The Fourth East Asian School of Knots
and Related Topics
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An infinite family of exotic 4-manifolds
and Rasmussen invariants of knots

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§1. Casson handles

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§1. Casson handles

A kinky handle:

\[ S^1 \times D^3 \]

The attaching region:

A kinky handle is a 2-handle modulo a finite number (≠ 0) of self-plumbings.
A Casson handle:
A Casson handle $\iff$ infinite-based signed tree
§2. **Known results**

**Theorem.** (M. Freedman (1983)) **Any Casson handle is homeomorphic to the standard open 2-handle.**

**Theorem.** (R. Gompf (1984)) **There exist countably many Casson handles.**

**Theorem.** (R. Gompf (1989)) **There exist uncountably many Casson handles.**
A Casson handle is **exotic** if the attaching circle does not bound a smooth 2-disc in the Casson handle.

**Problem.** Is any Casson handle exotic?

**Fact.** \( T, T' \): infinite-based signed trees.

\[
T \subset T' \Rightarrow CH_{T'} \subset CH_T.
\]

Thus if \( CH_T \) is exotic, then \( CH_{T'} \) is also exotic.
Ž. Bižaca showed an explicit example of an exotic Casson handle (1995).

: periodic Casson handles.
[Casson handle of boundary type]

$CH^+(\text{or } CH^-)$

Remark. Any Casson handle of boundary type is exotic.
Theorem. (T. Kato (2006)) There exists a Casson handle which is not boundary type.
§3. **Rasmussen’s $s$-invariant**

J. Rasmussen, *Khovanov homology and the slice genus*, math.GT/0402131, 2004  
(to appear in Inventiones Mathematicae.)

Lee’s variant of **Khovanov homology** $\implies$

a concordance invariant $s$ of a knot (combinatorially).
[Knot concordance group]
A knot $K$ is slice $\iff K$ bounds a smooth disc in $B^4$.
Knots $K_1$ and $K_2$ are concordant $\iff K_1^\# - K_2$ is slice.
The set $\{\text{concordance classes}\}$ forms a abelian group
under $\#$ (the knot concordance group).

[Slice genus]
$F$: a smooth conn. ori. surface properly embedded in $B^4$
with boundary $K$.
$g_s(K) := \min\{\text{the genus of } F\}$ (the slice genus of $K$).
Theorem (J. Rasmussen)

Let $K$ be a knot in $S^3$. Then

(1) $s$ induces a homomorphism from the knot concordance group to $\mathbb{Z}$;

(2) $|s(K)| \leq 2g_s(K)$;

(3) If $K$ is alternating, then $s(K) = \sigma(K)$, where $\sigma(K)$ is the classical knot signature of $K$;

(4) $s(T_{p,q}) = u(T_{p,q}) = (p - 1)(q - 1)/2$ (Milnor conjecture).
§4. **Results**

$CH$: a Casson handle.

$F$: a smooth conn. ori. surface properly embedded in $CH$ with boundary the attaching circle.

\[ g_s(CH) := \min \{ \text{the genus of } F \} \text{ (the slice genus of } CH) \].

**Problem.** Does there exists a Casson handle $CH$ satisfying $g_s(CH) > n$ for any positive integer $n$?
$CH_{m,n} =$
By using Rasmussen invariant, we show the following:

**Theorem 1.** Let $m_i$ and $n_i$ be non-negative integers with $m_i + n_i 
eq 0$ ($i = 1, 2$). If $m_1 + n_1 < |m_2 - n_2|$, then 

$$|m_1 - n_1| \leq g_s(CH_{m_1,n_1}) < g_s(CH_{m_2,n_2}) \leq m_2 + n_2.$$ 

**Remark.** Any $CH_{m,n}$ is exotic.
Proof of Theorem 1. It suffice to show that
\[ |m - n| \leq g_s(CH_{m,n}) \leq m + n \] if \( m + n \neq 0 \).

