On logarithmic knot invariant

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1. $\mathcal{U}_q(sl_2)$ quantum invariants

$\mathcal{U}_q(sl_2)$ quantum invariants

- For knots and links q : generic
  - (colored) Jones polynomial

- For 3-manifolds q : root of 1
  - Witten-Reshetikhin-Turaev inv.
  - Kuperberg-Hennings invariant
- For knots and links
  - Colored Alexander inariant 3^{rd} *EAS*
  - Logarighmic invariant *NEW*

Quantum group $\mathcal{U}_q(sl_2)$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$\mathcal{U}_q(sl_2) = \Big\langle K, E, F \;\Big|\; [E,F] = \frac{K - K^{-1}}{q - q^{-1}},$$
$$K\,E\,K^{-1} = q^2\,E, \qquad K\,F\,K^{-1} = q^{-2}\,F \Big\rangle$$

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highest weight representation h.t. $\lambda - 1 \in \mathbf{Z}_{\geqq 0}$

$$\begin{array}{lccccccccccc}
E & 0 \longleftarrow & & \overset{[1]}{\longleftarrow} & & \overset{[2]}{\longleftarrow} & & \cdots & & \overset{[\lambda-1]}{\longleftarrow} & & \\
K & & q^{\lambda-1} & & q^{\lambda-3} & & & & & & q^{-\lambda+1} & \\
 & & \diamondsuit & & \diamondsuit & & \diamondsuit & \cdots & \diamondsuit & & \diamondsuit & \\
F & & & \underset{[\lambda-1]}{\longrightarrow} & & \underset{[\lambda-2]}{\longrightarrow} & & \cdots & & \underset{[1]}{\longrightarrow} & & \longrightarrow 0
\end{array}$$

Colored Jones invariants $J_\lambda(K)$

Defined by the R-matrix corresponding to the irreducible representation of $\mathcal{U}_q(sl_2)$.

$$V_\lambda : \text{irreducible repr. of dim } \lambda.$$

WRT invariant $\tau_r(M)$

$q^{2r} = 1$

M: 3-manifold obtained by surgery along a framed link K in S^3.

$$\tau_r'(M) = \sum_\lambda d_\lambda \, J_\lambda(K)$$

$\tau_r(M)$: normalization of $\tau_r'(M)$ by the signature of the linking matrix and the number of the components of K.

2. Kuperberg-Hennings invariant

Restricted $\overline{\mathcal{U}}_q(sl_2)$

$$q^{2p} = 1$$

$$\overline{\mathcal{U}}_q(sl_2) = \Big\langle K, E, F \mid \ [E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$

$$\boxed{E^p = F^p = 0, \quad K^{2p} = 1},$$

$$K\,E\,K^{-1} = q^2\,E, \quad K\,F\,K^{-1} = q^{-2}\,F\Big\rangle$$

$\overline{\mathcal{U}}_q(sl_2)$ is $2\,p^3$ dimensional

Basis :

$$\{E^i\,K^j\,F^k : \quad 0 \leqq i, k \leqq p - 1, \quad 0 \leqq j \leqq 2p - 1\}$$

Finite dim. non-semisimple Hopf algebra

Integral invariant of $\overline{\mathcal{U}}_q(sl_2)$

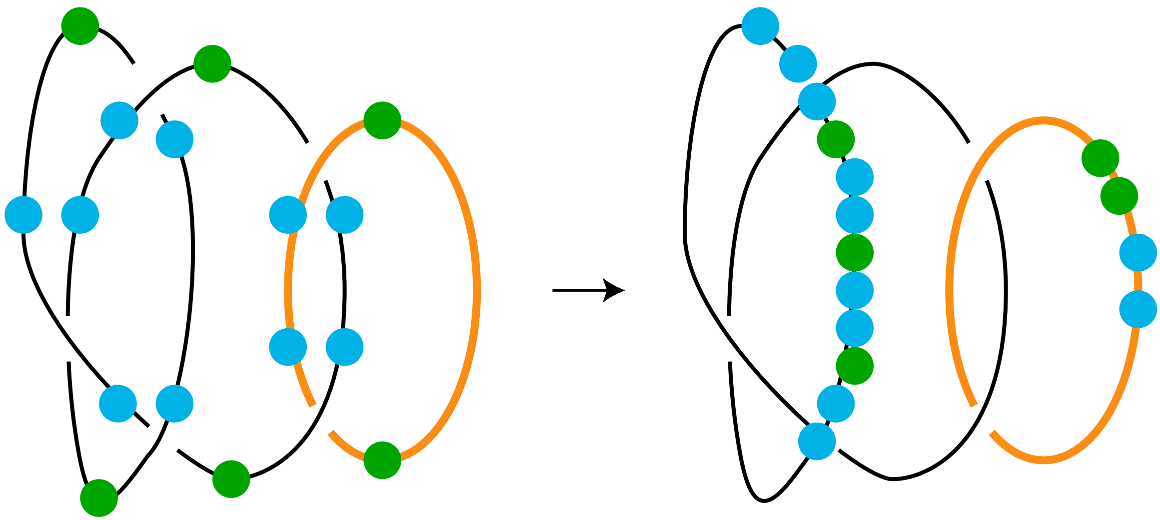
References:

- **G. Kuperberg:** *Involutory Hopf algebras and 3-manifold invariants.* Internat. J. Math. **2** (1991), 41–66.
- **M. Hennings:** *Invariants of links and 3-manifolds obtained from Hopf algebras.* J. London Math. Soc. (2) **54** (1996), 594–624. (preprint 1989)
- **L. Kauffman, D. Radford:** *Invariants of 3-manifolds derived from finite-dimensional Hopf algebras.* J. Knot Theory Ramifications 4 (1995), 131–162.
- **T. Ohtsuki:** *Invariants of 3-manifolds derived from universal invariants of framed links.* Math. Proc. Cambridge Philos. Soc. **117** (1995), 259–273.
- **J. Tian:** *On several types of universal invariants of framed links and 3-manifolds derived from Hopf algebras.* Math. Proc. Cambridge Philos. Soc. **142** (2007), 73–92.

