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Applications of braid group representations to dynamical systems

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1 Generalized Lefschetz number

X connected, finite cell complex

$f : X \to X$ continuous, $\mathrm{Fix}(f)$: fixed point set

Lefschetz number

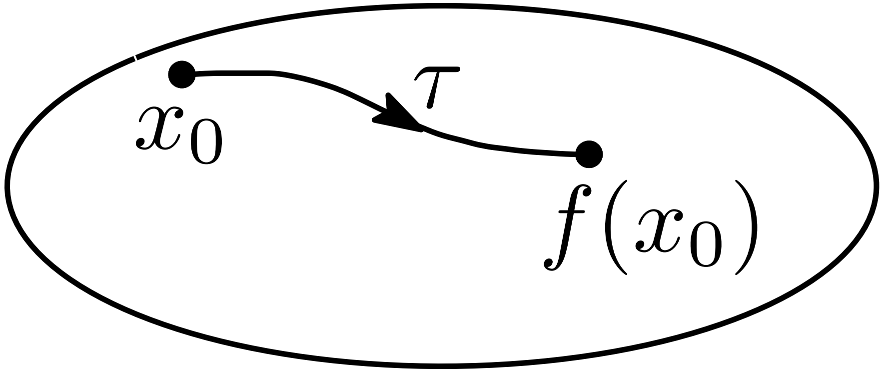
$L(f) = \sum_{q \geq 0} (-1)^q \mathrm{tr}\left[f_* : H_q(X) \to H_q(X)\right]$

$$= \mathrm{ind}(\mathrm{Fix}(f))$$

The generalized Lefschetz number $\mathcal{L}(f)$ is obtained by decomposing $\mathrm{Fix}(f)$ using the action of f on $\pi_1(X)$.

Choose a base point x_0, and a base path τ.

Let $\pi = \pi_1(X, x_0)$, $f_\pi = \tau_*^{-1} \circ f_* : \pi \longrightarrow \pi$.



Definition. $\lambda_1, \lambda_2 \in \pi$ are f_π-conjugate (or Reidemeister equivalent) if

$$^\exists \lambda \in \pi \text{ s.t. } \lambda_2 = f_\pi(\lambda)\lambda_1\lambda^{-1}$$

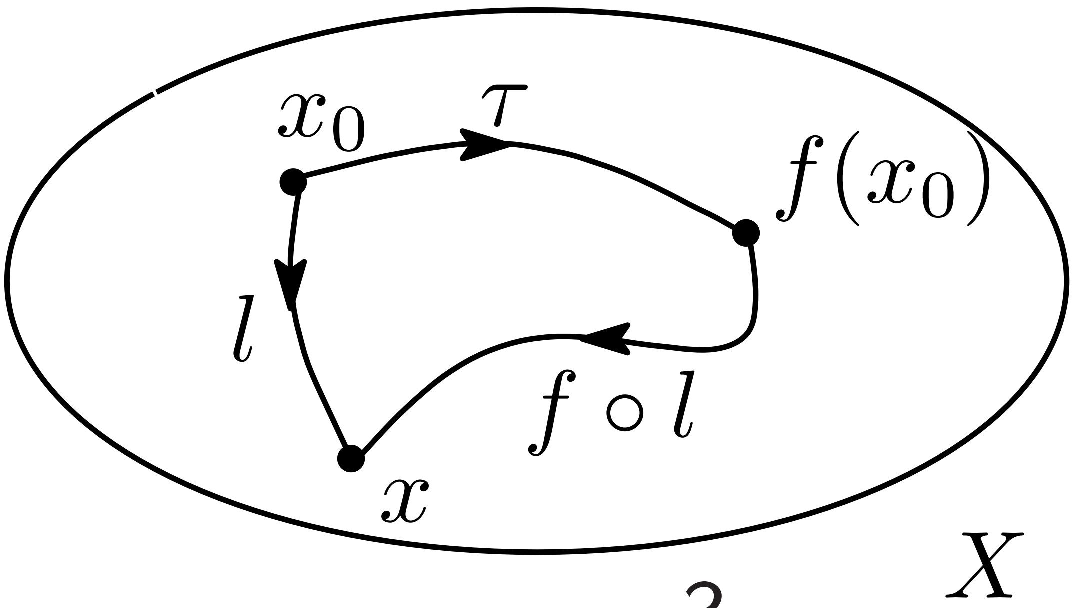
$\pi/f_\pi := \{f_\pi\text{-conjugacy classes}\}$

$\mathbb{Z}[\pi/f_\pi]$: free abelian group generated by π/f_π.

Define $R : \mathrm{Fix}(f) \to \pi/f_\pi$:

For $x \in \mathrm{Fix}(f)$, choose a path l from x_0 to x.

Let $R(x)$ be the f_π-conjugacy class represented by $[\tau(f \circ l)l^{-1}] \in \pi$ (Reidemeister class or coordinate of x) ($R(x)$ depends on x_0,τ.)



For $\alpha \in \pi/f_\pi$, let

$$\mathrm{Fix}_\alpha(f) = \{x \in \mathrm{Fix}(f) \mid R(x) = \alpha\}$$

: fixed point class determined by α.

$$\mathrm{Fix}(f) = \bigcup_{\alpha \in \pi/f_\pi} \mathrm{Fix}_\alpha(f) \quad \text{(disjoint union)}$$

There are only finitely many α with $\mathrm{Fix}_\alpha(f) \neq \emptyset$.

