{\chi_{\rho'_{triv}} \mid \rho'_{triv}(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

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#### Example

### In this case, we have

$$egin{array}{lll} \Phi &: S_0(K) 
ightarrow X(\Sigma_2) \ & \chi_{
ho_0} \mapsto \chi_{
ho'} \ & \chi_{
ho_{ab}} \mapsto \chi_{
ho'_{triv}} \end{array}$$

### Remark

Φ is bijective.

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### Example

# In this case, we have

$$egin{array}{lll} \Phi &: S_0({\cal K}) o X(\Sigma_2) \ & \chi_{
ho_0} \mapsto \chi_{
ho'} \ & \chi_{
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## Remark

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$$egin{aligned} \Phi &: S_0(\mathcal{K}) o \mathcal{X}(\Sigma_2) \ & \chi_{
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### Example

If K is  $8_5 = (3, 3, 2)$ -Pretzel knot, then  $\Phi$  is not injective but

If K is a Montesinous knot, then  $\Phi$  is surjective.

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## Remark

# If K is a two-bridge knot, then $\Phi$ is bijective.

If K is a Montesinous knot, then  $\Phi$  is surjective.

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## Remark

If *K* is a two–bridge knot, then  $\Phi$  is bijective.

## Remark

If K is  $8_5 = (3, 3, 2)$ -Pretzel knot, then  $\Phi$  is not injective but surjective.

### Remark

If *K* is a Montesinous knot, then  $\Phi$  is surjective.

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### Example

## Remark

Our results also hold a knot in a homology 3–sphere and  $SL_2(\mathbb{C})$ -representations of the knot group.

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