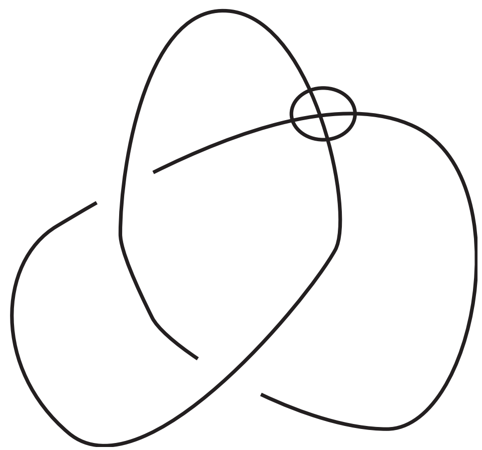
Bridge presentation of virtual knots

Seiichi Kamada

Joint work with Mikami Hirasawa and Naoko Kamada

January, 2008

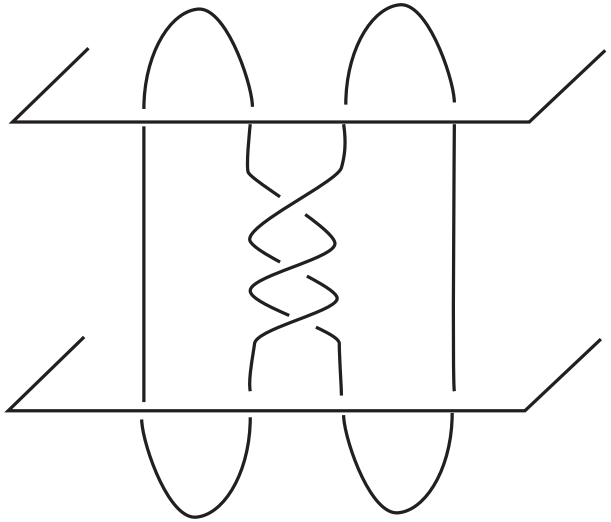
$\{\text{knots}\} \subset \{\text{virtual knots}\}$



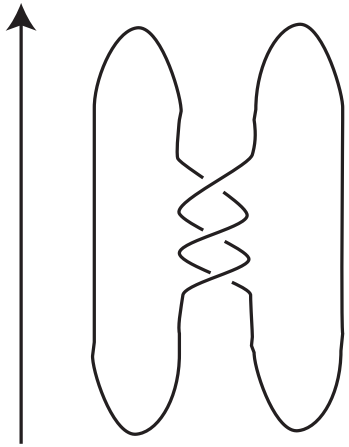
n-bridge knots $\Longrightarrow$ "n-bridge virtual knots"

Bridge presentation of classical knots

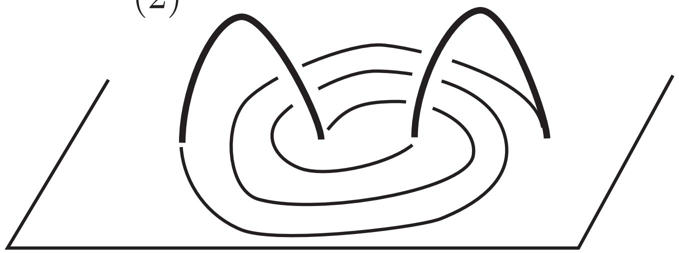
(1)



(3) diagram

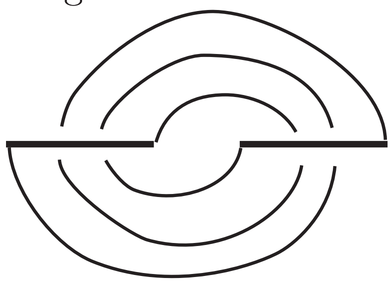


(2)



diagram

(4)



Remark. For classical knots, the following (A) and (B) are equivalent.

(A) K has a diagram with n over-paths and n under-paths.

(B) K has a diagram with n maximal points and n minimal points.

For virtual knots, (A) seems more natural to define "n-bridge presentation". We adapt our definition of bridge presentation to (A).

Definition. (Bridge presentation of a diagram D)
An n-bridge presentation of D is a division of D into n over-paths (paths without under-crossings) and n under-paths (paths without over-crossings) appearing alternately.

