On the geometry of certain slices of character varieties of knots

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University of Tokyo

The Fourth East Asian School of Knots and Related Topics

Outline

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Motivation

- $K \subset S^3$: a knot,
- $E_K = S^3 \setminus N(K)$: the knot exterior,
  ($\pi_1(E_K)$ is called the knot group of K)

A binary dihedral representation:
For a Wirtinger presentation,

$$\pi_1(E_K) = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_{k-1} \rangle$$

$$\pi_1(E_K) \to \mathrm{SU}(2)$$

$$x_i \mapsto \begin{pmatrix} 0 & e^{\theta_i \sqrt{-1}} \\ -e^{-\theta_i \sqrt{-1}} & 0 \end{pmatrix}$$

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Remark

Binary dihedral representations appear in various areas of 3-dimensional topology, concerning representations.

We focus on these special representations and the features. Our purpose is to describe the following things:

- the invariance of binary dihedral representations in the character variety.
- what kind of representations is related to the branched cover of S^3.

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Definitions of representations and characters

Definition (the SU(2)-representation space)

$$R(E_K) = \{\rho : \pi_1(E_K) \to \mathrm{SU}(2) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \text{homomorphism}\}$$

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$.

Definition (the SU(2)-character variety)

$$X(E_K) = \left\{ \begin{array}{ccc} \chi_\rho : \pi_1(E_K) & \to & \mathbb{R} \\ \gamma & \mapsto & \mathrm{tr}\,\rho(\gamma) \end{array} \,\middle|\, \rho \in R(E_K) \right\}$$

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Fact

- *Both of $R(E_K)$ and $X(E_K)$ have the structure of algebraic varieties.*
- *The following identification exists:*

$$X(E_K) = R(E_K)/conj$$

$$\rho \underset{conj}{\sim} \rho' \Leftrightarrow \exists A \in \mathrm{SU}(2),\ \rho' = A\rho A^{-1}$$

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Example of SU(2)-representation

$K =$

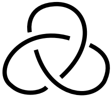

Figure: The trefoil knot

$$\pi_1(E_K) = \langle x, y \mid y^{-1}xy = xyx^{-1} \rangle$$

$$\rho_0(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_0(y) = \begin{pmatrix} 0 & \xi \\ -\xi^{-1} & 0 \end{pmatrix}, \quad \xi = e^{2\pi\sqrt{-1}/3}.$$

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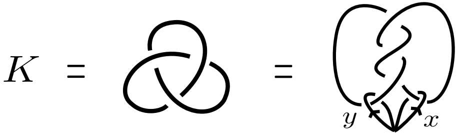

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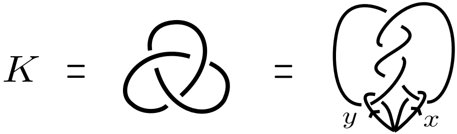

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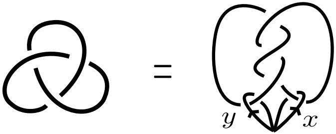

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1. Introduction
   - Motivation
2. Preliminaries
   - SU(2)-Representations & characters
   - Result concerning binary dihedral
3. Result & Example
   - Statement
   - Idea of the construction
   - Example

We want to show the following things.

- {binary dihedral rep.} forms certain fixed point set in the character variety.
- {binary dihedral rep.} is related to {abelian reps for the two-fold branched cover of S^3}.

We keep to prepare some notions.
(a subset in $X(E_K)$, concerning binary dihedral representations)

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Definition (Trace function)

μ : the meridian of K,

$$I_\mu : X(E_K) \to \mathbb{R}$$
$$\chi_\rho \mapsto \chi_\rho(\mu) = \mathrm{tr}\,\rho(\mu) (= 2\cos\theta)$$

Definition (Slice)

For $c \in [-2, 2]$,

$$S_c(K) := I_\mu^{-1}(c) \subset X(E_K)$$

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Example of $X(E_K)$

By E. Klassen,

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Figure: $X(E_K)$

abelian: the subset $\{\chi_\rho \mid \rho(\pi_1(E_K)) \subset \mathrm{SU}(2), \text{abelian}\}$,
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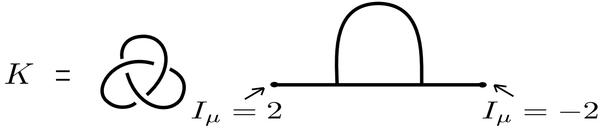

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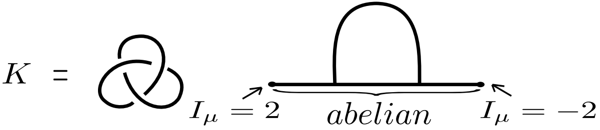

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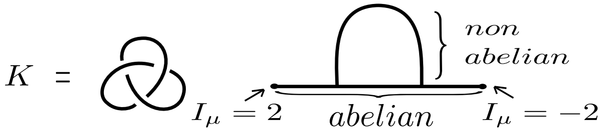