Universal invariant of $\overline{\mathcal{U}}_q(sl_2)$

R. Lawrence, T. Ohtsuki

Construction: $K = K_1 \cup \cdots \cup K_k$: framed link



Beads construction

$$u_1 \otimes u_2 \otimes \cdots \otimes u_k \in \overline{\mathcal{U}}_q(sl_2) \otimes \overline{\mathcal{U}}_q(sl_2) \otimes \cdots \otimes \overline{\mathcal{U}}_q(sl_2).$$

Universal invariant of $\overline{\mathcal{U}}_q(sl_2)$

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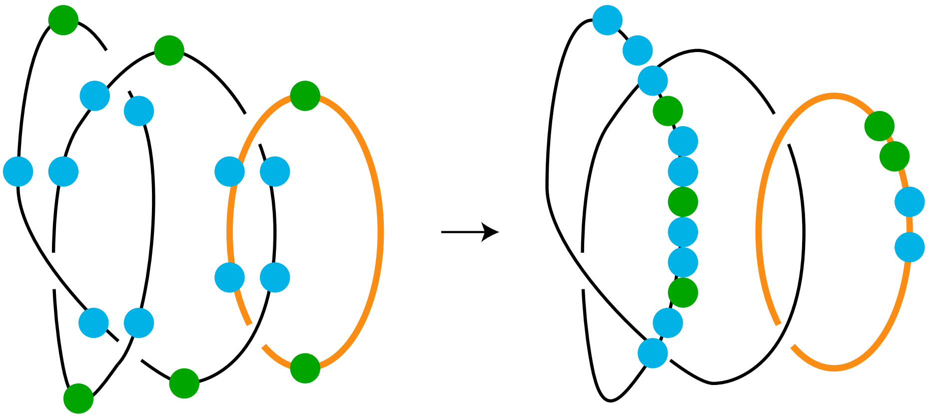


Beads construction

$$u_1 \otimes u_2 \otimes \cdots \otimes u_k \in \overline{\mathcal{U}}_q(sl_2) \otimes \overline{\mathcal{U}}_q(sl_2) \otimes \cdots \otimes \overline{\mathcal{U}}_q(sl_2).$$

$$f_i \in \left(\overline{\mathcal{U}}_q(sl_2) \big/ [\overline{\mathcal{U}}_q(sl_2), \overline{\mathcal{U}}_q(sl_2)] \right)^*$$

space of symmetric linear functions

$\prod_{i=1}^{k} f_i(u_i)$: inv. of K.

Integral invariant of $\overline{\mathcal{U}}_q(sl_2)$

Let

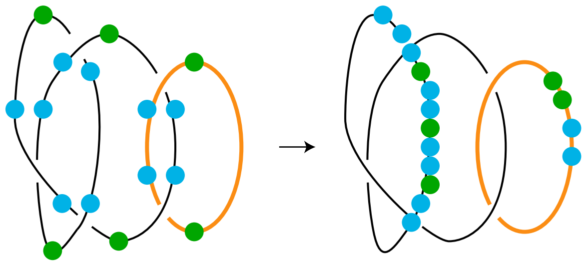
$$\phi : \overline{\mathcal{U}}_q(sl_2) \longrightarrow \mathbb{C}$$

be the integral of $\overline{\mathcal{U}}_q(sl_2)$ as a finite dim. Hopf algebra.

$(\phi = \int_{\overline{\mathcal{U}}_q(sl_2)})$ ϕ is a symmetric linear function on $\overline{\mathcal{U}}_q(sl_2)$.

Theorem (Kuperberg-Henning)

$\phi(u_1)\phi(u_2)\cdots\phi(u_k)$ is an invariant of 3-manifold obtained by the surgery along K.



$$u_1 \otimes u_2 \otimes \cdots \otimes u_k \in \overline{\mathcal{U}}_q(sl_2) \otimes \overline{\mathcal{U}}_q(sl_2) \otimes \cdots \otimes \overline{\mathcal{U}}_q(sl_2).$$

Integral of $\overline{\mathcal{U}}_q(sl_2)$

$q^{2p} = 1$

$$\overline{\mathcal{U}}_q(sl_2) = \Big\langle K, E, F \mid \; [E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$
$$E^p = F^p = 0, \quad K^{2p} = 1,$$
$$K\,E\,K^{-1} = q^2\,E, \quad K\,F\,K^{-1} = q^{-2}\,F \Big\rangle$$

Basis :

$$\{E^i\,K^j\,F^k : \quad 0 \leqq i, k \leqq p - 1, \quad 0 \leqq j \leqq 2p - 1\}$$

Integral :

$\phi(x) =$ coefficient of the term $E^{p-1}K^0F^{p-1}$

w.r.t. the above basis

3. Logarithmic invariant

Regular representation of $\overline{U}_q(sl_2)$

$$K_i := I + q^{-i} K + \cdots + q^{-ji} K^j + \cdots + q^{-(2p-1)i} K^{2p-1}$$

$$K K_i = q^i K_i$$

$$e_{ijk} := F^i E^j K_k$$

$$K e_{ijk} = q^{-2i+2j+k} e_{ijk}$$
$$E e_{ijk} = e_{ij+1k} + [i][-i+2j+k+1] e_{i-1jk}$$
$$F e_{ijk} = e_{i+1jk}$$

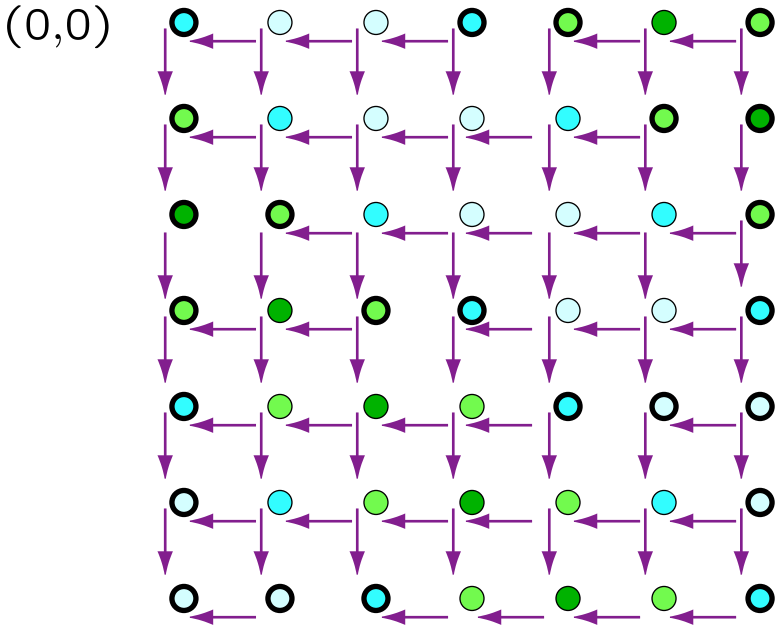
$$V^{(k)} := \left\langle e_{11k}, \ldots, e_{ijk}, \ldots, e_{r-1r-1k} \right\rangle$$

invariant subspace

E action on $V^{(k)}$

$$K\, e_{ijk} = q^{-2i+2j+k}\, e_{ijk}$$
$$E\, e_{ijk} = e_{ij+1k} + [i][-i+2j+k+1]\, e_{i-1jk}$$
$$F\, e_{ijk} = e_{i+1jk}$$