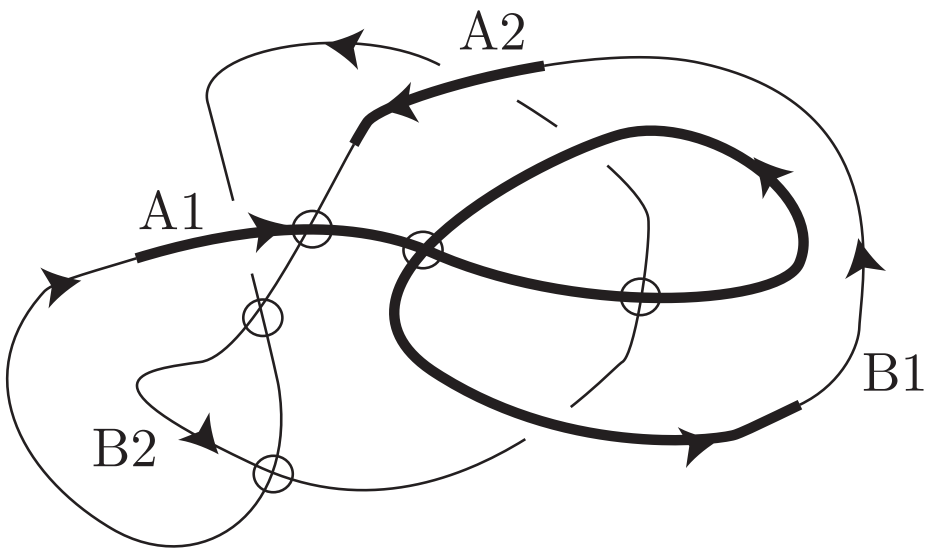


diagram with a 2-bridge presentation

Definition. (Virtual bridge index of K)
$b(K) = \min\{n \mid K \text{ has a diagram with an } n\text{-bridge presentation}\}$

Proposition 1 *If $b(K) = n$ then $Q_+(K)$ (upper knot quandle) and $Q_-(K)$ (lower knot quandle) can be generated by n elements. And hence $G_+(K)$ (upper knot group) and $G_-(K)$ (lower knot group) can be generated by n meridian elements.*

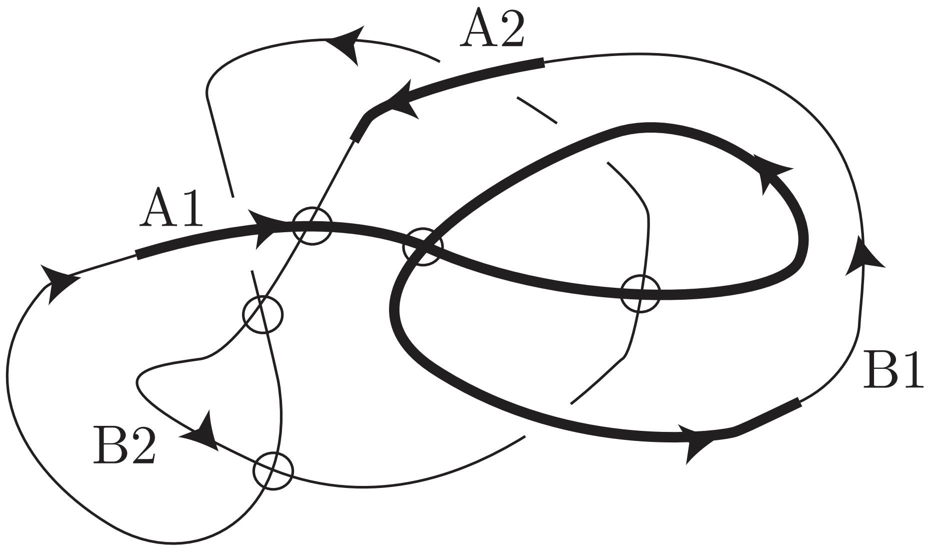


diagram with a 2-bridge presentation

$$Q_+(K) = \langle A_1, A_2 \mid A_1 = A_2, A_2^{A_1 A_1^{-1} A_2^{-1} A_1^{-1}} = A_1 \rangle \cong \langle A_1 \mid \ \rangle$$
$$Q_-(K) = \langle B_1, B_2 \mid B_1^{B_2^{-1}} = B_2, B_2^{B_2^{-1} B_2^{-1} B_2} = B_1 \rangle \cong \langle B_1 \mid \ \rangle$$

Corollary 2 *If $b(K) = 1$ then $Q_+(K) \cong Q_-(K) \cong \langle x \mid \ \rangle = \{x\}$, and $G_+(K) \cong G_-(K) \cong \langle x \mid \ \rangle \cong \mathbb{Z}$.*

Remark. For classical knots K, the followings are equivalent.
(i) K is a trivial knot.
(ii) $b(K) = 1$.
(iii) $Q_+(K) \cong Q_-(K) \cong \langle x \mid \ \rangle = \{x\}$
(iv) $G_+(K) \cong G_-(K) \cong \langle x \mid \ \rangle \cong \mathbb{Z}$

For virtual knots, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). But (iv) $\Rightarrow$ (i) does not hold in general.