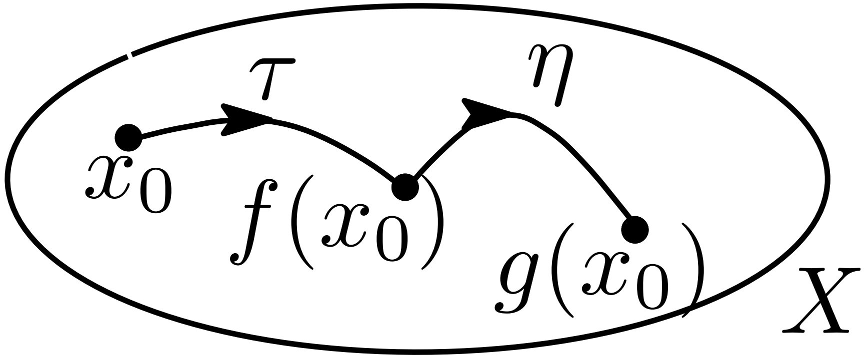
Definition (the gen. Lefschetz number)

$$\mathcal{L}(f) = \sum_{\alpha \in \pi/f_\pi} \mathrm{ind}(\mathrm{Fix}_\alpha(f)) \cdot \alpha \in \mathbb{Z}[\pi/f_\pi]$$

★Homotopy invariance

$f \sim g$, $\{h_t\}$: homotopy between f and g.

$\tau \cdot \eta$: a base path for g, where $\eta(t) = h_t(x_0)$.



Then $f_\pi = g_\pi$ and

$$\mathcal{L}(f) = \mathcal{L}(g) \in \mathbb{Z}[\pi/f_\pi] = \mathbb{Z}[\pi/g_\pi]$$

★ $\mathcal{L}(f)$ is a generalization (refinement) of $L(f)$. (The sum of coefficients in $\sum_\alpha \mathrm{ind}(\mathrm{Fix}_\alpha(f) \cdot \alpha$ is equal to $L(f)$.)

$$\because \sum_\alpha \mathrm{ind}(\mathrm{Fix}_\alpha(f)) = \mathrm{ind}\Big(\bigcup_\alpha \mathrm{Fix}_\alpha(f)\Big)$$
$$= \mathrm{ind}(\mathrm{Fix}(f)) = L(f).$$

Problem: *Compute* $\mathcal{L}(f)$!

★Reidemeister trace formula

(Reidemeister 1936, Wecken 1941, Husseini 1982**)**

$L(f) = \sum_{q \geq 0} (-1)^q \mathrm{tr}\,[f_{\sharp q} : C_q(X) \to C_q(X)]$

$\tilde{X}$: universal covering space

$C_q(\tilde{X})$: finitely gen. free $\mathbb{Z}[\pi]$-module

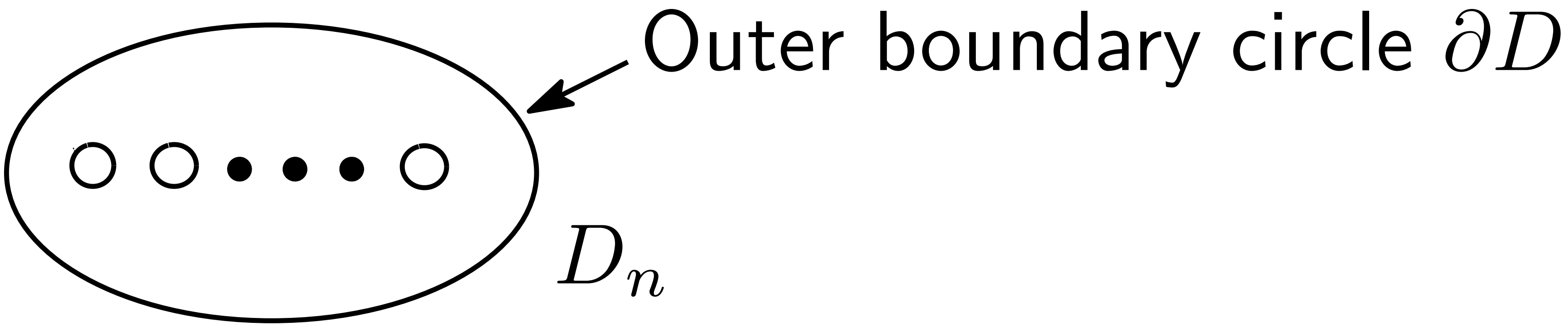
$$\mathrm{tr}[\tilde{f}_{\sharp q} : C_q(\tilde{X}) \to C_q(\tilde{X})] \in \mathbb{Z}[\pi / f_\pi]$$

is independent of the choice of a basis of $C_q(\tilde{X})$. (Reidemeister trace**)**

Trace formula: $\mathcal{L}(f) = \sum_{q \geq 0} (-1)^q \mathrm{tr} \tilde{f}_{\sharp q}$

2 Homeomorphisms on a punctured disk

Assume $n \geq 3$. Let D_n be a compact n-punctured disk. ($D_n = D - \{n \text{ open disks}\}$)



Consider a homeomorphism $f : D_n \to D_n$ preserving orientation and ∂D setwise.

