Eessential surfaces and torus knots with twists

by

Kanji Morimoto

Konan University, Japan

Abstract. In the present talk, we will show that torus knots with twists on 2-strands contain no closed essential surfaces in the exteriors. As an application, we will show that Hoidn-Morimoto's inequality for 1-bridge genus of knots is best possible. In addition, we will show that for any composite number r, there are infinitely many torus knots with twists on r-strands containing essential tori in the exteriors.

1. Definitions

K : a knot in the 3-sphere S^3

$N(K)$: the regular neighborhood of K in S^3

$E(K) = cl(S^3 - N(K))$: the exterior

F : a surface (i.e. a connected 2-manifold) $\subset E(K)$

F is closed essential in $E(K)$
if F is closed, incompressible and not parallel to the torus $\partial E(K)$.

F is meridionally essential in $E(K)$
if $\partial F \neq \emptyset$, each component of ∂F is a meridian of $N(K)$, F is incompressible and not parallel to an annulus in $\partial E(K)$.

A knot K is small
if $E(K)$ contains no closed essential surfaces

A knot K is meridionally small (m-small)
if $E(K)$ contains no meridionally essential surfaces.

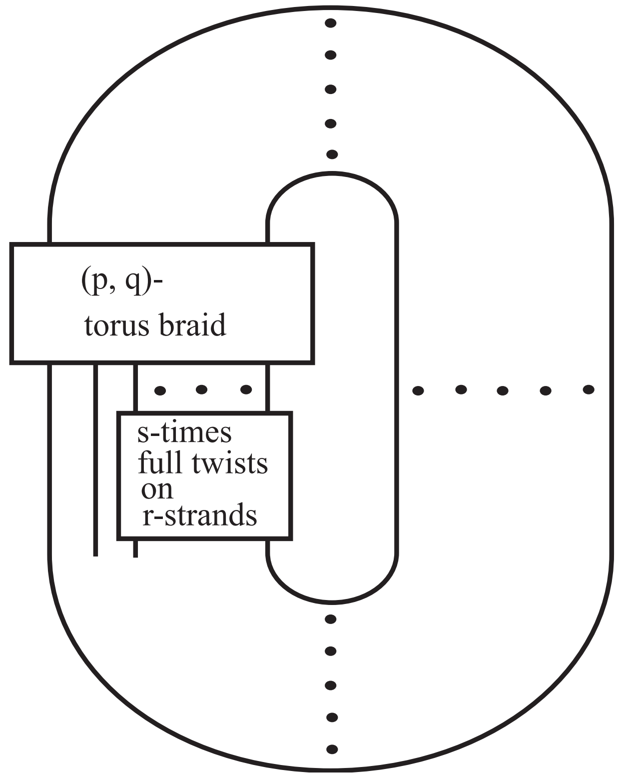
Note that :
if a knot K is small then it is m-small by [CGLS].

2. $K(p, q; r, s)$

$p, \ q$: coprime integers with $p > 2$ and $q > 1$
$T(p, q)$: the torus knot of type (p, q)
r : an integer with $p > r > 1$
s : an integer

Draw a picture of the torus knot $T(p, q)$.
Take r-strands in the parallel p-strings of $T(p, q)$.
Add s-times full twists on r-strands to the $T(p, q)$.
Then we get the knot denoted by $K(p, q; r, s)$ illustrated in the following figre :

K(p, q ; r, s)



