On spatial graph diagrams with at most three crossings

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joint work with

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Question.

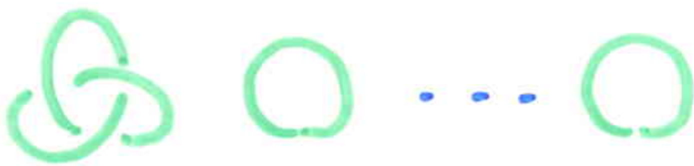
$$\begin{cases} L : \text{ n-comp. link} \\ \tilde{L} : \text{ regular diagram of } L \\ \quad\;\; \text{with at most 3 crossings} \end{cases}$$

$$\Longrightarrow \quad L \cong \; ?$$

Answer :

$L \cong$ 

or 

or 

> What does happen in the case of spatial graphs ?

§ 1. Preliminaries

G : finite graph

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sum_{i=1}^{4} x_i^2 = 1\}$$

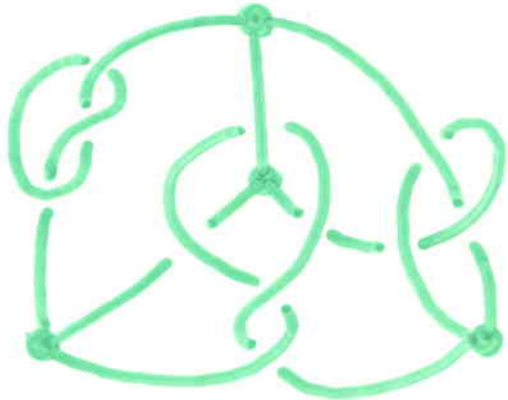
$$\cup$$

$$S^2 = \{(x_1, x_2, x_3, 0) \in S^3\}$$

Def 1.1. $\boxed{S^3}$

G :

$\xrightarrow[\text{emb.}]{f}$



: spatial emb. of G

$f(G)$: spatial graph

Def 1.2.

(1) Two sp. emb. $f, g : G \to S^3$

are **ambient isotopic** ($f \cong g$)

$\overset{\text{def}}{\Longleftrightarrow}$

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "g"'] & S^3 \arrow[d, "{}^{\exists}\Phi"] \arrow[dl, phantom, "\circlearrowright"] \\ & S^3 \end{tikzcd}$$

Φ : ori. pre. homeo.

(2) A sp. emb. $f : G \to S^3$

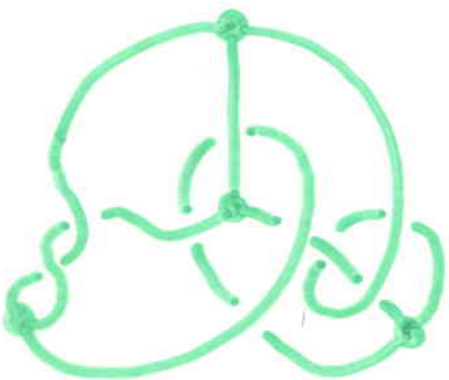
is **trivial**

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}h : G \to S^2 \subset S^3$ emb.

s.t. $f \cong h$

(Thus G must be **planar**.)

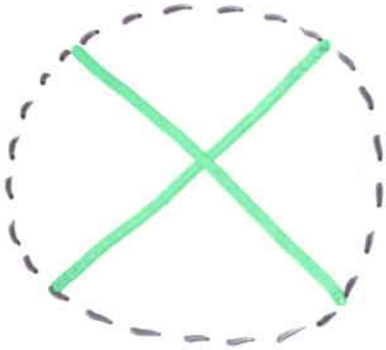
ex.



$\cong$

Def 1.3.