This knot has $b(K) = 1$, $Q_+(K) \cong Q_-(K) \cong \{x\}$, and $G_+(K) \cong G_-(K) \cong \mathbb{Z}$. But it is not a trivial knot. Jones polynomial $\neq 1$.

Theorem 3 *There exist infinitely many* 1*-bridge virtual knots.*

Proof. For $n \geq 2$, let K_n be a virtual knot with a diagram

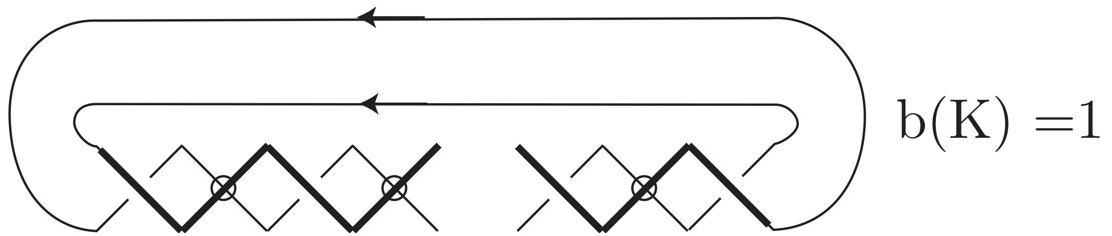


Figure 1: n positive crossings and $n-1$ virtual crossings

$$f_{K_n}(A) = \begin{cases} A^{-2n}\left(A^{-2}\dfrac{1-(-A^{-4})^n}{1+A^{-4}}+1\right) & n : \text{even} \\ -A^{-2n}\left(A^{-2}\dfrac{1-(-A^{-4})^n}{1+A^{-4}}-A^2-A^{-2}\right) & n : \text{odd} \end{cases}$$

Thus $K_i \neq K_j$ $(i \neq j)$. □

There are 99 pseudo-prime non-trivial virtual knots with crossing numbers less than 5 in Naoko Kamada's table.

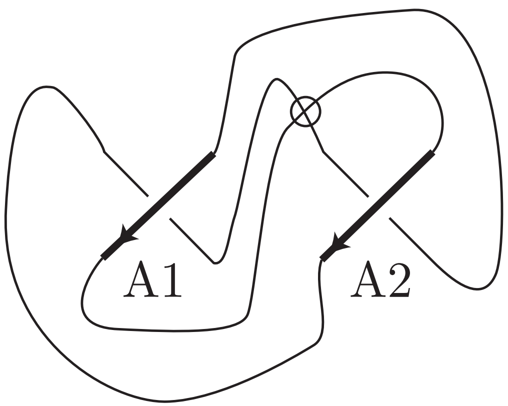
$c(K) = 2 \Longrightarrow \sharp 1,$

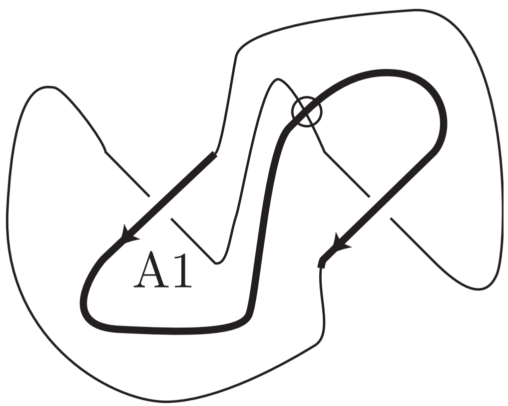
$c(K) = 3 \Longrightarrow \sharp 2, \cdots, \sharp 8,$

