Determinants of knots and Diophantine equations

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1. Main result

$$\prod_{i=1}^{n}(1+x_i z) =: \sum_{i=0}^{n} \sigma_{i,n}(x_1,\dots,x_n)\, z^i \quad \text{elementary symmetric polynomials}$$

Theorem $\sigma_{n-1,n}(x_1,\dots,x_n) = -1$ x_i odd $n \equiv 7\,(8)$
$\Rightarrow$ at least 3 x_i are negative

Exam $(1,1,1,1,-1,-1,-1)$ so claim is "best possible"

2. Knots and links

$S^1 \hookrightarrow S^3$

$\coprod_{i=1}^{n(K)} S^1 \hookrightarrow S^3$ represented by diagrams

$n = n(K)$ # of components

diagrams / Reidemeister moves $\cong$ links

(integer) invariant : $\{\text{diagrams}\} \to \mathbb{Z}$

links $\cong$ dg / Reid moves

3. Signature and determinant

K knot $\longmapsto K = \partial S$ compact oriented surface (Seifert surface) of genus g, Euler characteristic χ $\longmapsto M$ Seifert matrix

link of n components

$\mathbb{Z}$-entries

knots $2g \times 2g$

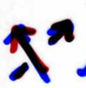
links $(1-\chi)\times(1-\chi)$

$$\left.\begin{aligned} \det(K) &= |\det(M+M^T)| \\ \sigma(K) &= \sigma(M+M^T) \quad \text{signature} \end{aligned}\right\} \text{ are invariants, i.e. indep. on } M, S$$

Properties: $\det(K)$ odd $\Leftrightarrow K$ is a knot (i.e. $n(K)=1$)

$2^{n(K)-1} \mid \det(K)$

for oriented links

(Murasugi) $\sigma($  $) - \sigma($  $) \in \{0, 1, 2\}$

$\sigma(O) = 0$ (unknot) $\qquad \sigma(!K) = -\sigma(K)$ (mirror image)

$|\sigma(K)| \le n(K) - \chi(K)$ $[\le 2g(K)$ K knot$]$

if $\det(K) \neq 0$ $\quad \sigma(K) + n(K) \equiv 1\ (2)$

$\Rightarrow$ if K knot, $\sigma(K)$ is even

$(\Rightarrow \sigma(\cdot) - \sigma(\cdot) \in \{0, 2\})$

K knot $\quad \sigma(K) \equiv 0\ (4) \Leftrightarrow \det(K) \equiv 1\ (4)$

$\sigma(K) \equiv 2\ (4) \Leftrightarrow \det(K) \equiv 3\ (4)$

$$\boxed{!!\ \text{if } \det(K) = 1 \Rightarrow 8 \mid \sigma(K)\ !!}$$

(because of σ of unimodular even quad forms)

4. Calculating det and σ

4.1. det

D diagram alternating :$\Leftrightarrow$ no $\perp\perp$ or $-|-|-$

det(D) = # spanning trees of G(D)

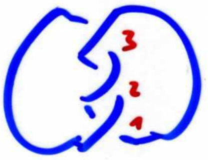
det = 3

braiding sequence

D diagram of n crossings $x_1, \dots, x_n$ odd integers

$D(x_1, \dots, x_n)$ replace crsg #i in D by  half-twists $x_i > 0$

$x_i < 0$

Ex  $D(1,-1,3) =$ 

actually two ways to twist in oriented diagrams distinguish them as 

reverse parallel

when D altern., $x_i > 0$ $\Rightarrow$ $D(x_1, \dots, x_n)$ alternating

$\det(D(x_1, \dots, x_n)) = P(x_1, \dots, x_n)$ is a polynomial in $(x_1, \dots, x_n) \in (2\mathbb{N}+1)^n$, linear in each x_i

(i.e. no monomial of P contains x_i^2 for any i)

for arbitrary $x_i \in 2\mathbb{Z}+1$ set

$\det(D(x_1, \dots, x_n)) = |\det' D(x_1, \dots, x_n)|$

$\tilde{P}$ is the unique extension of P to $(2\mathbb{Z}+1)^n$ $\longrightarrow$ $\tilde{P}(x_1, \dots, x_n)$

4.2. σ $K = K_1 \to K_2 \to \dots \to K_n = O$ for knots

K_i K_{i+1}  knowing $\det(K_i)$, $\det(K_{i+1})$ we know if σ changes

Murasugi's formulas for alternating diagrams

special al. :$\Leftrightarrow$ al & [all crossings or all crossings] $\Rightarrow$ $|\sigma| = 1 - \chi$ $= 2g$ for knots

e.g.  $\sigma = -2$

Lemma Consider

$\{ D_i = \underbrace{\ldots}_{i \text{ half-twists}} : i \text{ odd} \}$ D_i knot diagrams

1) twists are reverse.

if $\det(D_i) = \det(D_{i+2})$ for some i $\Rightarrow$ $\sigma(D_i) \equiv \text{const}\ \forall i$

if $\neq$ $\Rightarrow$ $\sigma(D_{i+2}) - \sigma(D_i) = \begin{cases} 0 & \forall i \text{ except } !1 \\ \pm 2 & \text{for this one } i \end{cases}$