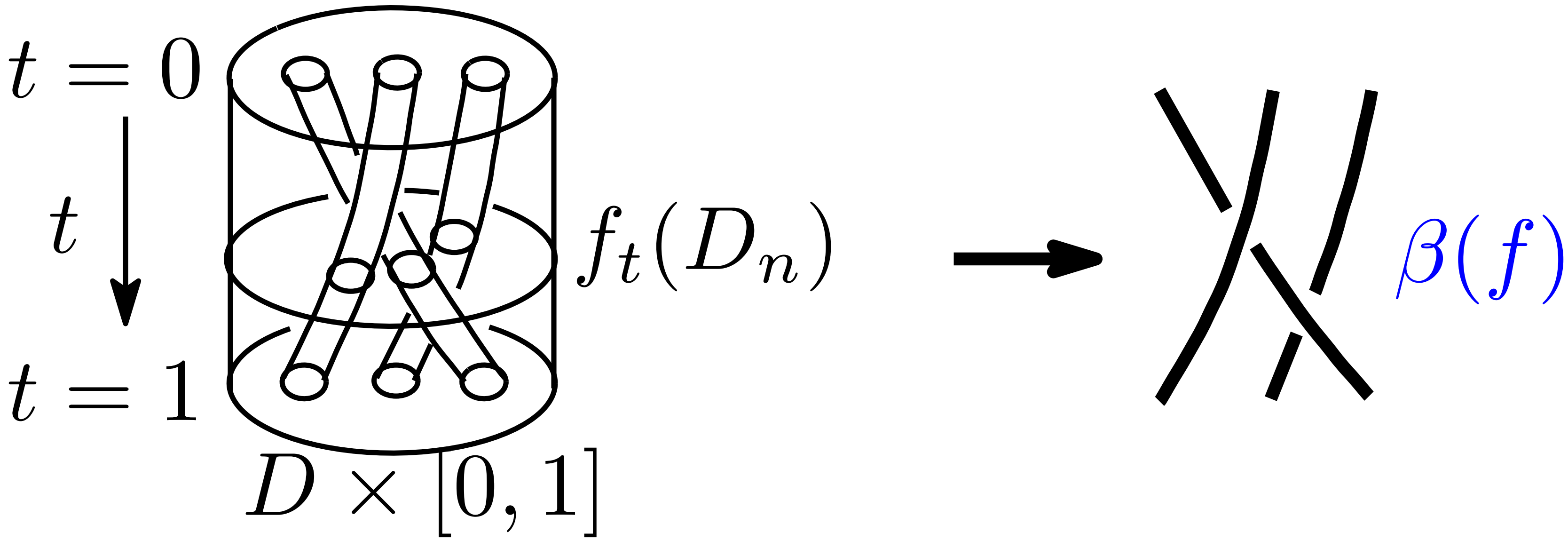
B_n : braid group, θ : full-twist braid

Center of $B_n = \{\theta^m \mid m \in \mathbb{Z}\}$

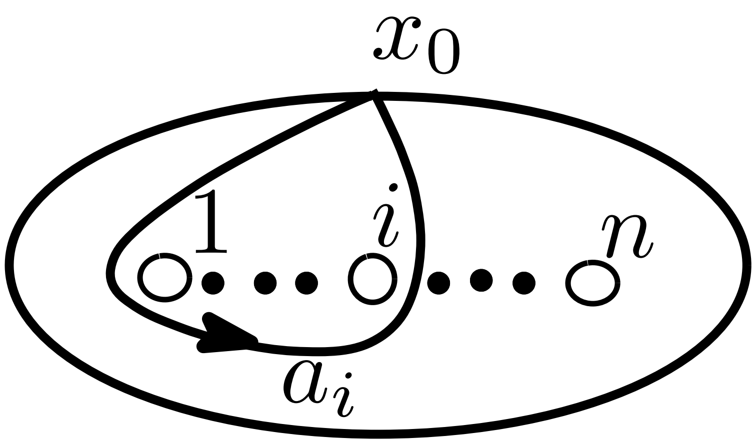
$$\mathrm{Iso}_+(D_n, \partial D) \longleftrightarrow B_n/\mathrm{Center}$$

($[f]$ corresponds to a braid $\beta(f)$ mod Center)

$\exists$ isotopy $f_t : D \to D$ s.t. $f_0 = \mathrm{id}$, $f_1|_{D_n} = f$



A surface with boundary is homotopy equiv. to a wedge of circles X. $\tilde{f}_{\sharp 1} : C_1(\tilde{X}) \to C_1(\tilde{X})$ coincides with the Jacobian matrix $J(f_\pi)$ w.r.t. Fox differential calculus., and hence $\mathcal{L}(f) = [1 - \mathrm{tr} J(f_\pi)]$ (Fadell & Husseini, 1983).



$x_0 \in \partial D$

$\pi_1(D_n) = F_n$: free group with generators $a_1, \ldots, a_n$

$$J(f_\pi) = (\partial f_\pi(a_i)/\partial a_j)$$

B_n acts on F_n by

$$\sigma_i : a_j \longrightarrow \begin{cases} a_{i+1} a_i^{-1} a_{i-1} & j = i \\ a_j & j \neq i. \end{cases}$$

Denote the image of $w \in F_n$ under β by w^β.