(1) An immersion $\varphi : G \longrightarrow S^2$ is a regular projection

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\forall}$ multipoint = 

(2) For a sp. emb. $f : G \longrightarrow S^3$,

φ is a regular projection of f

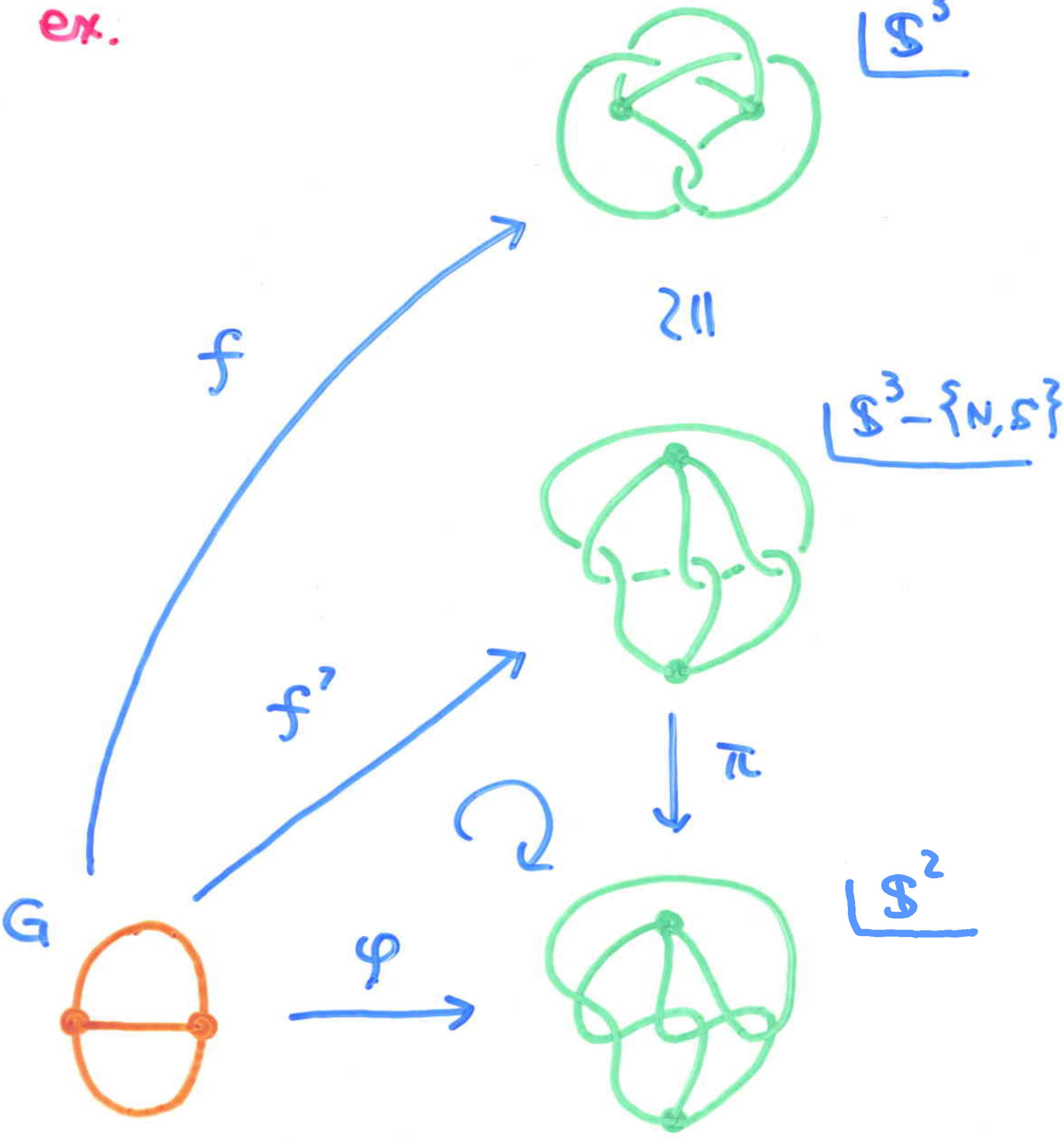
$$\overset{\text{def}}{\Longleftrightarrow} \ {}^{\exists} f' : G \xrightarrow{\text{emb.}} S^3 - \{\overset{N}{(0,0,0,1)}, \overset{S}{(0,0,0,-1)}\}$$

s.t. $f \cong f'$

$$\begin{tikzcd} & S^3 - \{N, S\} \arrow[d, "\pi"] \\ G \arrow[ur] \arrow[r, "\varphi"'] \arrow[r, phantom, shift left=3ex, "\circlearrowright"] & S^2 \end{tikzcd}$$

$\boxed{\text{natural projection}}$

ex.