Claim 1. \( g_s(CH_{m,0}) \leq m + n \).

Claim 2.
\( T_m \): the connected sum of \( c_i(> 0) \)-fold untwisted positive doubles \( (1 \leq i \leq m) \) of the positive trefoil knot.
\( T_n \): the connected sum of \( c_i(> 0) \)-fold untwisted negative doubles \( (m + 1 \leq i \leq n) \) of the negative trefoil knot.

Let \( T_{m,n} = T_m \# T_n \).

\[ |m - n| \leq g_s(T_{m,n}) \Rightarrow |m - n| \leq g_s(CH_{m,n}). \]
Claim 3. $|m - n| \leq g_s(T_{m,n})$.

By using Rasmussen invariant, we can prove this claim.
Corollary 2. If \( m_1 + n_1 < |m_2 - n_2| \), then \( CH_{m_1, n_1} \) does not embed in \( CH_{m_2, n_2} \).

Corollary 3. For any positive integer \( n \), there exist countably many Casson handles \( \{CH_i\}_{i=0}^{\infty} \) such that \( g_8(CH_i) \geq n \).
[Kinkiness of a knot]

$K$: a knot in $S^3$.

$D$: a normally immersed disc in $B^4$ which span $K$.

$k_{\pm} = \min \# \{\text{positive (or negative) kinks in } D\}$

$\iff$ the kinkiness of $K$. 
[Kinkiness of a smooth 4-manifold]

$V$: a smooth 4-manifold.

$C$: a smoothly embedded circle in $\partial V$ with $C$ null-homotopic in $V$.

(e.g. $(V, C) = a$ Casson handle.)

$D$: a normally immersed disc in $V$ which span $C$.

$k_\pm = k_\pm(V, C) = \min\{ \# \text{ positive (or negative) kinks in } D \}$

$\iff$ the kinkiness of $(V, C)$. 
Theorem 4. For any non-negative integers $m$ and $n$ with $m + n \neq 0$, we have $k(CH_{m,n}) = (m, n)$. 

\[ CH_{m,n} = \]
Proof of Theorem 4.

Claim 1. $k_+(CH_{m,0}) = k_+(CH_{m,n})$; $k_-(CH_{0,n}) = k_-(CH_{m,n})$.

Claim 2. $k_+(CH_{m,0}) \leq m$; $k_-(CH_{0,n}) \leq n$.

Claim 3. $T_m$: the connected sum of $c_i(>0)$-fold untwisted positive doubles ($1 \leq i \leq m$) of the positive trefoil knot.

$m \leq k_+(T_m)$; $n \leq k_-(T_n) \Rightarrow$

$m \leq k_+(CH_{m,0})$; $n \leq k_-(CH_{0,n})$.  

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Claim 4. \( m \leq k_+(T_m); \quad n \leq k_-(T_n). \)

By using C. Bohr’s inequality (2002) and Rasmussen invariant, we can show this claim.
§5. **Other research**

By using Rasmussen invariant, we show the following:

**Theorem 5.** Every non-compact, connected, oriented, smooth 4-submanifold of $\mathbb{R}^4$ admits at least two smooth structures.

By using a consequence of gauge theory (Donaldson’s Theorem), we show the following:

**Theorem 6.** Every non-compact, connected, oriented, smooth 4-submanifold of $\#_{i=1}^{\infty} \mathbb{C}P^2$ admits at least two smooth structures.
Corollary 7. For any positive integer $n$, every non-compact, connected, oriented, smooth 4-submanifold of $\#_{i=1}^{n} \mathbb{C}P^2$ admits at least two smooth structures.

Problem (A. Kawauchi (1984)).

Does $\#_{i=1}^{\infty} \mathbb{S}^2 \times \mathbb{S}^2$ have at least two smooth structures?
§6. Problem

\[ \text{diff.} \]
§6. Problem

Thank you very much.