Since $a_n^\beta = a_n$, we have $\partial a_n^\beta / \partial a_j = \delta_{n,j}$.

Therefore, $J(\beta)$ has the form $\begin{pmatrix} J_r(\beta) & * \\ 0 & 1 \end{pmatrix}$.

$J_r : B_n \to GL_{n-1}(\mathbb{Z}[F_n])$ gives a twisted representation.

$$(J_r(\beta\beta') = J_r(\beta)^{\beta'} J_r(\beta'))$$

Relationship between J_r and fixed points

As a base path, choose $\tau(t) = f_t(x_0)$.
Then, $f_\pi = \beta(f) : F_n \to F_n$, $\pi/f_\pi = F_n/\beta(f)$.

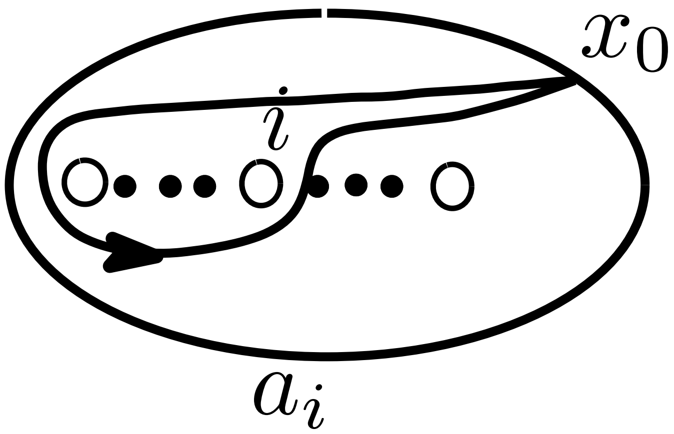
$$\begin{aligned}\mathcal{L}(f) &= [\mathrm{tr}\tilde{f}_{\sharp 0} - \mathrm{tr}\tilde{f}_{\sharp 1}] = [1 - \mathrm{tr}J(\beta(f))] \\ &= -[\mathrm{tr}J_r(\beta(f))] \in \mathbb{Z}[F_n/\beta(f)]\end{aligned}$$

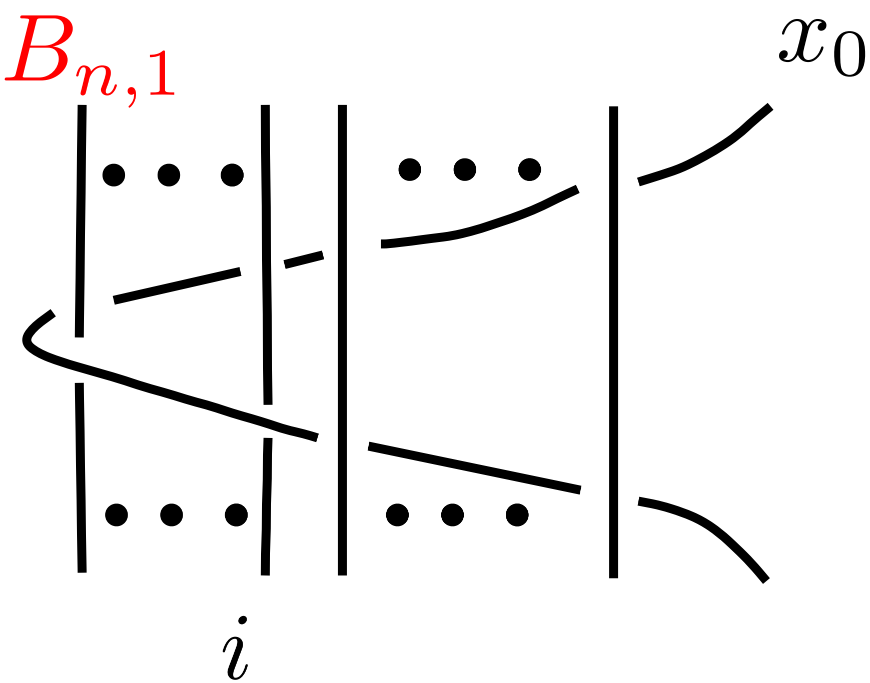
$\mathcal{L}(f)$ depends on the choice of $\{f_t\}$, but is uniquely determined up to multiples of a_n.
(Note that $J_r(\theta^\mu \beta) = a_n^\mu J_r(\beta)$) .

★ A genuine representation can be made from J_r (B. Jiang) .

$B_{n,1}$: the subgroup of B_{n+1} consisting of braids with associated permutation fixing $n+1$.

$B_n \subset B_{n,1}$, $\quad F_n \subset B_{n,1}$





For ${}^{\forall} w \in F_n, {}^{\forall} \beta \in B_n$, $\quad w\beta = \beta w^{\beta} \in B_{n,1}$.

(i.e., $w^{\beta} = \beta^{-1} w \beta$)

Define $\zeta : B_n \to GL_{n-1}(\mathbb{Z}[B_{n,1}])$ by

$$\zeta(\beta) = \beta J_r(\beta).$$

ζ is a representation.