“ f projects on φ. ”

§ 2. Results

Def 2.1.

$f: G \to S^3$ sp. emb.

(1) f is **free**

$\overset{\text{def}}{\Longleftrightarrow}$ $\pi_1(S^3 - f(G))$ is a free group.

(2) f is **totally free**

$\overset{\text{def}}{\Longleftrightarrow}$ $f|_H$ is free for $^{\forall} H \subset G$.

Remark.

If G is planar, then

f is totally free $\Longleftrightarrow$ f is trivial.

[Scharlemann - Thompson]

ex.

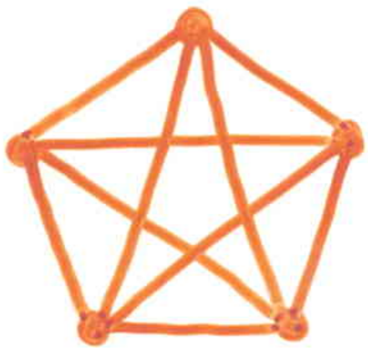
(1)

$\xrightarrow{f}$

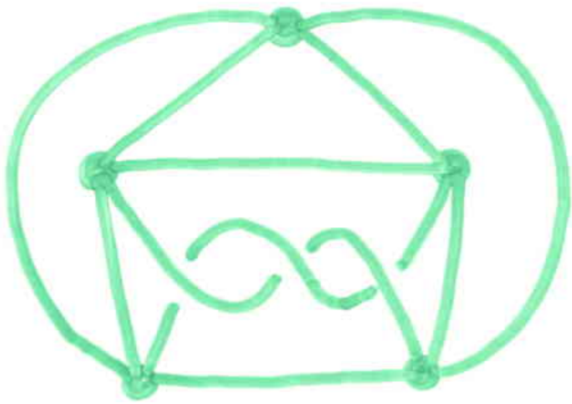
free, but not totally free.

(2)

K_5



$\xrightarrow{f}$



free, but not totally free.

Thm 2.2. [Huh − N]

$$\begin{cases} \varphi : G \longrightarrow S^2 & \text{reg. proj.} \\ cr(\varphi) \overset{\text{def}}{=} \#\{\text{double pts. of } \varphi\} \leqq 3 & \\ f : G \longrightarrow S^3 & \text{sp. emb.} \end{cases}$$

which projects on φ.

If $f(G) \not\supset$ ,  or 

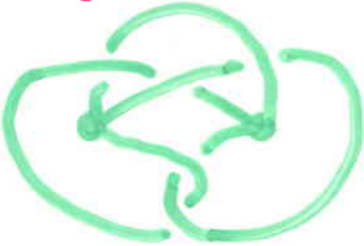
$\Longrightarrow$ f is totally free.

↑ "trivial"

if G is planar.

Def 2.3.

Kinoshita's θ-curve



$$\begin{cases} G : \text{planar graph} \\ f : G \to S^3 \ \text{sp. emb.} \end{cases}$$

f is **minimally knotted**

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} f \text{ is non-trivial} \\ f|_H \text{ is trivial for } {}^\forall H \subsetneqq G. \end{cases}$$

Cor 2.4.

$$\begin{cases} G : \text{planar graph} \\ \varphi : G \to S^2 \ \text{reg. proj.} \\ f : G \to S^3 \ \text{sp. emb. projects on } \varphi \end{cases}$$