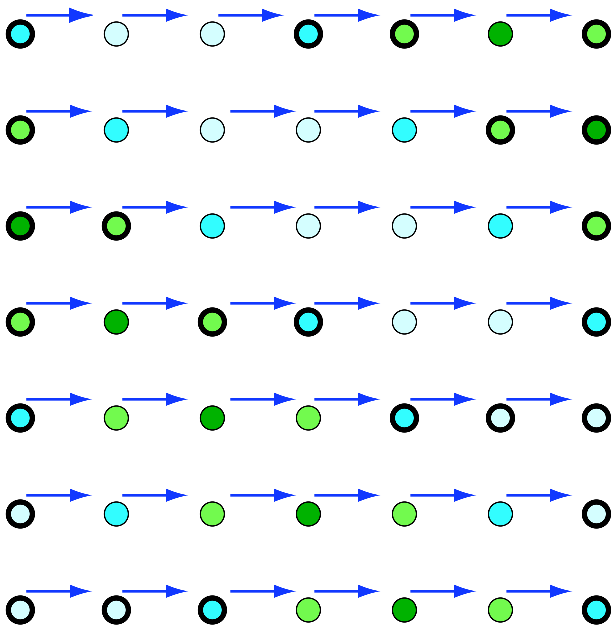
$p = 7, k = 3$
E action



F action on $V^{(k)}$

$$K\, e_{ijk} = q^{-2i+2j+k}\, e_{ijk}$$
$$E\, e_{ijk} = e_{ij+1k} + [i][-i+2j+k+1]\, e_{i-1jk}$$
$$F\, e_{ijk} = e_{i+1jk}$$

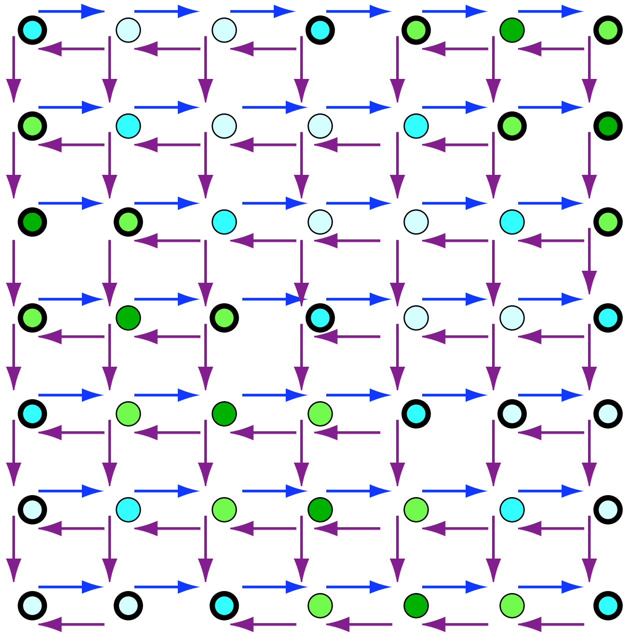
$p = 7, k = 3$
F action



E, F action on $V^{(k)}$

$$K\, e_{ijk} = q^{-2i+2j+k}\, e_{ijk}$$
$$E\, e_{ijk} = e_{ij+1k} + [i][-i+2j+k+1]\, e_{i-1jk}$$
$$F\, e_{ijk} = e_{i+1jk}$$