$(\because \zeta(\beta\beta') = \beta\beta' J_r(\beta\beta') = \beta\beta'(J_r(\beta)^{\beta'} J_r(\beta'))$
$= \beta J_r(\beta)\beta' J_r(\beta').)$

$(B_{n,1})_c$: the set of congugacy classes of $B_{n,1}$

★ $F_n/\beta \subset (B_{n,1})_c$ can be defined by

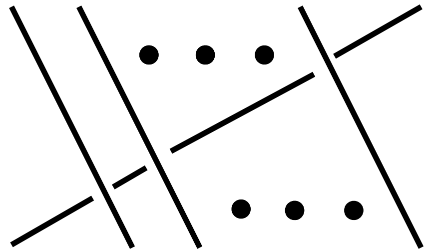
$$[w] \to [\beta w].$$

Note that $[\mathrm{tr} J_r(\beta)] \mapsto [\mathrm{tr} \beta J_r(\beta)] = [\mathrm{tr}\zeta(\beta)]$.

$$\mathcal{L}(f) = -[\mathrm{tr} J_r(\beta(f))] \in \mathbb{Z}[F_n/\beta(f)]$$

$$= -[\mathrm{tr}\zeta(\beta(f))] \in \mathbb{Z}[(B_{n,1})_c]$$

An expression of braids

$\rho = \sigma_{n-1} \cdots \sigma_2 \sigma_1 =$ 

For $I = (i_1, \ldots, i_d) \in \mathbb{N}^d$, let

$$\beta(I) = \beta(i_1, \ldots, i_d) = \sigma_1^{i_1} \rho \cdots \sigma_1^{i_d} \rho.$$

Then, $^\forall$braid is conjugate to

$$\theta^\mu \beta(I) \quad (\mu \in \mathbb{Z}, I \in \mathbb{N}^d) \text{ (M,1993)}$$

Example. $\sigma_1 \sigma_2^{-1} \sim \beta(4) \in B_3$.

Remark. This expression is not unique.
(Ex. $\beta(i, \underbrace{1, \ldots, 1}_{n-2}, j) = \theta\beta(i + j - 1)$.)

$$I \to \beta(I) \longleftrightarrow [f] \to \mathcal{L}(f)$$

(up to Center)

Problem: Determine $\mathcal{L}(f)$ directly by I.

3 Abelianization of $\mathcal{L}(f)$

$\mathcal{E} : F_n \to \mathbb{Z} \quad (\mathcal{E}(a_i) = i)$ induces

$\mathcal{E} : \mathbb{Z}[F_n] \to \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[q, q^{-1}]$, and then

$\mathcal{E} : \mathbb{Z}[F_n/\beta] \to \mathbb{Z}[q, q^{-1}]$ for ${}^\forall \beta \in B_n$.

$\mathcal{E}(J_r(\beta)) = \mathrm{Bur}(\beta)$: the reduced Burau matrix

Since $\mathcal{L}(f) = -[\mathrm{tr} J_r(\beta(f))]$,

we have $\quad \mathcal{E}(\mathcal{L}(f)) = -\mathrm{tr}\,\mathrm{Bur}(\beta(f))$.

If $\beta(f) = \theta^\mu \gamma^{-1} \beta(I) \gamma$,

$$\mathcal{E}(\mathcal{L}(f)) = -q^{n\mu} \mathrm{tr}\,\mathrm{Bur}(\beta(I)).$$

★ $\operatorname{tr}\operatorname{Bur}(\beta(I))$ has been computed

(M, Contemp. Math. Vol. 152(1993)).

$\operatorname{Bur}(\beta(i)) =$

$$\begin{pmatrix} -\sum_{\ell=2}^{i}(-q)^\ell & (-q)^i & \sum_{\ell=0}^{i-1}(-q)^\ell & 0 & \cdots & 0 \\ -q^2 & 0 & 1 & & & \vdots \\ -q^3 & 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 1 \\ -q^{n-1} & 0 & 0 & \cdots & & 0 \end{pmatrix}$$

$\operatorname{tr}\operatorname{Bur}(\beta(i)) = -\sum_{\ell=2}^{i}(-q)^\ell.$

Next, consider $\beta(i)^d = \beta(\underbrace{i, \ldots, i}_{d})$.

For $d \in \mathbb{N}$, let $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z} = \{1, \ldots, d\}$.
For $1 \leq p, q \leq d$, define a sequence $[p, q]$ of consecutive elements of $\mathbb{Z}_d$ by

$$[p, q] = \begin{cases} (p, \ldots, q) & p \leq q \\ (p, \ldots, d, 1, \ldots, q) & p > q \end{cases}$$

: a block in $\mathbb{Z}_d$

Definition

(1) A set of blocks $\{B_1, \cdots, B_s\}$ is a partition of $\mathbb{Z}_d$

$\Longleftrightarrow \quad \mathbb{Z}_d = \underline{B}_1 \cup \cdots \cup \underline{B}_s,$ (disjoint)

where $\underline{B}_r$ is the set of integers contained in B_r.