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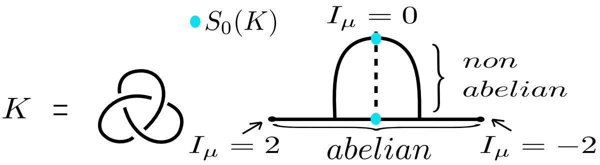

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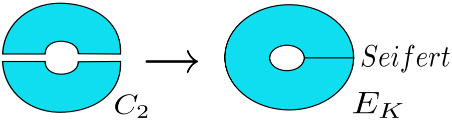
C_2:two–fold cover of E_K



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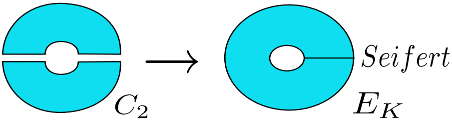
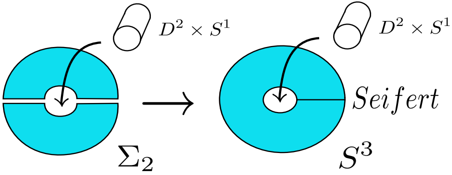


Σ_2:two–fold branched cover of S^3 along K

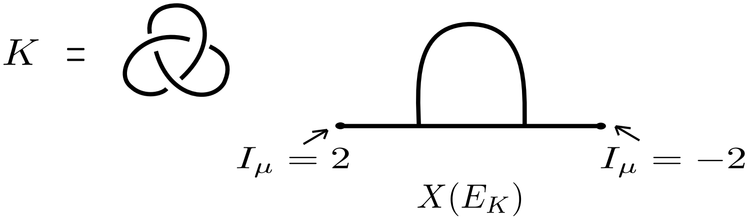


Lemma

$$\exists \iota : X(E_K) \to X(E_K) \quad \textit{involution}$$

$$\textit{i.e.}, \ \iota^2 = id.$$

Example of ι



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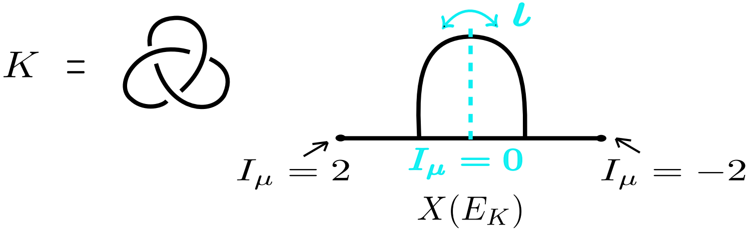

Figure: Involution ι

Theorem

$$\exists \Phi : S_0(E_K) \to X(\Sigma_2)$$

and $\Phi : S_0(E_K) \to \operatorname{Im}\Phi$ *two–fold branched covering such that* ι *acts as the covering transformation. Moreover the branched set is given as follows:*

$$S_0(E_K)^\iota = \{\chi_\rho \mid \rho : \textit{binary dihedral}\} \\ \cup \{\chi_\rho \mid \rho : \textit{abelian}, \rho(\mu) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}\}$$

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Outline

The construction of the map Φ

$$\pi_1(C_2) \xrightarrow{\pi} \pi_1(\Sigma_2)$$

Figure: Maps among the character varieties

Intersection $X(E_K)$ with $X(\Sigma_2)$ in $X(C_2)$ gives a correspondence.

The construction of the map Φ

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\begin{array}{ccc}
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Idea of the construction

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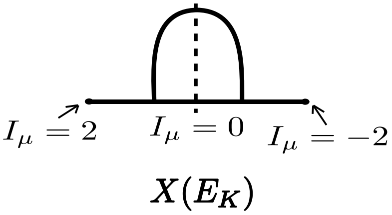

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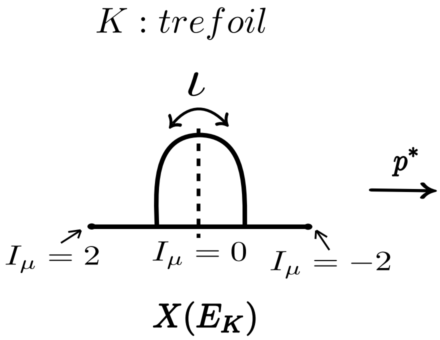

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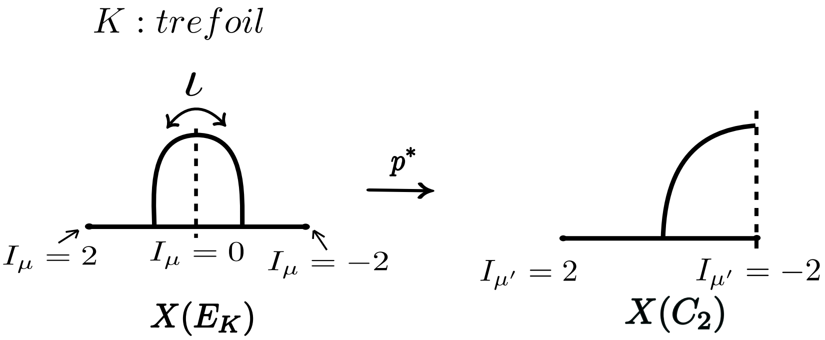