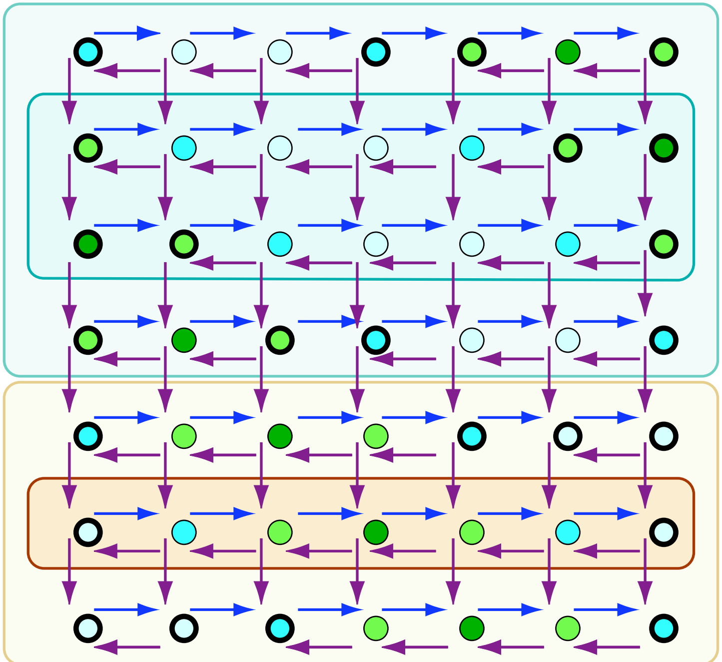
$p = 7, k = 3$
E, F action



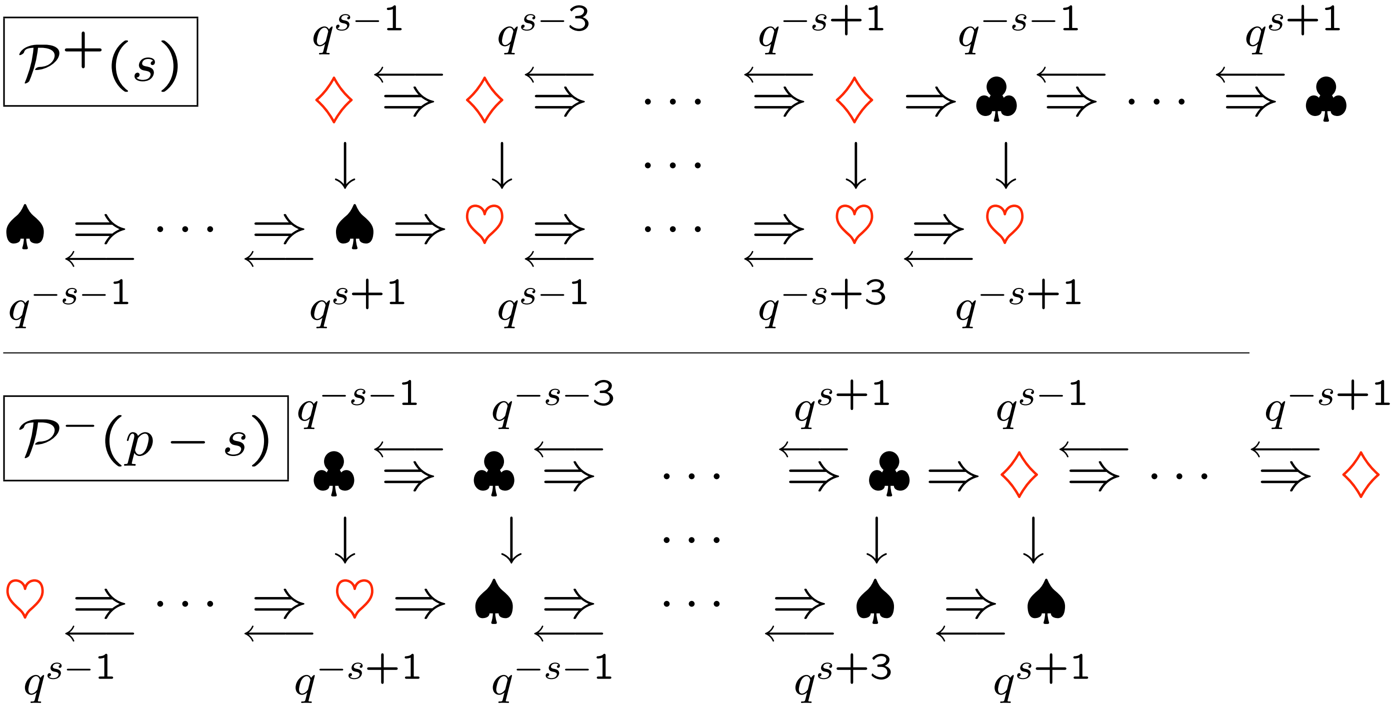
Invariant subquotients of $V^{(k)}$

$$K\,e_{ijk} = q^{-2i+2j+k}\,e_{ijk}$$
$$E\,e_{ijk} = e_{ij+1k} + [i][-i+2j+k+1]\,e_{i-1jk}$$
$$F\,e_{ijk} = e_{i+1jk}$$

$p = 7, k = 3$



Projective module representation of $\overline{\mathcal{U}}_q(sl_2)$



$\mathcal{P}^+(0)$: p-dim. representation of h.w. -1
$\mathcal{P}^-(0)$: p-dim. representation of h.w. $p-1$
($\Longleftrightarrow$ Kashaev inv. corresponding to volume)

Projective module representation of $\overline{\mathcal{U}}_q(sl_2)$

Reference:

- **M. Jimbo, T. Miwa, Y. Takeyama:** *Counting minimal form factors of the restricted sine-Gordon model.* Mosc. Math. J. **4** (2004), 787–846, 981.

Center of $\overline{\mathcal{U}}_q(sl_2)$

The action of the Kasimir element on $\mathcal{P}^+(s)$ and $\mathcal{P}^-(p-s)$ are the same. (act by the same scalar)

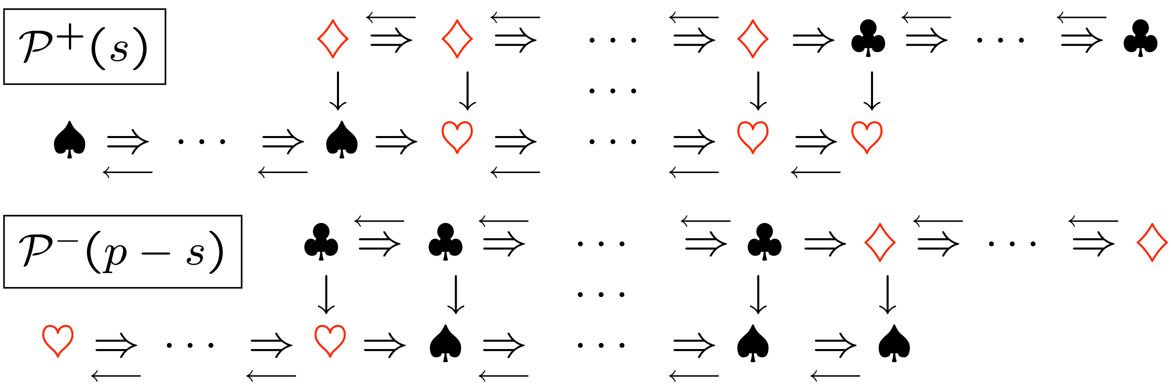
Basis of the center (e_i : principal idempotent):

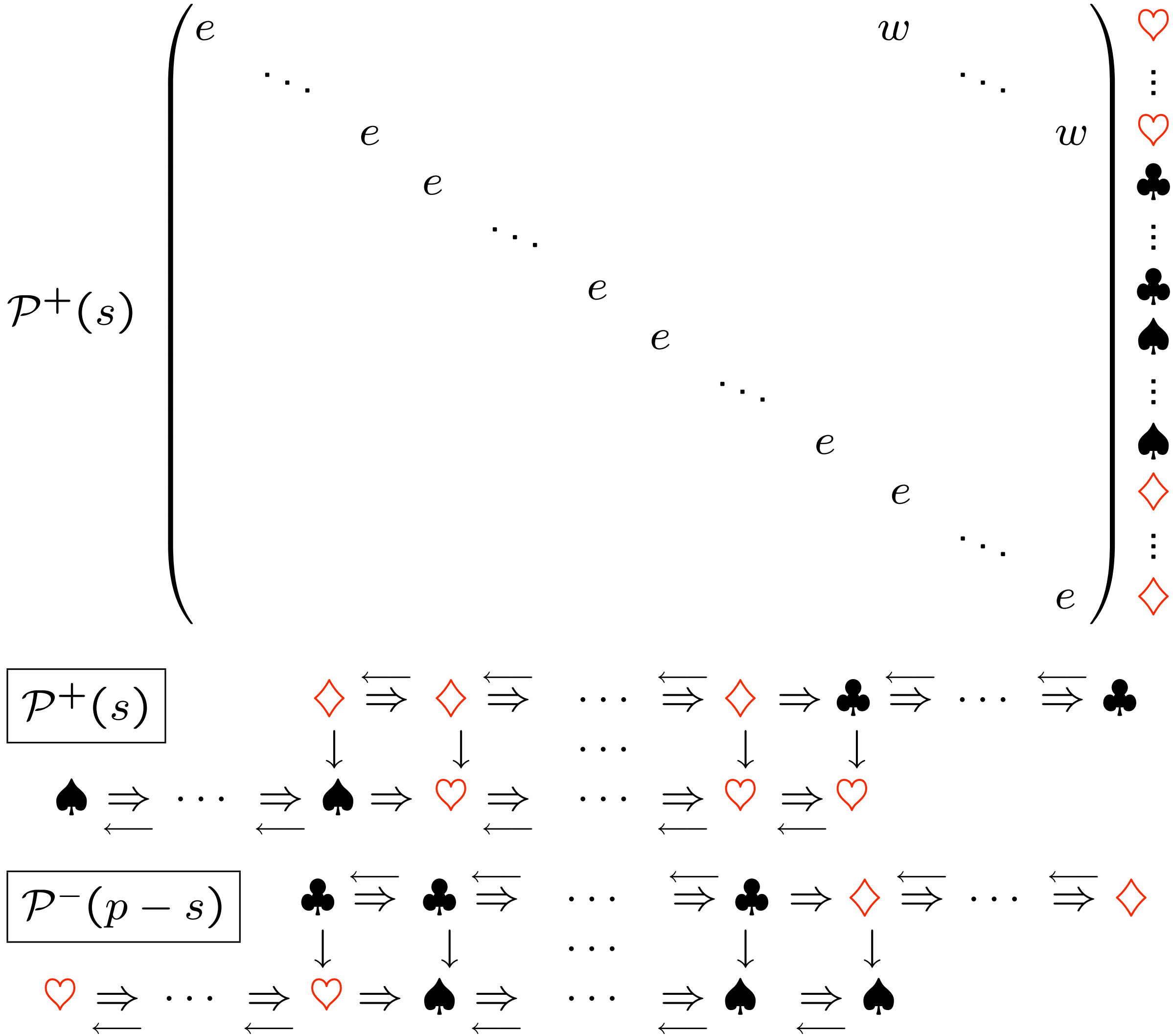
e_s : acts on $\mathcal{P}^+(s)$ and $\mathcal{P}^-(p-s)$ as the id. $(1 \le s \le p-1)$

e_0 : acts on $\mathcal{P}^+(0)$ as id. w_s^+ : maps $\diamondsuit$ of $\mathcal{P}^+(s)$ to $\heartsuit$

e_p : acts on $\mathcal{P}^-(0)$ as id. w_s^- : maps $\clubsuit$ of $\mathcal{P}^-(s)$ to $\spadesuit$