(2) $\mathcal{P}(d) = \{$partitions of $\mathbb{Z}_d\}$

Example **:**

$\mathcal{P}(4)$ consists of 15 partitions:

$$\{(1),(2),(3),(4)\},$$
$$\{(1,2),(3),(4)\},\quad \{(1),(2,3),(4)\},$$
$$\{(1),(2),(3,4)\},\quad \{(2),(3),(4,1)),$$
$$\{(1,2),(3,4)\},\quad \{(2,3),(4,1)\},$$
$$\{(1,2,3),(4)\},\quad \{(1),(2,3,4)\},$$
$$\{(2),(3,4,1)\},\quad \{(3),(4,1,2)\},$$
$$\{(1,2,3,4)\},\ \{(2,3,4,1)\},\ \{(3,4,1,2)\},\ \{(4,1,2,3)\}$$

Fact on linear algebra

Let $A \in M_\nu(R)$, $\nu \in \mathbb{N}$, R: commutative ring.

Let $\mathrm{PM}(A;k) = 0$ for $k > \nu$, and for $1 \leq k \leq \nu$

$$\mathrm{PM}(A;k) = \sum_{1 \leq j_1 < \cdots < j_k \leq \nu} \det A \begin{pmatrix} j_1, \ldots, j_k \\ j_1, \ldots, j_k \end{pmatrix}$$

: the sum of principal minors of order k.

$$\mathrm{tr} A^d$$
$$= \sum_{\mathcal{B} \in \mathcal{P}(d)} (-1)^{d + \sharp \mathcal{B}} \mathrm{PM}(A; |B_1|) \cdots \mathrm{PM}(A; |B_s|),$$

where $\mathcal{B} = \{B_1, \ldots, B_s\}$.

Example.

$\underline{d=2}$. Since $\mathcal{P}(2) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$, where
$\mathcal{B}_1 = \{(1),(2)\}, \mathcal{B}_2 = \{(1,2)\}, \mathcal{B}_3 = \{(2,1)\}$,

$$\begin{aligned}\mathrm{tr}A^2 &= (-1)^4\mathrm{PM}(A;1)^2 + 2(-1)^3\mathrm{PM}(A;2)\\ &= (\mathrm{tr}A)^2 - 2\,\mathrm{PM}(A;2).\end{aligned}$$

$\underline{d=3}$

$\mathrm{tr}A^3 = (\mathrm{tr}A)^3 - 3\,\mathrm{PM}(A;2)\mathrm{tr}A + 3\,\mathrm{PM}(A;3).$

Let $R = \mathbb{Z}[q, q^{-1}]$ and $A = \mathrm{Bur}(\beta(i))$.
Define a polynomial $P(i; k)$ by

$$P(i,k) = \mathrm{PM}(\mathrm{Bur}(\beta(i)); k)$$
$$= \begin{cases} -\sum_{\ell=k+1}^{i+k-1} (-q)^{\ell} & \text{if} \quad k < n-1 \\ (-q)^{i+n-1} & \text{if} \quad k = n-1. \end{cases}$$

$$\mathrm{tr}\,\mathrm{Bur}(\beta(i)^d) = \mathrm{tr}(\mathrm{Bur}(\beta(i)))^d$$
$$= \sum_{\mathcal{B} \in \mathcal{P}(d)} (-1)^{d + \sharp \mathcal{B}} P(i; |B_1|) \cdots P(i; |B_s|).$$

$$\begin{aligned}&\operatorname{tr}\mathrm{Bur}(\beta(i,\ldots,i))\\&\quad=\sum_{\mathcal{B}\in\mathcal{P}(d)}(-1)^{d+\sharp\mathcal{B}}P(i;|B_1|)\cdots P(i;|B_s|).\end{aligned}$$

Theorem 1. (M,1993)
For ${}^{\forall}I=(i_1,\ldots,i_d)\in\mathbb{N}^d$,

$$\begin{aligned}&\operatorname{tr}\mathrm{Bur}(\beta(I))\\&=(-1)^{d+\sharp\mathcal{B}}\sum_{\mathcal{B}\in\mathcal{P}(d)}P(i_{p_1};|B_1|)\cdots P(i_{p_s};|B_s|),\end{aligned}$$

where $\mathcal{B}=\{B_1,\ldots,B_r\}$ and p_r is the initial element of B_r $(r=1,\ldots,s)$.

4 Computation of $\mathcal{L}(f)$

For $\beta \in B_n$, let $\Phi_\beta : \mathbb{Z}F_n \to \mathbb{Z}[F_n/\beta]$ be the projection. Assume $\beta(f) = \beta(I)$.

Find element $\in \mathbb{Z}[F_n]$

$$\downarrow \qquad \downarrow \Phi_{\beta(f)}$$

$$\left(\text{element} = \sum_{\mathcal{B} \in \mathcal{P}(d)} W_I(\mathcal{B}) \right) \qquad \mathcal{L}(f) \in \mathbb{Z}[F_n/\beta(f)]$$

$$\downarrow \qquad \downarrow \mathcal{E}$$

$$\sum_{\mathcal{B} \in \mathcal{P}(d)} P_I(\mathcal{B}) = -\mathrm{trBur}(\beta(I)) \ni \mathbb{Z}[q, q^{-1}]$$