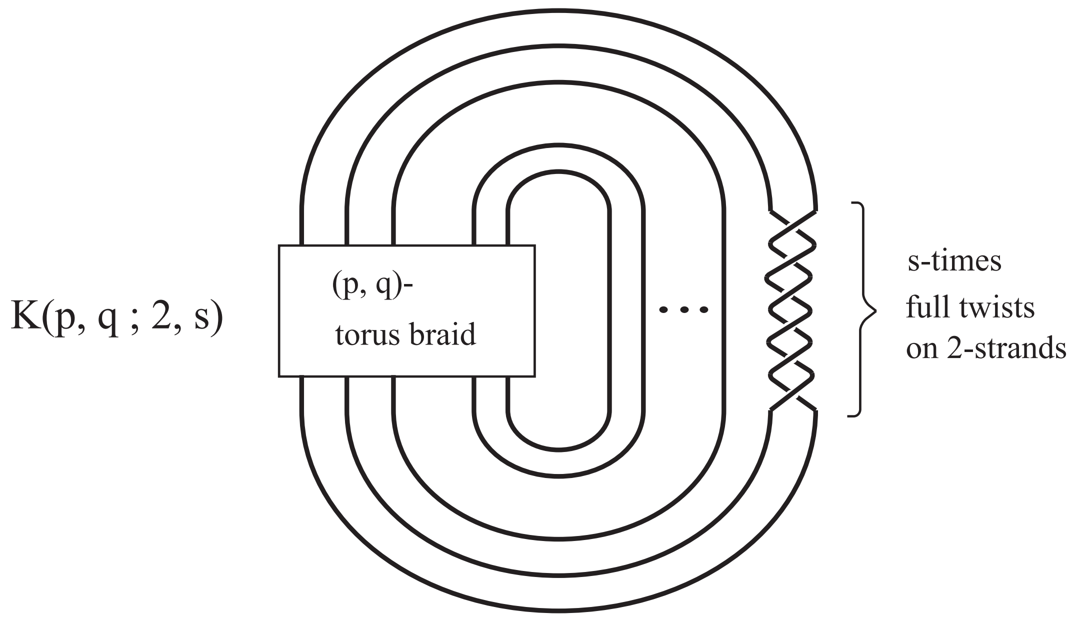
This knot is quoted as a twisted torus knot.

3. Results

Theorem 1

Put $r = 2$.

Then $K(p, q; 2, s)$ is small and m-small for any p, q and s.



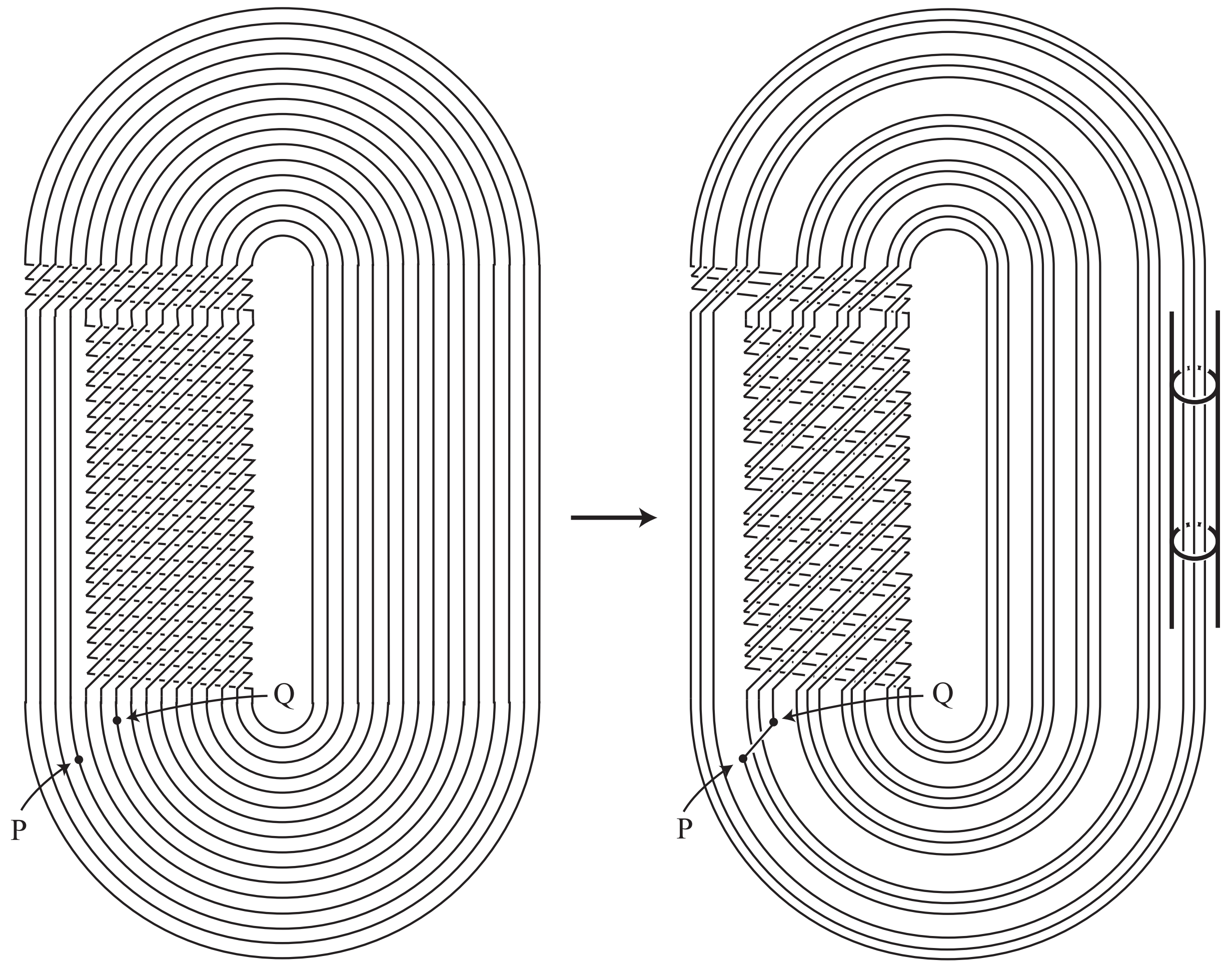
This knot has been denoted by $K(p, q; r)$ in [M].