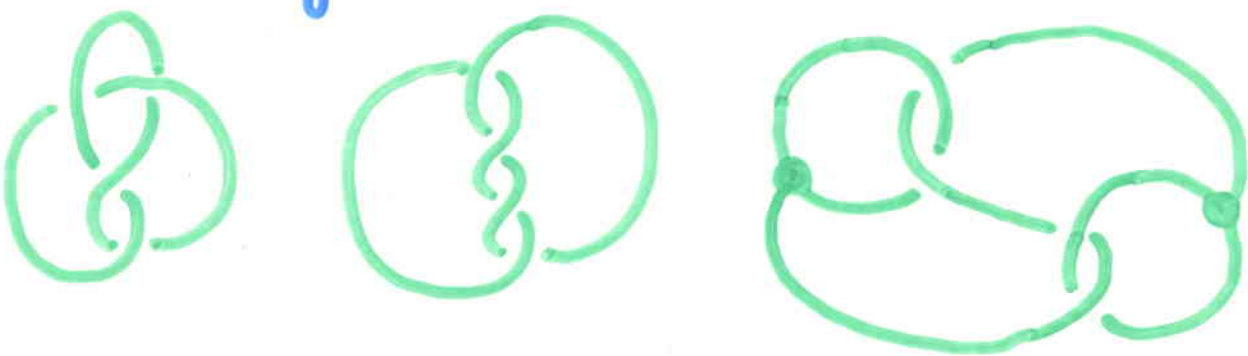
If f is minimally knotted

and $f(G) \not\cong$  ,  or 

$$\Longrightarrow \quad cr(\varphi) \geqq 4.$$

Remark.

This inequality is best possible :



Proof of Thm 2.2

By applying the following :

Thm 2.5. [Wu, Robertson-Seymour-Thomas]

f is totally free

$\Longleftrightarrow$ $^{\forall}$cycle γ of G,

$\exists D_\gamma \subset S^3$ 2-disk ,

s.t. $f(G) \cap D_\gamma = f(G) \cap \partial D_\gamma$

$= f(\gamma)$.

§3. Knotted projection

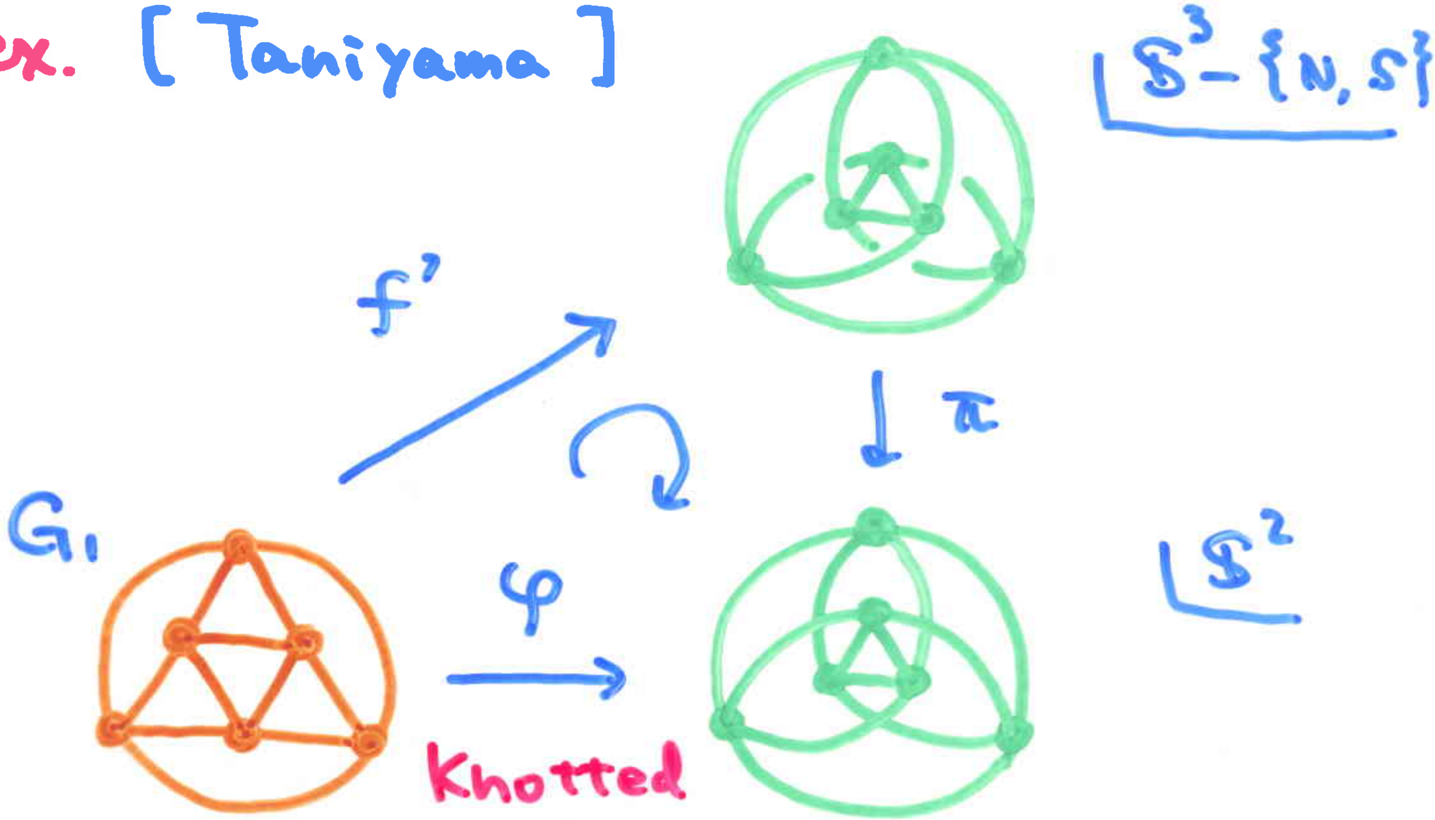
Def 3.1.