$c(K) = 4 \Longrightarrow \sharp 9, \cdots, \sharp 99.$

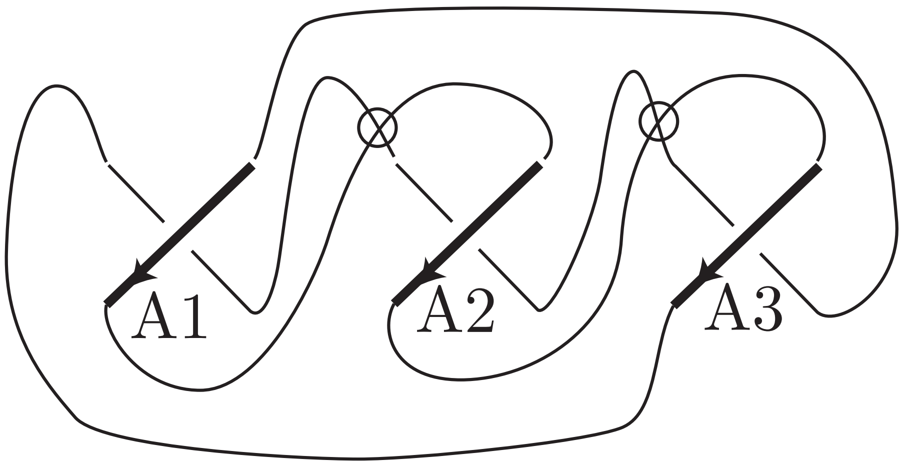
Here $c(K)$ is the minimal crossing number of K. K is pseudo-prime if K does not have a composite diagram with $c(K)$ crossings.