Theorem 2 (with Y.Yamada)
For any composite number $r = km$ with $k > 1,\ m > 1$,
put $p = kn + 1$ with $n \geq m$ and $q = k$.
Then $K(p, q; r, s)$ is a satellite knot whose companion is the torus knot $T(m, ms + 1)$,
and the pattern is the torus knot $T(k, k(n + m^2 s) + 1)$.
Hence $K(p, q; r, s)$ is a k-cable knot along $T(m, ms + 1)$.

The follwoing figure is in the case when $k = 3, m = 4, n = 5$ and $s = 2$. Hence

$p = kn + 1 = 16$, $q = k = 3$, $r = km = 12$ and $s = 2$, i.e.,

$K(p, q; r, s) = K(16, 3; 12, 2)$.



K(16, 3 ; 12, 2) = a 3-cable knot along T(4, 9)

4. Applications

$t(K)$: the tunnel number of K

i.e., $t(K)$ is the minimal number of arcs in $E(K)$ whose complementary space is a handlebody.

$g_1(K)$: the 1-bridge genus of K

i.e., $g_1(K)$ is the minimal genus of Heegaard splittings of S^3 so that each handlebody of the Heegaard splitting interesects K in a single trivial arc.

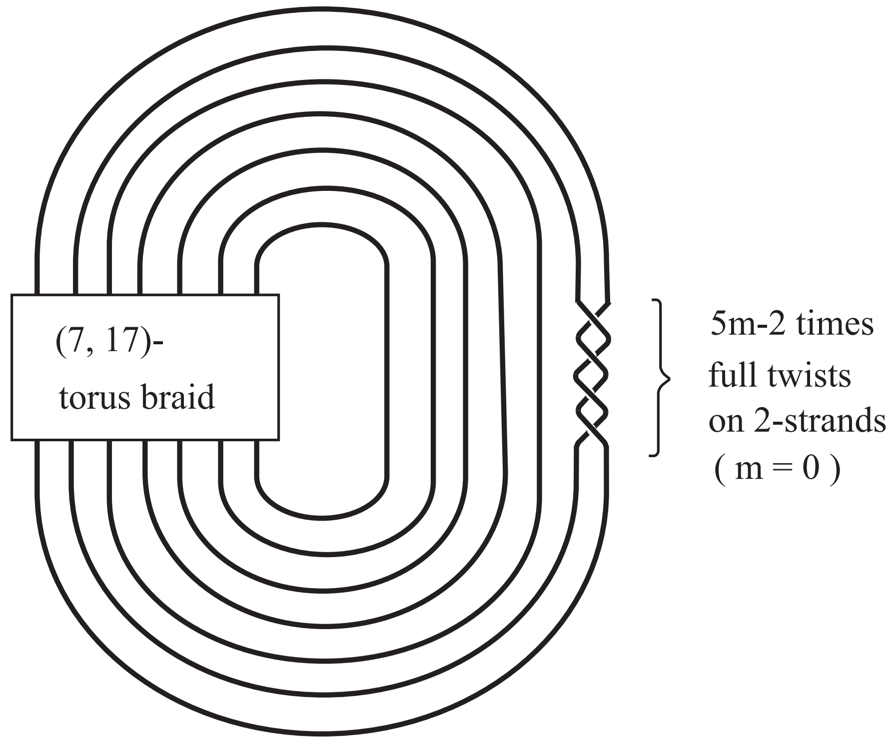
Then there is the following relation between $t(K)$ and $g_1(K)$.