2) twists are parallel

$\sigma(D_{i+2}) - \sigma(D_i) = \begin{cases} \pm 2 & \forall i \text{ except } !1 \\ 0 & \text{for that one } i \end{cases}$

Proof. $\det'(D_{i+2}) - \det'(D_i) = 2\det(\ldots) = 2\det(\underbrace{)(}_{D_\infty}) =: \tilde{d}$ independent on i

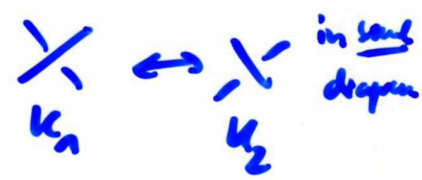
(det without forgetting sign)

arithmetic progression

1) D_∞ is 2-component link $\Rightarrow$ $4 \mid \tilde{d}$ and σ changes $!$ once, when $\mathrm{sgn}(\det')$ changes (if $\tilde{d} \neq 0$)

2) D_∞ is a knot, so $4 \nmid \tilde{d}$ and σ changes always, except only once, when $\mathrm{sgn}(\det')$ changes.

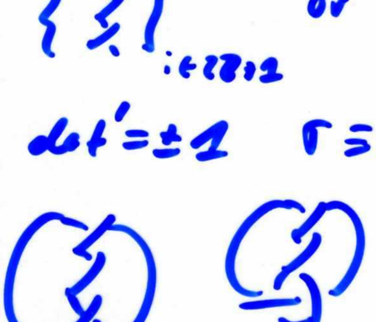
5. Knot distance, weighted knots



consider $\{ \ldots \}_{i \in 2\mathbb{Z}+1}$ or $\{ \ldots \}_{i \in 2\mathbb{Z}+2}$.

If $\exists$ a $\det' = \pm 1$ $\sigma \equiv 4\,(8)$ then ↯ so distance$(K_{12}) > 1$.

ex 1



K_1 $\sigma = 2$ $\det = 3$ K_2 $\sigma = 0$ $\det = 5$

$\Rightarrow$ dist() = 2

[also CGLS cyclic surgery]

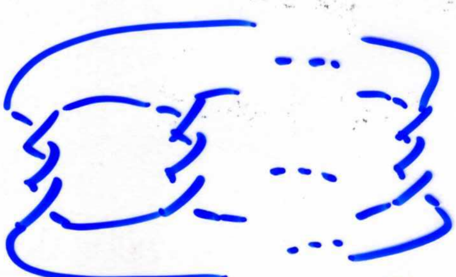
	u_1	u_2	
1	3	5	det
4	2	0	σ

ex 2

$\not\to$ det = 9

u_1 $\sigma = 4$ det = 5 $\qquad u_2$

Proof of main theorem.



$Pr(x_1, \dots, x_n)$ pretzel knot

$\det = |\sigma_{n-1,n}(x_1, \dots, x_n)|$

$n-1-2\#\{\text{negative } x_i\} \leq \sigma \leq n-1$ $\square$

6. Open problems

A) How to apply to a specific polynomial

B) Is a pair (d, s) with Murasugi's condition $(\det, \sigma)$ of some knot K (First studied by Shinohara in the 70's but then forgotten.)

mostly yes, difficult case $\sigma \equiv 4 (8)$.

Suffices to find K when d prime $\equiv 1 (4)$

Theorem If $d > 1$ $d \equiv 1\,(4)$ then $\exists K$ $\begin{matrix} \det(K) = d \\ \sigma(K) \equiv 4\,(8) \end{matrix}$

unless all prime factors of d are $\equiv 1\,(24)$ and $> 4 \cdot 10^9$.

Proof. Use connected sum and find arithmetic progressions $\{a + bk\}_{k \in \mathbb{N}} \subset \{d : \exists K \ \begin{matrix} \det(K) = d \\ \sigma(K) \equiv 4\,(8) \end{matrix}\}$ ...

Question: Does every sufficiently large $\# \ d \equiv 1\,(24)$ have a realization as $d = \sigma_{n-1,n}(x_1, \dots, x_n)$ $x_i \geq 1$ odd $n \equiv 5\,(8)$?

c) Regularities in the distribution of maximal $\{a + bk\}$:
as subsets of $\mathbb{N}$

clearly $4 \mid b$, $a \neq 1$, but it seems also that

(c.1) $b/4$ is always a prime (makes sense but always so?)

(c.2) # of a for given $b > 8$ are $\dfrac{b/4 + 1}{2}$

(c.3) a never a perfect square except in $(a, b) = (9, 12)$

(c.4) $(5, b)$ occurs, i.e. $a = 5 \iff 5 \nmid b/4 \pm 1$.

b	a
8	5
12	5, 9
20	5, 13, 17
28	5, 13, 17, 21
44	13, 17, 21, 29, 33, 41

END