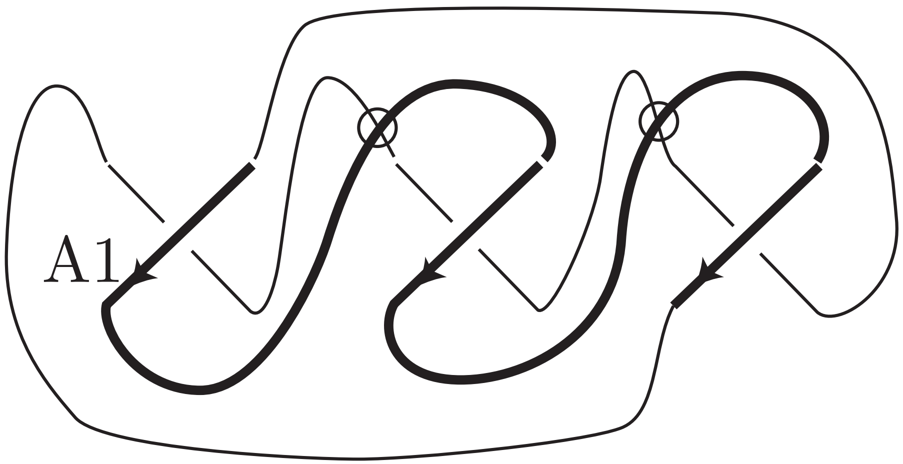
#1 (02)(13)/(1,1) b(K) =1



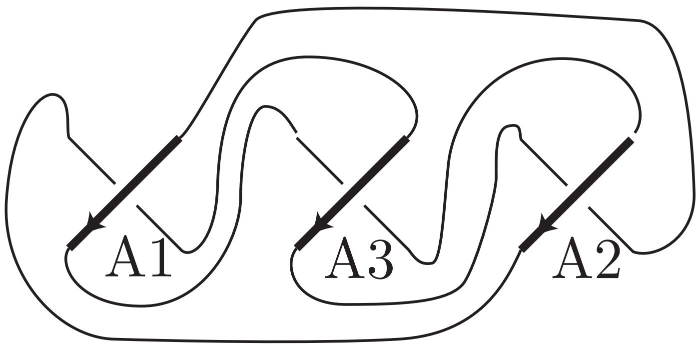


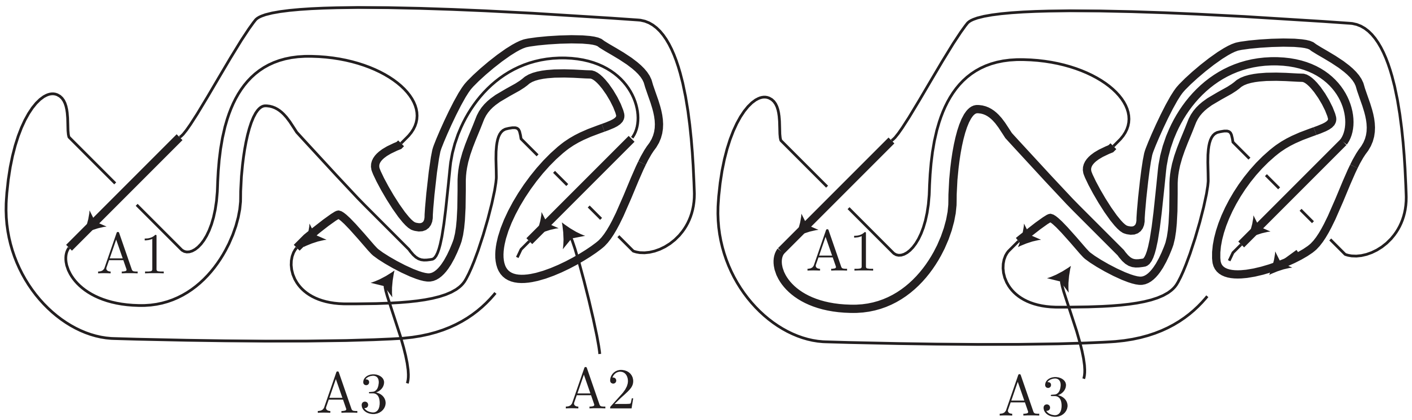
#2 (03)(14)(25)/(1,1,1) b(K) =1





#3 (03)(41)(25)/(1,1,1) b(K) =2





This knot has a 2-bridge presentation. $G_+(K) \neq \mathbb{Z}$. Thus $b(K) = 2$.

Lemma 4 (Bridge Reduction Lemma) *Let A_i and A_{i+1} be over-paths and B_i an under-path such that A_i, B_i and A_{i+1} appear in this order.*

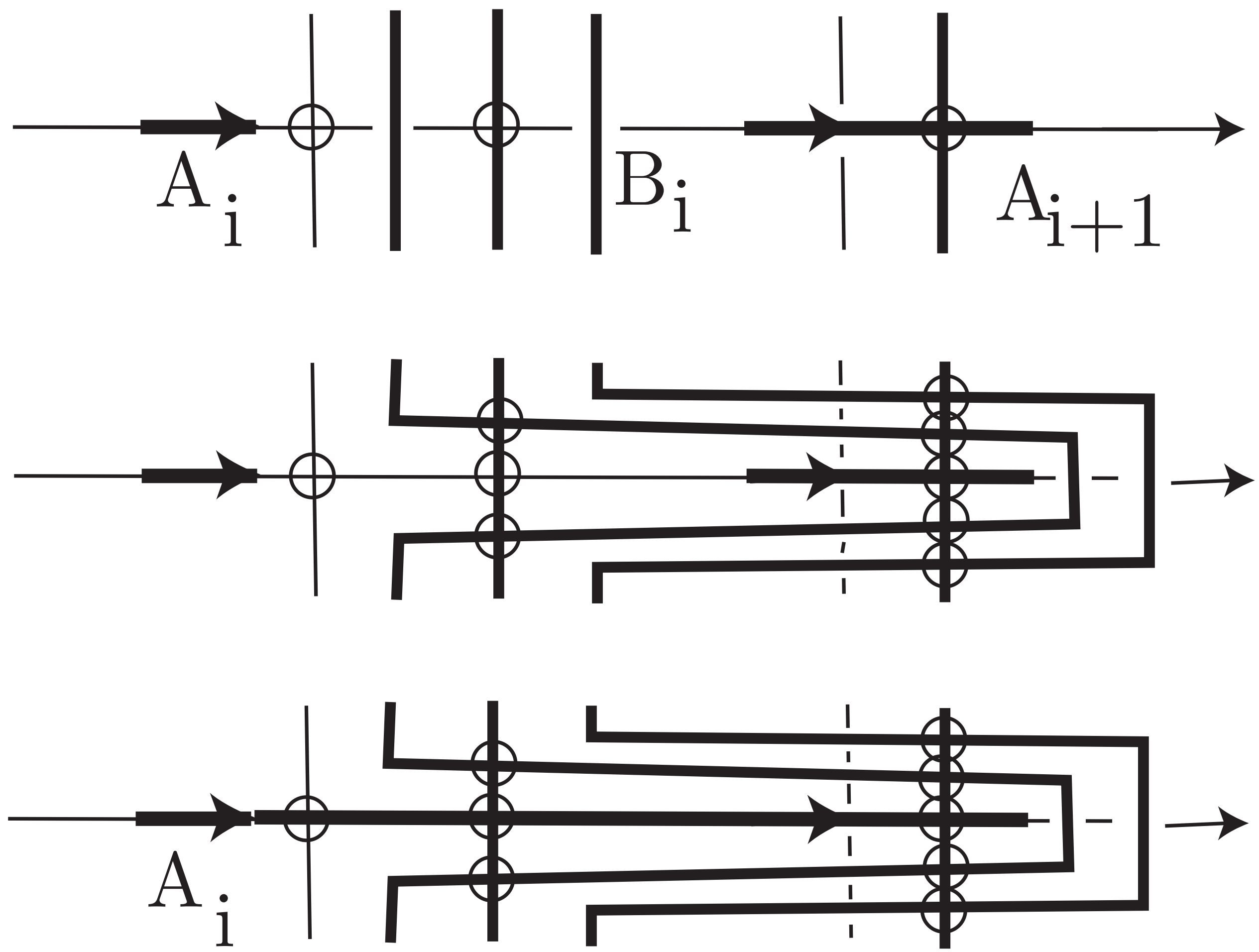
(1) (Forward Bridge Reduction) *If there exist no real crossings between B_i and A_{i+1}, then we can reduce the number of bridges.*