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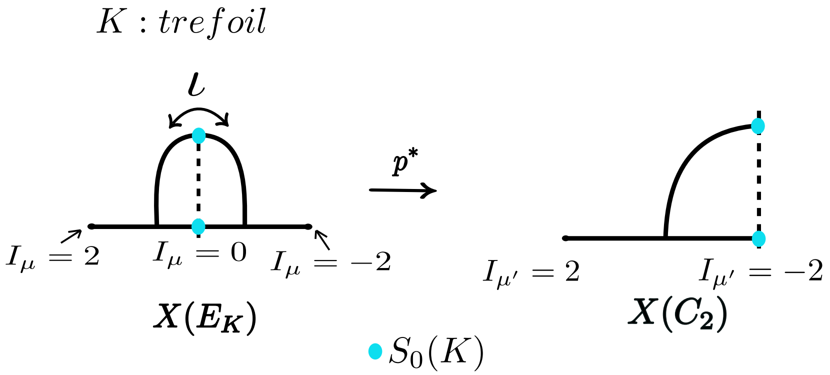

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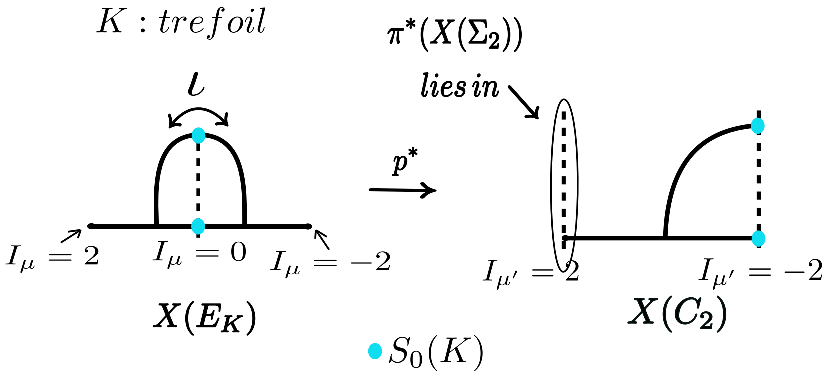

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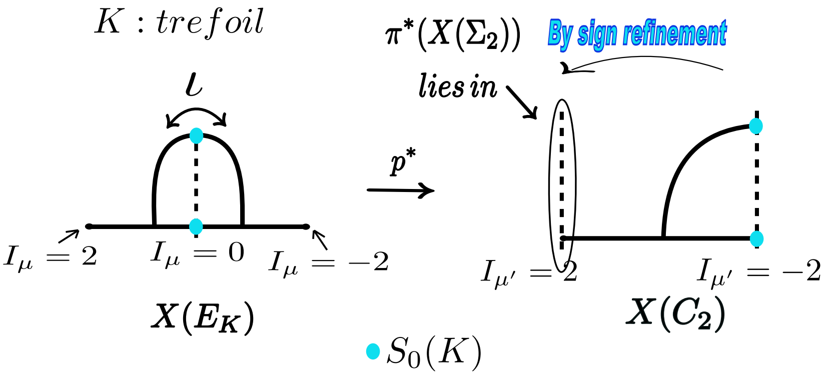

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Example of Φ

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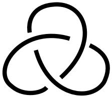

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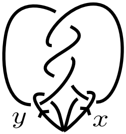


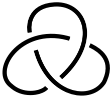
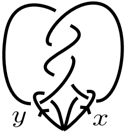
$$\pi_1(E_K) = \langle x, y \mid y^{-1}xy = xyx^{-1} \rangle$$

A binary dihedral representation ρ_0 is given by

$$\rho_0(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_0(y) = \begin{pmatrix} 0 & \xi \\ -\xi^{-1} & 0 \end{pmatrix}$$

where $\xi = e^{2\pi\sqrt{-1}/3}$.

Example of Φ

$K =$  $=$ 

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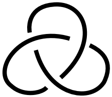
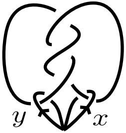
K =  = 

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$$S_0(K) = \{\chi_{\rho_0}\} \cup \left\{\chi_{\rho_{ab}} \,\middle|\, \rho_{ab}(x) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \rho_{ab}(y) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}\right\}$$

$\Sigma_2 = L(3,1)$:Lens space,
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Example

In this case, we have

$$\Phi : S_0(K) \to X(\Sigma_2)$$
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Φ is bijective.

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If K is a two–bridge knot, then Φ is bijective.

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If K is $8_5 = (3,3,2)$-Pretzel knot, then Φ is not injective but surjective.

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