On the Alexander polynomials of alternating knots of genus 2

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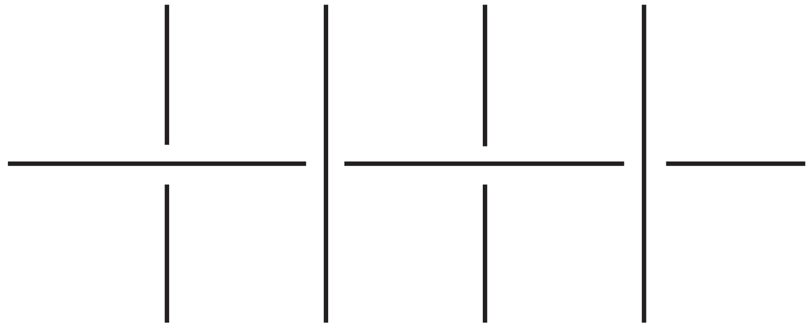
in The 4th East Asian School of Knots and Related Topics; The University of Tokyo

Contents

§1. Introduction

Alternating links

An *alternating diagram* is a link diagram such that an overcrossing and an undercrossing appear alternately along every component.



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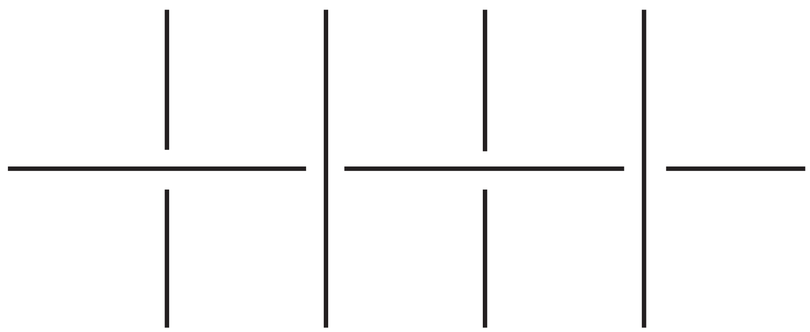

An *alternating link* is a link with an alternating diagram.

The Alexander polynomial

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Characterization of Δ_K

- K: a knot $\Rightarrow \Delta_K(t^{-1}) \doteq \Delta_K(t)$ and $\Delta_K(1) = \pm 1$.
- $f(t) \in \mathbb{Z}[t, t^{-1}]$ with $f(t^{-1}) \doteq f(t)$ and $f(1) = \pm 1$
  $\Rightarrow \exists$ a knot K such that $\Delta_K(t) = f(t)$.

My motivation and known results

Motivation

Characterize $\Delta_K(t)$ of an **alternating knot** K.

Known results

- Crowell-Murasugi's theorem
- Trapezoidal conjecture & Log-concavity conjecture
- Ozsváth-Szabó's theorem

The Alexander polynomial of an alternating knot

Notation

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Proposition 1.1. ['59 R. H. Crowell, K. Murasugi]

Let K be an alternating knot. Then

- $\deg \Delta_K = 2g(K)$.
- $[\Delta_K]_i > 0$ for $i = 0, 1, \ldots, 2g(K)$.

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$\Delta_K = [\Delta_K]_0 + [\Delta_K]_1 t + [\Delta_K]_2 t^2 + [\Delta_K]_1 t^3 + [\Delta_K]_0 t^4$,

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Trapezoidal conjecture & Log-concavity conjecture

Definition $f(t) = \sum_{i=0}^{m} a_i t^i \in \mathbb{Z}[t, t^{-1}]$: *trapezoidal* $\Leftrightarrow$
$f(t^{-1}) \doteq f(t)$ and $0 < a_0 < \cdots < a_j = a_{j+1} = \cdots = a_{[\frac{m}{2}]}$
for some $0 \leq j \leq [\frac{m}{2}]$.

Trapezoidal conjecture ['62 R. H. Fox]

L : a non-split alternating link $\Rightarrow \Delta_L(-t)$: trapezoidal

Example

$$\Delta_{5_1}(-t) = 1 + t + t^2 + t^3 + t^4$$
$$\Delta_{6_3}(-t) = 1 + 3t + 5t^2 + 3t^3 + t^4$$
$$\Delta_{7_3}(-t) = 2 + 3t + 3t^2 + 3t^3 + 2t^4$$
$$\Delta_{8_5}(-t) = 1 + 3t + 4t^2 + 5t^3 + 4t^4 + 3t^5 + t^6$$
$$\Delta_{8_7}(-t) = 1 + 3t + 5t^2 + 5t^3 + 5t^4 + 3t^5 + t^6$$

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Definition A polynomial $f \in \mathbb{Z}[t, t^{-1}]$ is *log-concave* $\Leftrightarrow [f]_{i-1}[f]_{i+1} \leq [f]_i^2$ for all i.

Log-concavity conjecture [05' A. Stoimenow]

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Remark

- Log-concavity conjecture "$\supset$" Trapezoidal conjecture.

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- Log-concavity conjecture "$\supset$" Trapezoidal conjecture.
- Log-concavity conjecture is true for alternating knots of genus 2. ['07 J.]

Ref. I. D. Jong, "*Alexander polynomials of alternating knots of genus two*" (submitted to OJM)

Proposition 1.2. ['03 P. Ozsváth-Z. Szabó]

K: an alternating knot, $\sigma = \sigma(K)$: the signature of K.