(2) (Backward Bridge Reduction) *If there exist no real crossings between A_i and B_i, then we can reduce the number of bridges.*

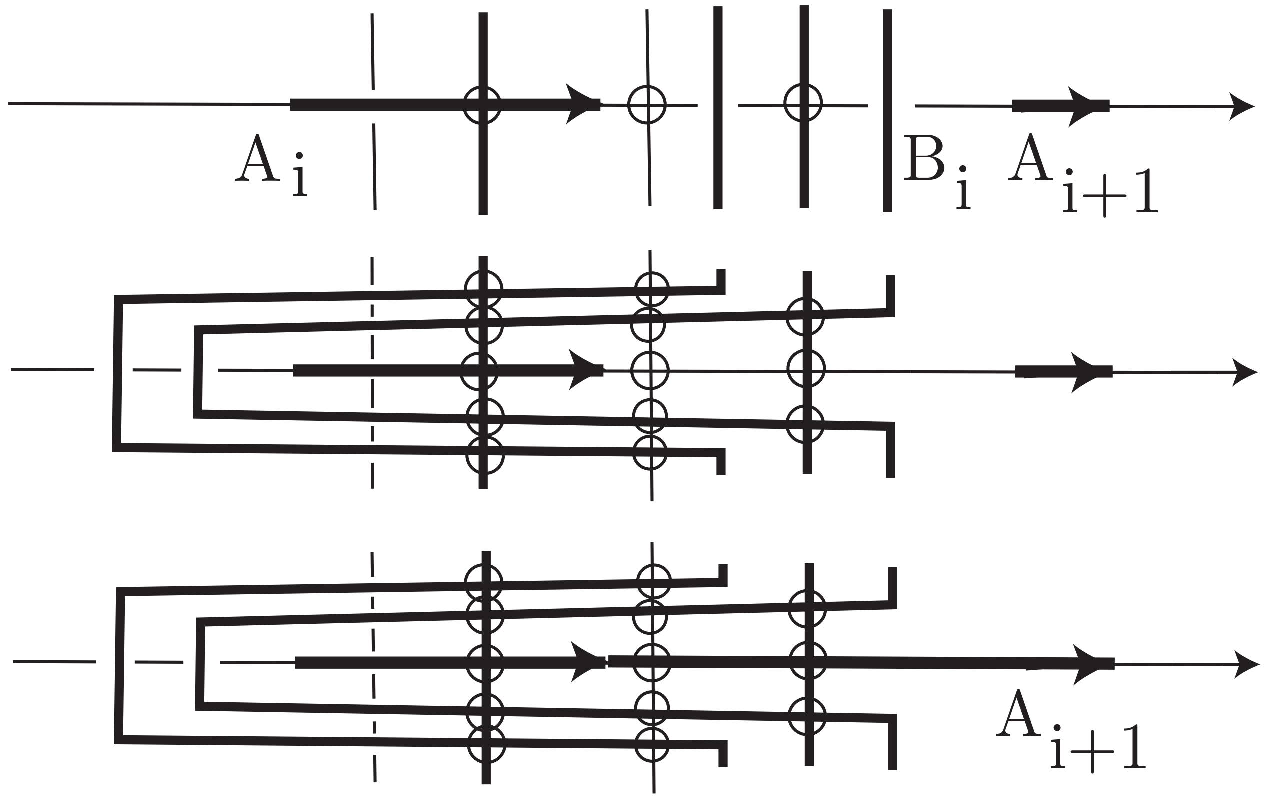
Proof. See Figures. ∎

This lemma is very useful to reduce the number of bridges.