$(1 \le s \le p-1)$




P+(s)
e
w
e
w
e
e
e
e
e
e
P+(s)
P−(p − s)

Sym. Linear Fun. of $\overline{\mathcal{U}}_q(sl_2)$

(Y. Arike (2007))

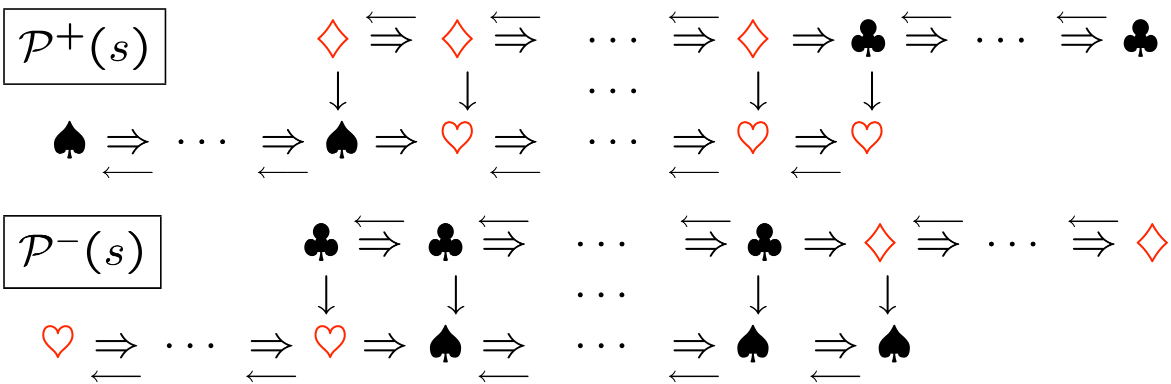
χ_s^+ : trace of $\diamondsuit$ block of $\mathcal{P}^+(s)$

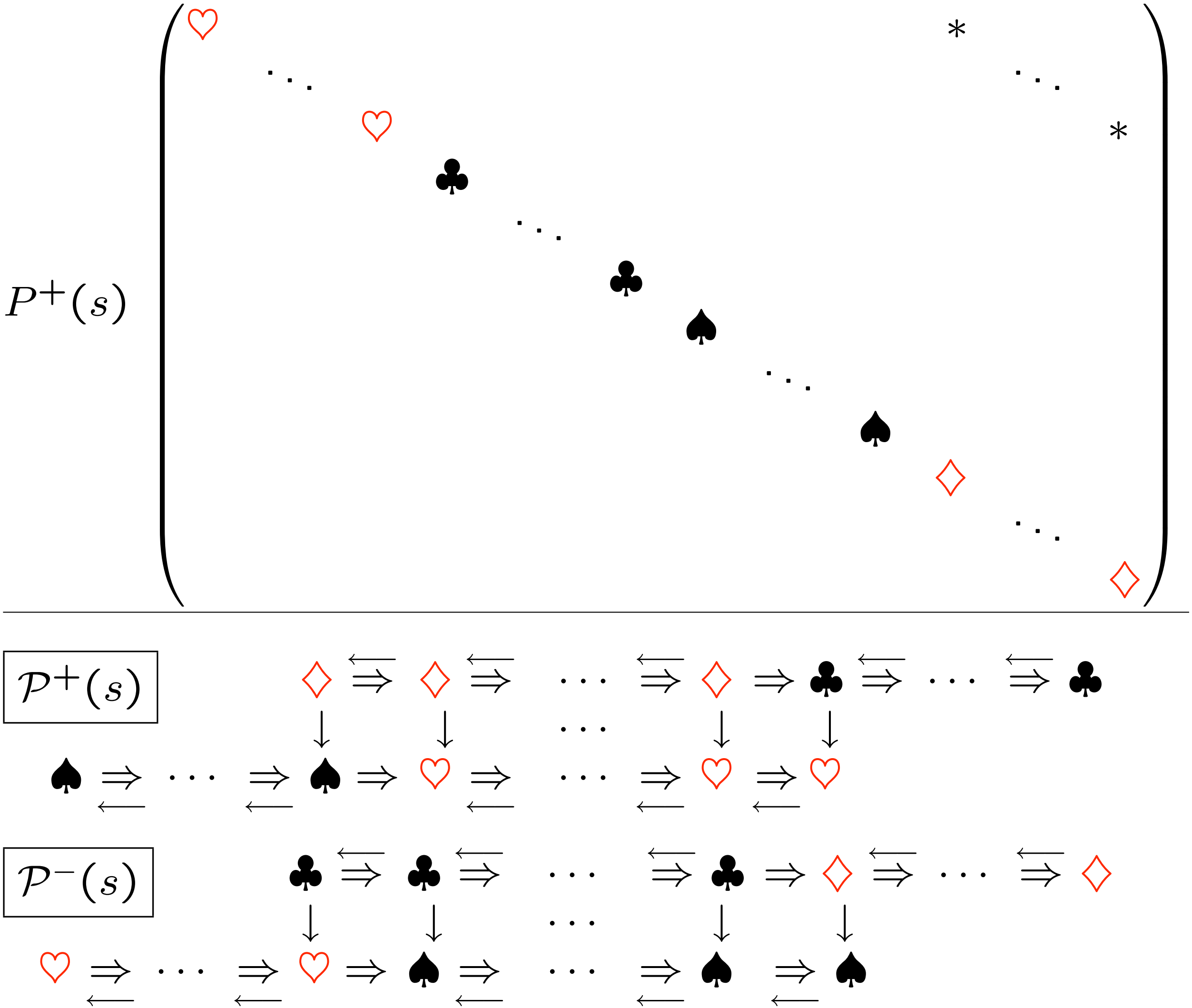
χ_s^- : trace of $\clubsuit$ block of $\mathcal{P}^-(s)$ $(1 \leq s \leq p-1)$