Fact

$t(K) \leq g_1(K) \leq t(K) + 1$

Theorem 3 (M-Sakuma-Yokota '96)
$K = K(7, 17; 2, 5m - 2)$. Then $t(K) = 1$ and $g_1(K) = 2$.

K(7, 17 ; 2, -2)



The origin of the study in the present talk is in this theorem. This knot has been denoted by $K(7, 17; 10m - 4)$ in [M-S-Y].

On the additivity problem of 1-bridge genus of knots under connected sum, we have shown the following.

Theorem 4 (Hoiden-M '00-'05)
Suppose both K_1 and K_2 are m-small, then we get

$$g_1(K_1 \# K_2) \geq g_1(K_1) + g_1(K_2) - 1.$$

Question : Is the above inequality best possible ?

As an application of Theorem 1, we can show that the above inequality is best possible as follows.

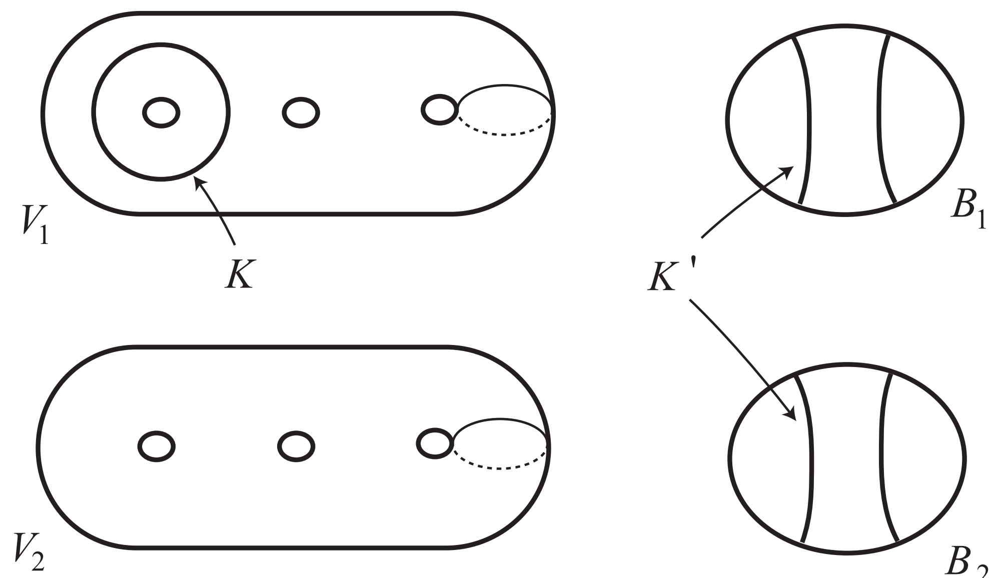
To porve the best possibility, we need the following lemma.

Lemma 5 (M '05)

Let K be a knot with $g_1(K) = t(K) + 1$.

Then we have $g_1(K\#K') \leq g_1(K)$ for any 2-bridge knot K'

Sketch proof of Lemma 5



Proof of the best possibility of Theorem 4

$K_1 : K(7,17;2,5m-2)$ for any integer m.

Then K_1 is m-small by Theorem 1.

$K_2 :$ a 2-bridge knot. Then $g_1(K_2) = 1$ and K_2 is m-small (well-known).

By Theorem 3, $g_1(K_1) = 2$ and $t(K_1) = 1$, i.e., $g_1(K_1) = t(K_1) + 1$.

Then by Lemma 5, $g_1(K_1 \# K_2) \leq g_1(K_1) = 2$.

On the other hand,

By Theorem 4, $g_1(K_1 \# K_2) \geq g_1(K_1) + g_1(K_2) - 1 = 2 + 1 - 1 = 2$.

Thus $g_1(K_1 \# K_2) = 2 = g_1(K_1) + g_1(K_2) - 1$.

This shows the best possibility of the inequality in Theorem 4.

On the additivity problem of tunnel number of knots under connected sum, we have shown the following.

Theorem 6 (M '00)
Let K_1 *and* K_2 *be both m-small. Then* $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ *if and only if* $g_1(K_i) = t(K_i) + 1$ *for both* $i = 1, 2$.