The Jacobian matrix $J_r(\beta(I))$ can be computed by

$$a_k^{\beta(i)} = \begin{cases} (a_3a_1^{-1})^{\frac{i-1}{2}} a_3a_2^{-1}(a_3a_1^{-1})^{-\frac{i-1}{2}} & (k = 1, i\ odd) \\ (a_3a_1^{-1})^{\frac{i}{2}} a_2a_1^{-1}(a_3a_1^{-1})^{-\frac{i}{2}} & (k = 1, i\ even) \\ a_{k+1}a_1^{-1} & (2 \leq k \leq n-1) \\ a_n & (k = n) \end{cases}$$

$$\mathbb{Z}[F_n] \ni g_j = \begin{cases} -a_2^{j/2} & j \text{ even} \\ a_1 a_2^{(j-1)/2} & j \text{ odd} \end{cases}$$

For $1 \leq \ell \leq d$, let $\beta_\ell(I) = \beta(i_\ell, \ldots, i_d) \in B_n$.

For a block $B = [p, q]$, define $\alpha(B), \omega(B) \in \mathbb{Z}[F_n]$ by

$$\alpha(B) = \beta_p(I),$$

$$\omega(B) = \begin{cases} \beta_q(I) & p \leq q \\ \beta_q(I)\beta(I)^{-1} & p > q. \end{cases}$$

For a block B, define $W_I(B) \in \mathbb{Z}[F_n]$ by

$$W_I(B) = \begin{cases} (g_0 + \cdots + g_{i_p - 2})^{\alpha(B)} a_{|B|+1}^{\omega(B)} & \text{if} \quad |B| < n-1 \\ g_{i_p}^{\alpha(B)} a_{n-1}^{\omega(B)} & \text{if} \quad |B| = n-1 \\ 0 & \text{if} \quad |B| \geq n. \end{cases}$$

Define $W_I : \mathcal{P}(d) \to \mathbb{Z}[F_n]$ by

$$W_I(\mathcal{B}) = W_I(B_1) \cdots W_I(B_s)$$

for $\mathcal{B} = \{B_1, \ldots, B_s\}$ ($1 \leq p_1 \leq \cdots \leq p_s \leq d$)

Theorem 2.

$$\beta(f) = \theta^{\mu}\gamma^{-1}\beta(I)\gamma \qquad (\mu \in \mathbb{Z}, \gamma \in B_n)$$

$$\Longrightarrow \quad \mathcal{L}(f) = -\Phi_{\beta(f)}\left(a_n^{\mu} \sum_{\mathcal{B}\in\mathcal{P}(d)} W_I(\mathcal{B})^{\gamma}\right).$$

$$\in \mathbb{Z}[F_n/\beta(f)].$$

5 Applications

A surface homeomorphism φ is a canonical homeomorphism if it is finite order, pseudo-Anosov, or reducible (decomposed into finite order and pseudo- Anosov components).

A canonical homeomorphism has the "simplest" dynamical complexity in its isotopy class.

Let $\varphi : D_n \to D_n$ be a canonical homeo. preserving orientation and ∂D (setwise).

Problem 1 **:** *Determine the period and rotation number of periodic points of* φ *on* ∂D**.**

The following lemma shows Theorem 2 can be applied to solve this problem.

Lemma. $m \in \mathbb{N}, \nu \in \mathbb{Z}$: rel. prime. Assume $\Phi_{\beta(\varphi^m)}(a_n^\nu)$ has non-zero coefficient in $\mathcal{L}(\varphi^m)$ Then, the periodic points on ∂D has period m and rotation number ν/m.

Suppose $\beta(\varphi) = \beta(I)$, and
$i_1, \ldots, i_d \geq 2$ if $n \geq 4$, $i_1, \ldots, i_d \geq 3$ if $n = 3$.

Proposition 1. ${}^{\exists}\mathcal{P}'(d) \subset \mathcal{P}(d)$, ${}^{\exists}W'_I(\mathcal{B}) \in \mathbb{Z}[F_n]$
s.t. (i) $\mathcal{L}(f) = -\Phi_{\beta(I)}\left(\sum_{\mathcal{B}\in\mathcal{P}'(d)} W'_I(\mathcal{B})\right)$
(ii) If $w \in F_n$ has non-zero coefficient in $\sum_{\mathcal{B}\in\mathcal{P}'(d)} W'_I(\mathcal{B})$, then $[w] \in F_n/\beta(I)$ has non-zero coefficient in $\mathcal{L}(f)$.

Proposition 2. period $= \mathrm{LCM}\{d, n-2\}/d$,
rotation number $= d/(n-2)$

Problem 2 **:** *Given f*, *determine the finite-order and pseudo-Anosov components of the canonical homeomorphism φ in $[f]$.*

There are several algorithms (Bestvina-Handel(1995), Benardete, Gutierrez, and Nitecki(1995), Hamidi-Tehrani and Chen (1996) etc.). But, the computation is still difficult.

As an application of the computation of the Burau matrices, we have

Theorem 3(M, 1993). If $n \geq 4$, $i_1, \ldots, i_d \geq 2$, $(i_1, \ldots, i_d) \neq (2, \ldots, 2)$, then φ has a pseudo-Anosov component.