χ_0^+ : trace of $\mathcal{P}^+(0)$

χ_0^- : trace of $\mathcal{P}^-(0)$

r_s : sum of traces of off-diagonal blocks of $\mathcal{P}^+(s)$ and $\mathcal{P}^-(s)$ $\qquad (1 \leq s \leq p-1)$



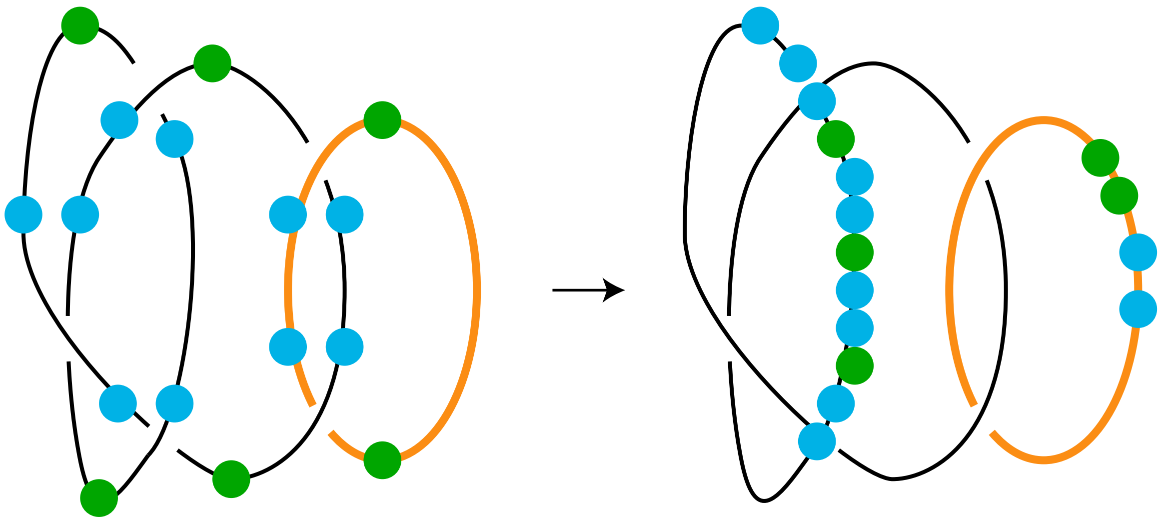


$P^{+}(s)$
$\mathcal{P}^{+}(s)$
$\mathcal{P}^{-}(s)$

Logarithmic invariant

Theorem.

$\prod_s \chi_s^{\pm}(u_i)$, $\prod_s r_i(u_i)$ are invariants of knots and links.

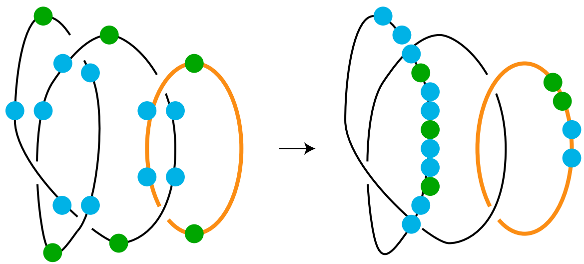


$$u_1 \otimes u_2 \otimes \cdots \otimes u_k \in \overline{\mathcal{U}}_q(sl_2) \otimes \overline{\mathcal{U}}_q(sl_2) \otimes \cdots \otimes \overline{\mathcal{U}}_q(sl_2).$$

Logarithmic invariant

Theorem.

$\prod_s \chi_s^{\pm}(u_i)$, $\prod_s r_s(u_i)$ are invariants of knots and links.

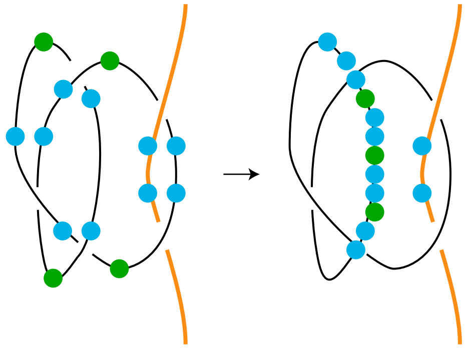


$$u_1 \otimes u_2 \otimes \cdots \otimes u_k \in \overline{\mathcal{U}}_q(sl_2) \otimes \overline{\mathcal{U}}_q(sl_2) \otimes \cdots \otimes \overline{\mathcal{U}}_q(sl_2).$$

Remark.

1. $\chi_s^{\pm}(u_i) = \pm$ colored Jones invariant.
2. $\prod r_s(u_i)$ corresponds logarithmic CFT.
3. Integral ϕ is a linear combination of r_s and $\chi_0^{\pm}$.

Correspondence to center



(1,1)-tangle $\longrightarrow$ center of $\overline{\mathcal{U}}_q(sl_2)$ $\quad e_s$, $w_s^{\pm}$

Quantum trace of e_s is equal to 0.

Basis of the center (e_i : principal idempotent):

e_s : acts on $\mathcal{P}^+(s)$ and $\mathcal{P}^-(p-s)$ as the id. $\quad (1 \le s \le p-1)$

e_0 : acts on $\mathcal{P}^+(0)$ as id. $\quad w_s^+$: maps $\diamondsuit$ of $\mathcal{P}^+(s)$ to $\heartsuit$

e_p : acts on $\mathcal{P}^-(0)$ as id. $\quad w_s^-$: maps $\clubsuit$ of $\mathcal{P}^-(s)$ to $\spadesuit$

4. Colored Alexander invariant

Semi-restricted $\widetilde{\mathcal{U}}_q(sl_2)$

$q^{2p} = 1$

$$\widetilde{\mathcal{U}}_q(sl_2) = \Big\langle K, E, F \mid \; [E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$

$$E^p = F^p = 0, \quad \cancel{K^{2p} = 1},$$

$$K\,E\,K^{-1} = q^2\,E, \quad K\,F\,K^{-1} = q^{-2}\,F \Big\rangle$$

$\widetilde{\mathcal{U}}_q(sl_2)$ is infinite dimensional

basis :

$$\{E^i\,K^j\,F^k : \quad 0 \leqq i, k \leqq p-1, \quad j \in \mathbb{Z}_{\geq 0}\}$$

non-semisimple Hopf algebra

Repserentation of $\widetilde{\mathcal{U}}_q(sl_2)$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$\widetilde{\mathcal{U}}_q(sl_2) = \Big\langle K, E, F \mid \; [E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \quad E^p = F^p = 0, \quad K\,E\,K^{-1} = q^2\,E, \quad K\,F\,K^{-1} = q^{-2}\,F \Big\rangle$$