$$\begin{cases} G : \text{planar graph} \\ \varphi : G \longrightarrow S^2 \text{ reg. proj.} \end{cases}$$

φ is **knotted**

$\overset{\text{def}}{\Longleftrightarrow}$ $\nexists f : G \longrightarrow S^3$ **trivial** sp. emb. which projects on φ.

ex. [Taniyama]

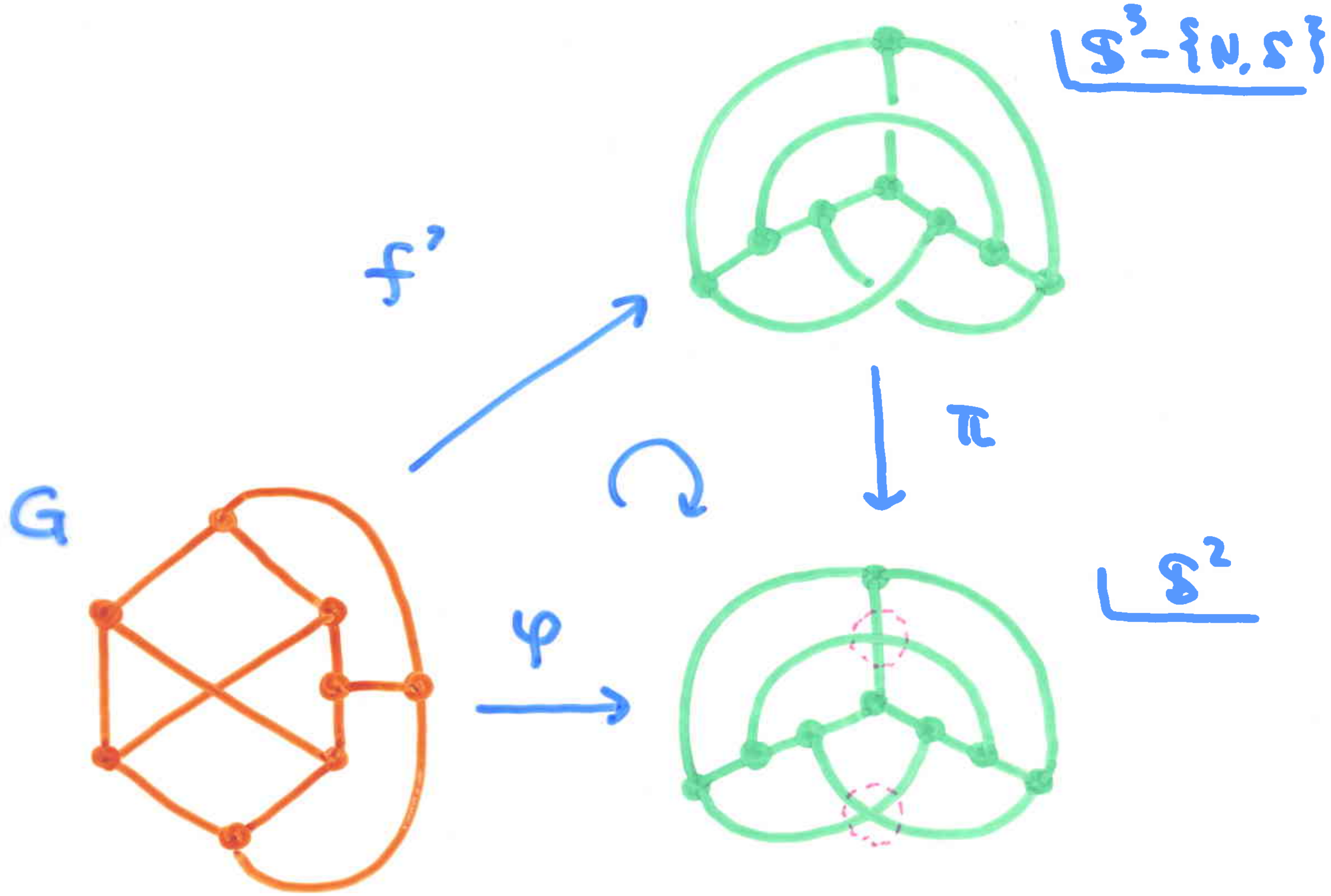


Thm 3.2. [Huh – N]

φ is knotted $\Longrightarrow$ $cr(\varphi) \geqq 3$.

Remark.

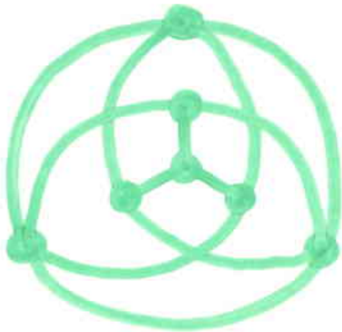
If G is non-planar,
then the following example exists :

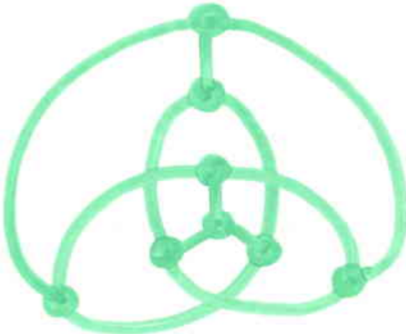


"Hopf link-inevitable"

< Knotted proj. φ with $cr(\varphi) = 3$ >

ex.

G_2  $\xrightarrow{\varphi}$ 

G_3  $\xrightarrow{\varphi}$ 

Prop 3.3. [Huh - N]

φ : knotted proj. with $cr(\varphi) = 3$

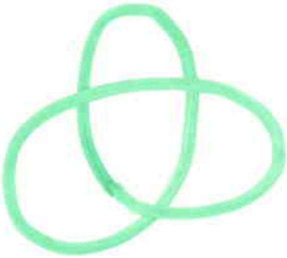
(1) $^{\forall}$double pt. of φ is produced by two disjoint edges.

(2) $^{\forall}f : G \to S^3$ sp. emb. projects on φ contains ,  or .

(partial answer for Ozawa's question)

Conjecture 1.

φ is knotted and $cr(\varphi) = 3$

$\Longrightarrow$ $\varphi(G) \supset$ 

Prop 3.4. [Huh - N]

If $G = G_1, G_2, G_3,$

then Conj. 1 is true.

Conjecture 2.

G has a knotted projection φ with $cr(\varphi) = 3$

$\Longrightarrow$ $G > G_1, G_2$ or G_3

minor