Concerning the above theorem, the following question occurs.

Do there exist such knots K that satisfies the hypothesis in Theorem 6, i.e., "m-small" and "$g_1(K) = t(K) + 1$" ?

Theorem 1 and Theorem 3 show the affirmative answer to the above question. In fact, $K(7, 17; 2, 5m - 2)$ satisfies the hypothesis, and this shows that Theorem 6 is not vanity.

5. Open problems

By Theorems 1 and 2, the following problems arise.

1. Do there exist $K(p,q;r,s)$ containing essential surfaces in the exteriors for prime number $r > 2$?

I want to conjecture that $K(p,q;r,s)$ contains no essential surfaces in the exteriors for any prime number r.

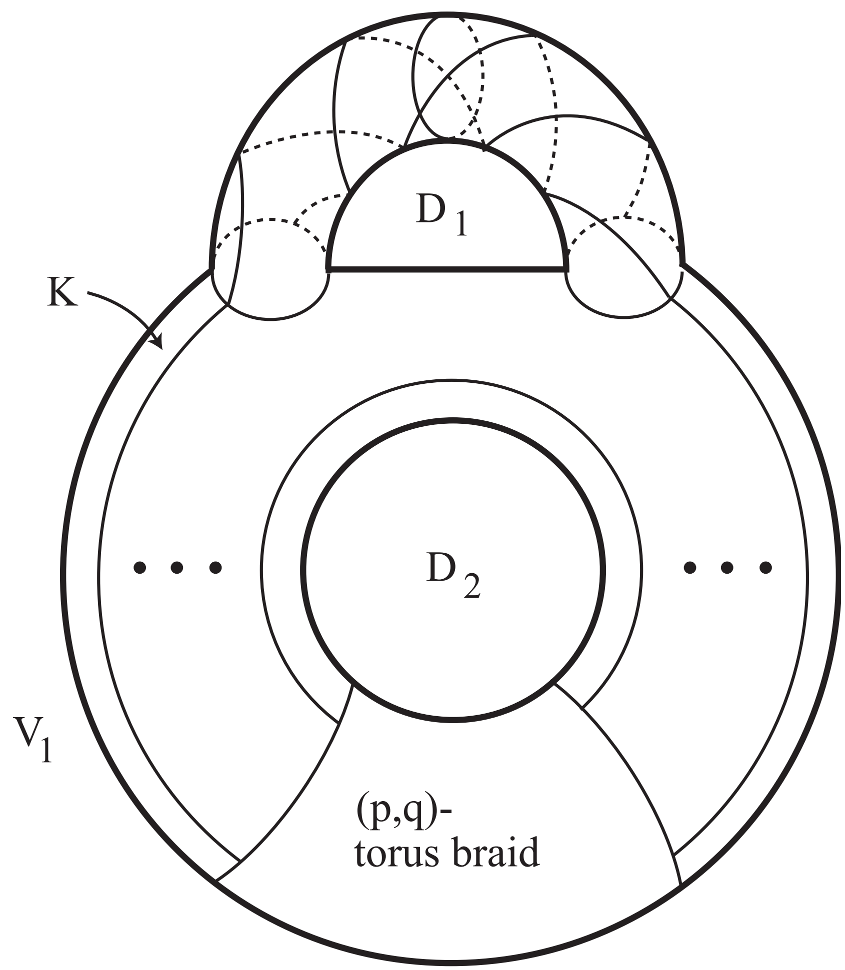
2. Do there exist $K(p,q;r,s)$ containing closed essential surfaces (not tori) in the exteriors for composite number r ?

5. Outline of the proof of Theorem 1

Put $K = K(p, q; 2, s)$.
K can not be put in the boundary of the standard solid torus in S^3, because K is not a torus knot.

But by regarding the twisted part as a 1-handle, we can put K in the boundary of the standard genus two handlebody, say V_1, in S^3.

Hence $K \subset \partial V_1$ as the following figure.


D 1
K
D 2
V 1
(p,q)-
torus braid

Suppose $E(K)$ contains a closed essential surface or
a meridionally essential surface, say F.

D_1, D_2 : two disks as in the figure,

i.e., two meridian disks of the complementary handlebody.

Then by considering the intersections of F and $D_1 \cup D_2$, and the intersections of F and ∂V_1, we can show that there is no such a surface F. The arguments are so complicated but elementary.