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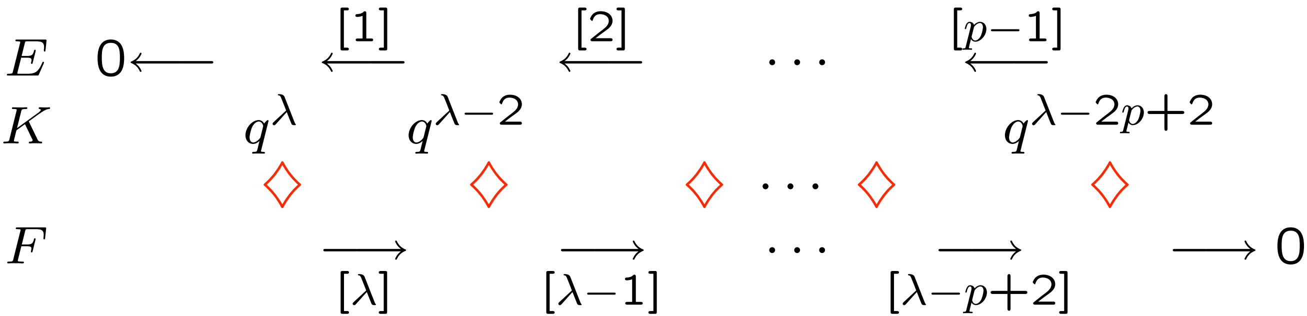
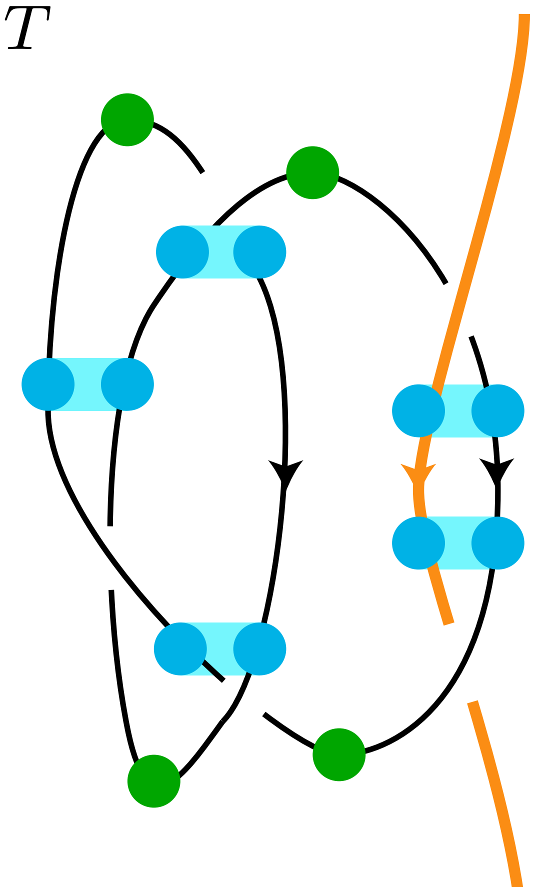
highest weight repr. V^λ h.t. $\lambda \in \mathbb{C}$





$$O^p_\lambda(T) : V^\lambda \to V^\lambda$$

$$
\begin{array}{c}
V^\lambda \\
\downarrow n \otimes 1 \\
(V^\mu)^* \otimes V^\mu \otimes V^\lambda \\
\downarrow 1 \otimes 1 \otimes n \otimes 1 \\
(V^\mu)^* \otimes V^\mu \otimes (V^\mu)^* \otimes V^\mu \otimes V^\lambda \\
\downarrow 1 \otimes R \otimes 1 \otimes 1 \\
(V^\mu)^* \otimes (V^\mu)^* \otimes V^\mu \otimes V^\mu \otimes V^\lambda \\
\downarrow 1 \otimes 1 \otimes 1 \otimes R \\
(V^\mu)^* \otimes (V^\mu)^* \otimes V^\mu \otimes V^\lambda \otimes V^\mu \\
\downarrow R \otimes 1 \otimes 1 \otimes 1 \\
(V^\mu)^* \otimes (V^\mu)^* \otimes V^\mu \otimes V^\lambda \otimes V^\mu \\
\downarrow 1 \otimes 1 \otimes 1 \otimes R \\
(V^\mu)^* \otimes (V^\mu)^* \otimes V^\mu \otimes V^\mu \otimes V^\lambda \\
\downarrow 1 \otimes R \otimes 1 \otimes 1 \\
(V^\mu)^* \otimes V^\mu \otimes (V^\mu)^* \otimes V^\mu \otimes V^\lambda \\
\downarrow 1 \otimes 1 \otimes u \otimes 1 \\
(V^\mu)^* \otimes V^\mu \otimes V^\lambda \\
\downarrow u \otimes 1 \\
V^\lambda
\end{array}
$$

Colored Alexander invariant

Theorem. (Akutsu-Deguchi-Ohtsuki)

Let $\hat{T}$ be the closure of T and let

$$\Phi^p(T) = O^p_\lambda(T)/\sin p\,\lambda_0 .$$

Then $\Phi^p_\lambda(T)$ is an invariant of $\hat{T}$ with colored components.

Remark (Volume Conjecture for colored Alexander)

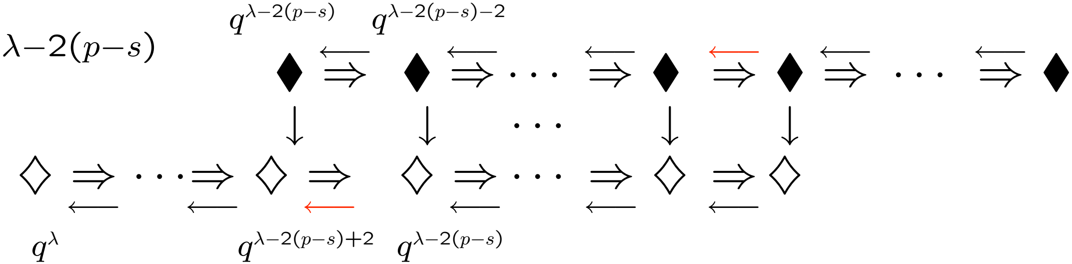
$$\lim_{p\to\infty} \Phi^p_\lambda(K) = 2\,\pi\,\mathrm{Vol}(\mathrm{S}^3_{\mathrm{K},\lambda})$$

where $S^3_{K,\lambda}$ is the cone manifold along K with cone angle $\pi\,p\,\lambda_i$ along K_i.