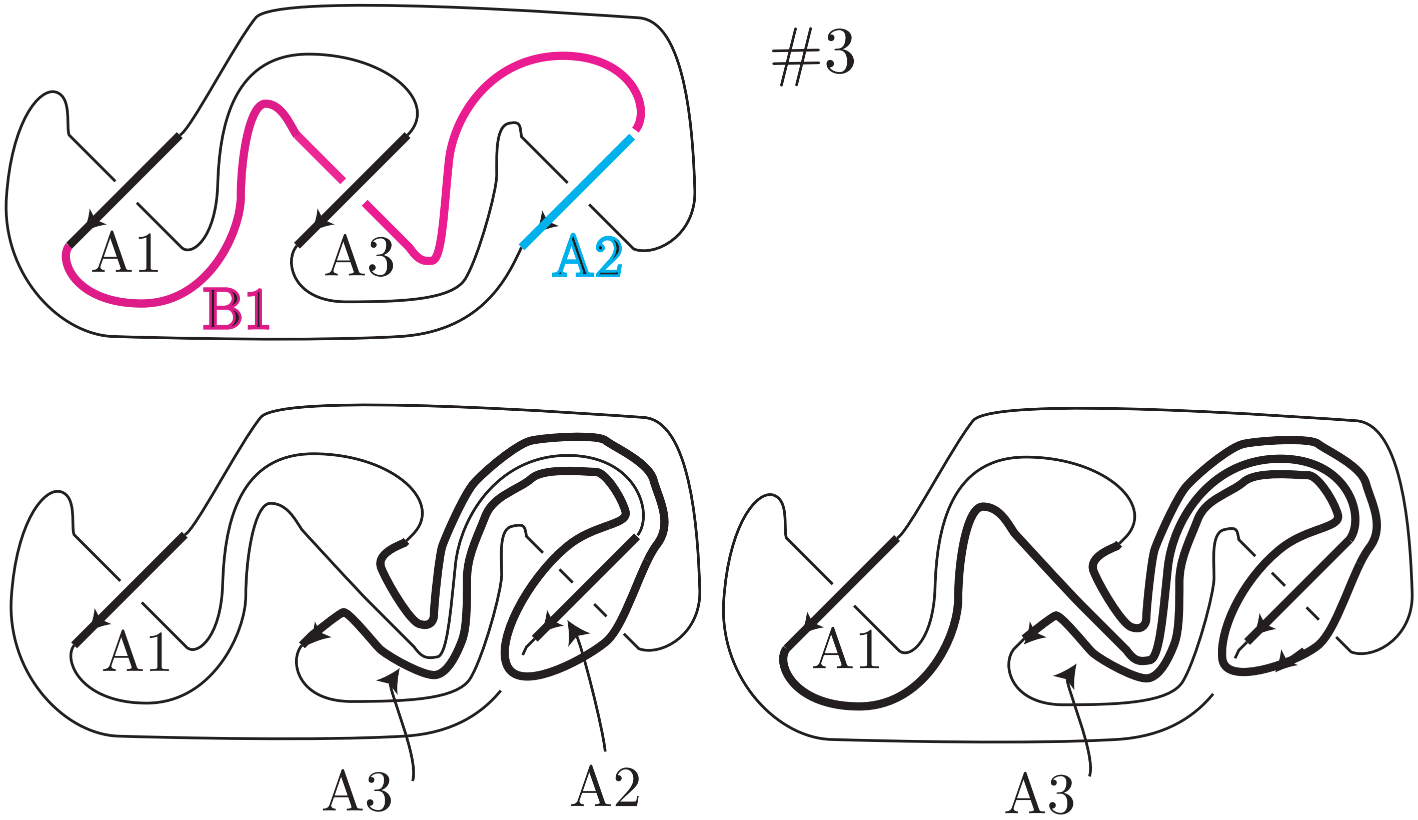
Bridge Reduction (Forward)



Bridge Reduction (Backward)



#3



Bridge Reduction (Forward)

(1) No real crossings between B1 and A2.