As an application of Proposition 2 (period $= \mathrm{LCM}\{d, n-2\}/d$, rot. number $= d/(n-2)$ on ∂D), we have

Proposition 3. $n \geq 5$, $i_1, \ldots, i_d \geq 2$ all even or all odd. Then, φ is pseudo-Anosov with foliations having no interior singularities.

Estimations of the Nielsen number

(Corollary of Proposition 1)

$N(f)$: the number of non-zero terms in $\mathcal{L}(f)$.

(Nielsen number)

In the case of $n = 3$, $i_1, \ldots, i_d \geq 3$, we have

$$\sharp S(I) - 4 \leq N(f) \leq \sharp S(I),$$

where $S(I)$ is the set of $(j_1, \ldots, j_d) \in \mathbb{N}^d$ with $2 \leq j_\ell \leq i_\ell$, $(j_\ell, j_{\ell+1}) \neq (i_\ell, 2)$ for $1 \leq^\forall \ell \leq d$.

Also, a similar estimation is obtained if $n \geq 4$, $i_1, \ldots, i_d \geq 2$.

6 Braid representations for periodic orbits

H. Zheng(J. Knot Th. Ramif. (2005))generalized $\zeta : B_n \to GL_{n-1}(\mathbb{Z}[B_{n,1}])$ to representations $\zeta_{n,m} : B_n \to GL_N(\mathbb{Z}[B_{n,m}])$ for any $m \geq 2$,

where $N = \binom{n+m-2}{n-2}$ and
$B_{n,m}$ is the subgroup of B_{n+m} consisting of braids with associated permutation fixing the subset $\{n+1, \ldots, n+m\}$.

$f : D_n \to D_n$ induces a homeomorphism $f_m : C_m(D_n) \to C_m(D_n)$, where $C_m(D_n)$ is the m-th configuration space of D_n.

$$\{m\text{-periodic orbits }\} \subset \mathrm{Fix}(f_m).$$

$B_m(D_n) \subset B_{n,m}$. $\Gamma_{\beta,m} = \langle \beta, B_m(D_n) \rangle \subset B_{n,m}$.

$$\pi_1(C_m(D_n))/\beta = B_m(D_n)/\beta \subset (\Gamma_{\beta,m})_c$$
$$([w] \mapsto [\beta w])$$

$$\mathcal{L}(f_m) \in \mathbb{Z}[\pi_1(C_m(D_n))/\beta] \subset \mathbb{Z}[(\Gamma_{\beta(f),m})_c]$$

$\zeta_{n,m} : B_n \to GL_N(\mathbb{Z}[B_{n,m}])$ satisfies

$$\zeta_{n,m}(\beta) \in \mathbb{Z}[\Gamma_{\beta,m}] \text{ for any } \beta.$$

Theorem 4.(Jiang and Zheng, Topology(2007))

$$\mathcal{L}(f_m) = (-1)^m \mathrm{tr}\zeta_{n,m}(\beta(f)) \in \mathbb{Z}[(\Gamma_{\beta(f),m})_c]$$

up to collapsible terms.

Case of $m = 2$.

$B_{n,2} = \langle \sigma_1, \ldots, \sigma_{n-1}, \sigma_n^2, \sigma_{n+1} \rangle$.

Consider $\rho : \mathbb{Z}[B_{n,2}] \to \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ defined by

$\rho(\sigma_i) = 1 (i < n), \ \ \rho(\sigma_n^2) = q, \ \ \rho(\sigma_{n+1}) = t.$

Then,

$\rho \circ \zeta_{n,2} = \mathrm{LK} : B_n \to GL_{\frac{n(n-1)}{2}}(\mathbb{Z}[q^{\pm 1}, t^{\pm 1}])$.

(the Lawrence-Krammer representation)

As a corollary of Jiang-Huang(2007) and Zheng(2005), we have

Corollary. For any integers i,j with i odd, the coefficient of $q^i t^j$ in $\rho(\mathcal{L}(f_2))$ coincides with that of $\operatorname{tr}\mathrm{LK}(\beta(f)^{-1})^*$. where $q^* = q^{-1}, t^* = t^{-1}$.

Problem. *Compute* $\operatorname{tr}\mathrm{LK}(\beta)$.

Clearly,

$$\operatorname{tr} \mathrm{LK}(\beta(i,\ldots,i)) = \sum_{\mathcal{B}\in\mathcal{P}(d)} (-1)^{d+\sharp\mathcal{B}} Q(i;|B_1|)\cdots Q(i;|B_s|),$$

where $Q(i;k) = \mathrm{PM}(\mathrm{LK}(\beta(i));k)$.

How to generalize this to any $I \in \mathbb{N}^d$?

Let

$$Q_{i,j} = (-1)^{j+1}t(1-t)\sum_{k=j+6}^{i+j+4}(-q)^k\sum_{l=0}^{j-2}(tq)^l$$
$$+(-1)^{j+1}tq^{j+2}Q(i;1)$$
$$+(-1)^{i+j+1}t^2q^{i+j+4} + (-1)^j t^{j+1}q^{2j+4}$$

Note that $Q_{i,i} = Q(i;2) = \mathrm{PM}(\mathrm{LK}(\beta(i);2)$.

Then, $\mathrm{tr}\,\mathrm{LK}(\beta(i,j))$

$$= Q(i;1)Q(j;1) - Q_{i,j} - Q_{j,i}.$$