Relation to proj. repr. of $\overline{\mathcal{U}}_q(sl_2)$

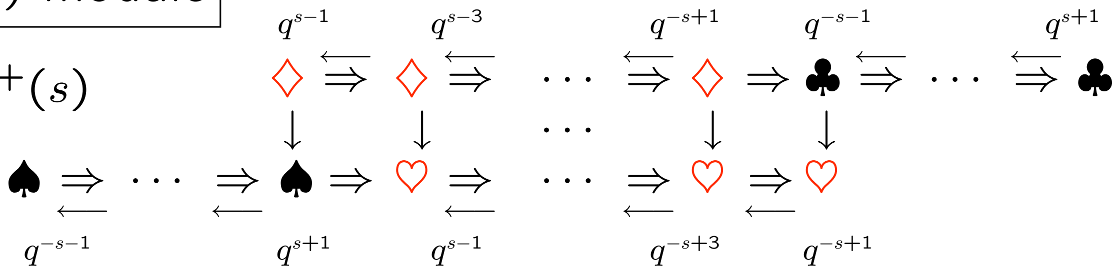
$\widetilde{\mathcal{U}}_q(sl_2)$ module

$V^{\lambda} \oplus V^{\lambda-2(p-s)}$



$\widetilde{\mathcal{U}}_q(sl_2)$ module

$\mathcal{P}^{+}(s)$



Let $\lambda = 2p - s - 1$, then $\longleftarrow$ vanished and the above two modules become isomorphic.

Relation to logarithmic invariant

T : $(1,1)$-tangle $\quad\quad \hat{T}$: closure of T

Theorem. (J.M.-K.Nagatomo)

$$r_s(\hat{T}) = \text{const.} \left(\frac{\partial\, O^p_\lambda(T)}{\partial\, \lambda}\bigg|_{\lambda=2p-s-1} - \frac{\partial\, O^p_\lambda(T)}{\partial\, \lambda}\bigg|_{\lambda=-s-1} \right)$$

C.f.

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \underset{x\to 1}{\longleftarrow} \begin{pmatrix} a\,x & 0 \\ 0 & a/x \end{pmatrix} \sim \begin{pmatrix} a\,x & 1 \\ 0 & a/x \end{pmatrix} \underset{x\to 1}{\longrightarrow} \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

5. Conclusion

Conclusion

$\mathcal{U}_q(sl_2)$ $\overline{\mathcal{U}}_q(sl_2)$ $\widetilde{\mathcal{U}}_q(sl_2)$

q : generic $q^{2p} = 1$

$V_\lambda \longrightarrow J_\lambda$

colored Jones

$\swarrow$ $\nwarrow$

Φ_λ^p colored Alexander

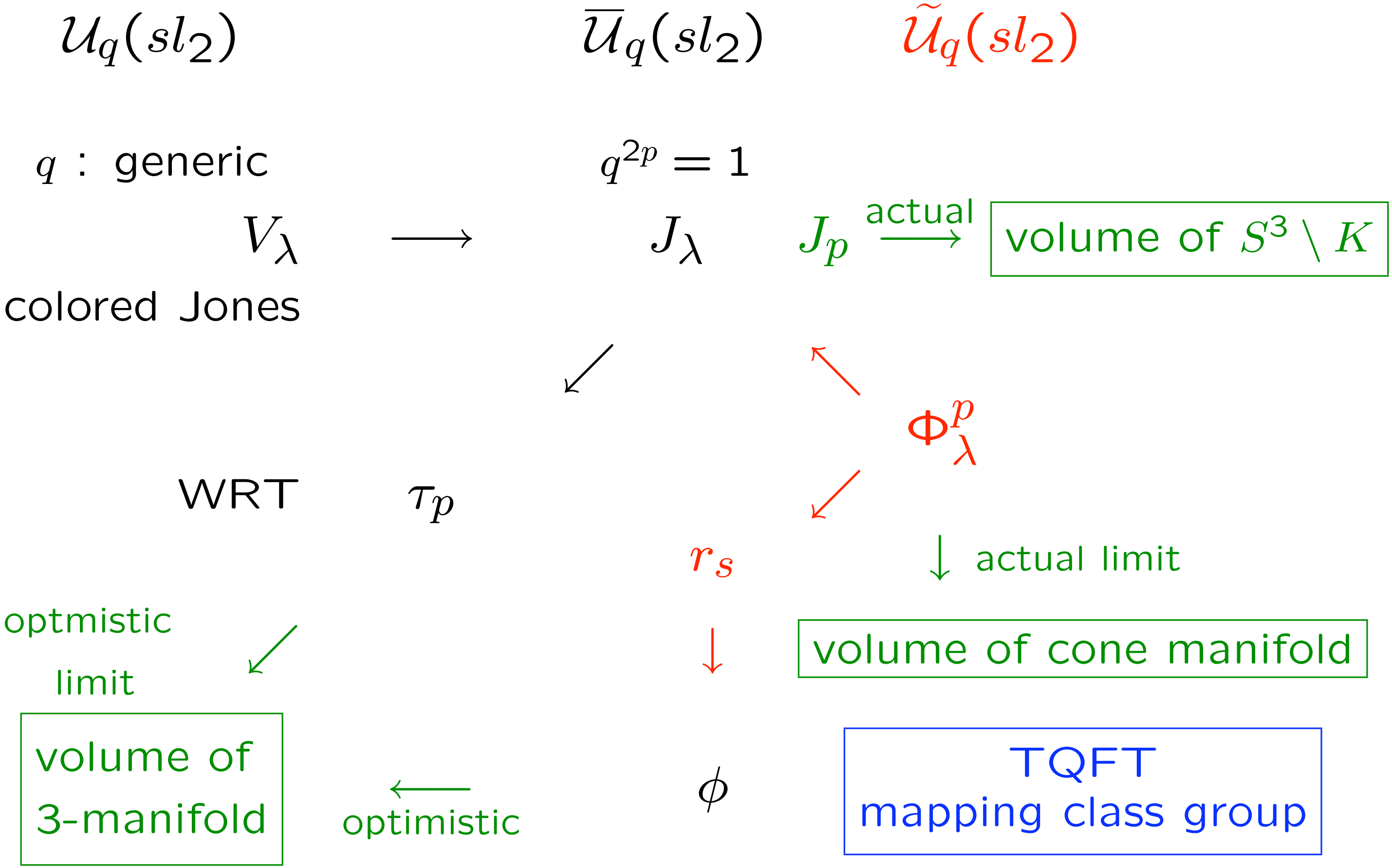
WRT τ_p $\swarrow$

r_s logarithmic invariant

$\downarrow$

ϕ Kuperberg-Hennings

Conclusion



$\mathcal{U}_q(sl_2)$
$\overline{\mathcal{U}}_q(sl_2)$
$\widetilde{\mathcal{U}}_q(sl_2)$
q : generic
$q^{2p} = 1$
V_λ
J_λ
J_p
actual
volume of $S^3 \setminus K$
colored Jones
Φ_λ^p
WRT
τ_p
r_s
actual limit
optmistic
limit
volume of cone manifold
volume of
3-manifold
optimistic
ϕ
TQFT
mapping class group

Reference for $SL(2,\mathbb{Z})$ action

- **V. Lyubashenko:** *Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity.* Comm. Math. Phys. **172** (1995), 467–516.
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- **V. Lyubashenko, S. Majid:** *Braided groups and quantum Fourier transform.* J. Algebra **166** (1994), 506–528.

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- **B. Feigin, A. Gainutdinov, A. Semikhatov, I. Tipunin:** *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center.* Comm. Math. Phys. **265** (2006), 47–93.