(2) Move real crossings on B1 beyond A2.

(3) Connect A1 and A2 to make a single over-path.

Theorem 5 *For the* 99 *virtual knots in the table, we have the following.*

(1) The following knots have bridge indices 2*:*
$3, 4, 19, 20, 21, 44, 45, 46, 47, 49, 50, 51, 58, 59, 60,$
$61, 62, 81, 82, 83, 84, 85, 86, 87, 88.$

(2) The knot ♯57 *has bridge index* 1 *or* 2*.*

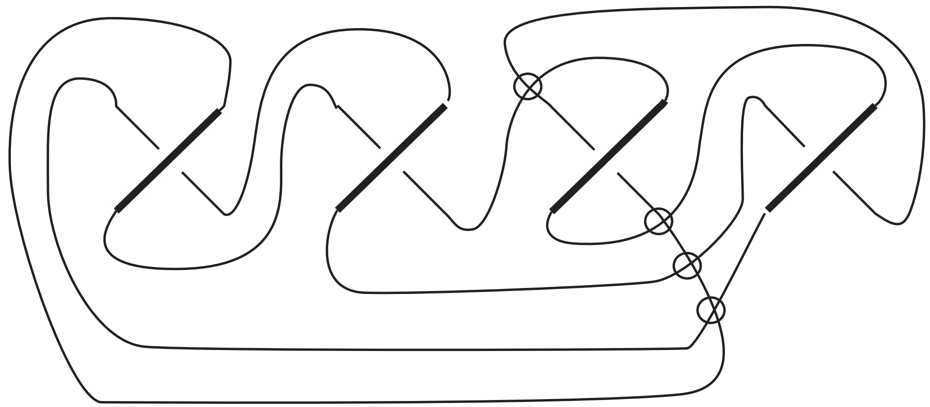
(3) The other knots have bridge indices 1*.*

Proof. By Bridge Reduction Lemma, all knots that are not listed in (1) and (2) can be presented by a diagram with 1-bridge presentation. Thus we have (3).

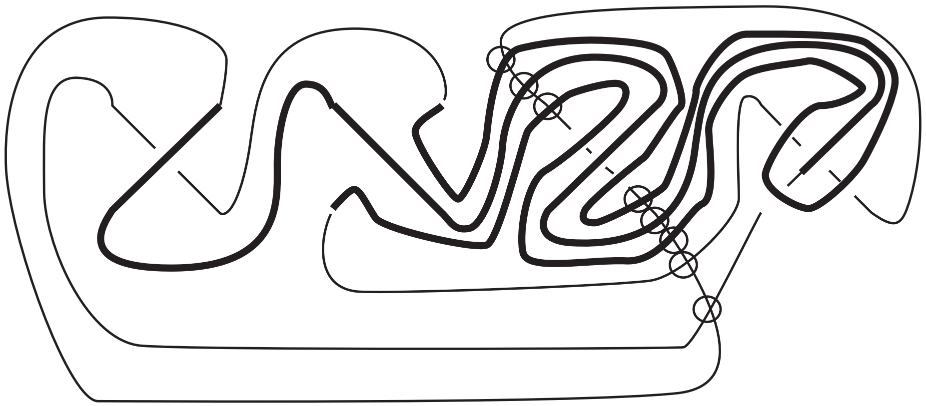
By Bridge Reduction Lemma, the knots listed in (1) and (2) can be presented by a diagram with 2-bridge presentation. Thus we have (2).

If $b(K) = 1$ then $G_+(K) \cong G_-(K) \cong \mathbb{Z}$. However, knots listed in (1) do not satisfy $G_+(K) \cong G_-(K) \cong \mathbb{Z}$. Thus we have (1). ∎

Problem 1: Decide the bridge index of ♯57.



#57



2-bridge presentation   b(K) =1 or 2?

Problem 2: For a classical knot K, define

$$b^{\mathrm{real}}(K) = \min\left\{ n \;\middle|\; \begin{array}{l} K \text{ has a diagram without virtual crossings} \\ \text{and with } n\text{-bridge presentation} \end{array} \right\}.$$

By definition, $b(K) \leq b^{\mathrm{real}}(K)$.
Is $b(K) = b^{\mathrm{real}}(K)$?

Problem 3: $b(K_1 \sharp K_2) = b(K_1) + b(K_2) - 1$? (Yes, for $b^{\mathrm{real}}(K)$ of classical knots.)