$\Delta_K(t)$ is normalized so that $\Delta_K(1) = 1$.

Then, for each $s = 0, 1, \ldots, g(K)$,

$$(-1)^{s+\frac{\sigma}{2}}\left(\sum_{j=1}^{g(K)-s} j[\Delta_K(t)]_{g(K)-s-j} - \max(0, \lceil \frac{|\sigma| - 2|s|}{4} \rceil)\right) \leq 0.$$

Ozsváth-Szabó's inequality

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In particular, for an alternating knot with $g(K) = 2$,

$$2[\Delta_K]_0 \leq [\Delta_K]_1 \text{ if } \sigma(K) = 0,$$
$$2[\Delta_K]_0 + 1 \leq [\Delta_K]_1 \text{ if } |\sigma(K)| = 2,$$
$$2[\Delta_K]_0 - 1 \leq [\Delta_K]_1 \text{ if } |\sigma(K)| = 4.$$

Remark For $\forall$ knot K, $|\sigma(K)| \leq 2g(K)$.

Main Theorem

Let K be an alternating knot of genus 2.

Then the following inequalities hold ($[\Delta]_0 \geq 1$):

$$3[\Delta_K]_0 - 1 \leq [\Delta_K]_1 \leq 6[\Delta_K]_0 + 1 \text{ if } \sigma(K) = 0,$$

$$2[\Delta_K]_0 + 1 \leq [\Delta_K]_1 \leq 6[\Delta_K]_0 - 1 \text{ if } |\sigma(K)| = 2,$$

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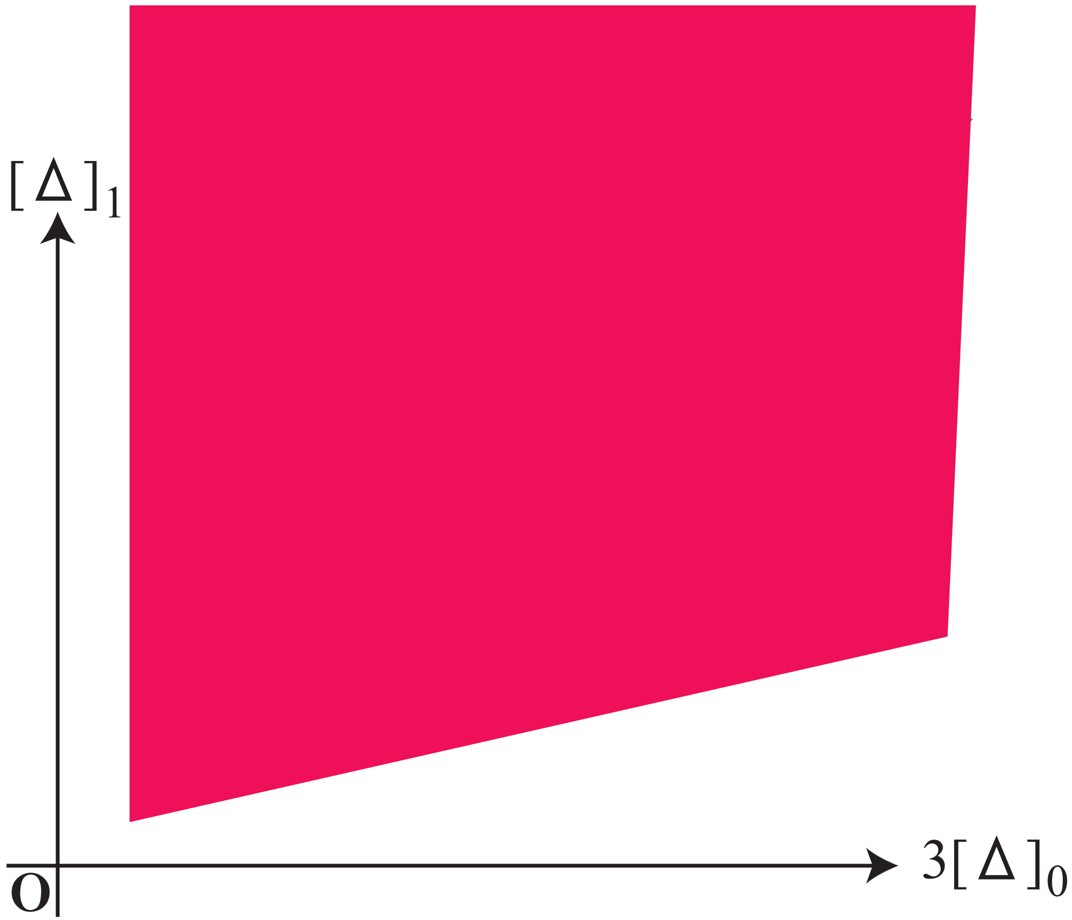
$$2[\Delta_K]_0 - 1 \leq [\Delta_K]_1 \leq 4[\Delta_K]_0 - 2 \text{ if } |\sigma(K)| = 4.$$

Moreover, any other linear inequality on $[\Delta]_0$ and $[\Delta]_1$ for all alternating knots of genus 2 is a consequence of our inequalities. (Completeness)

Main Theorem vs trapezoidal property & Ozsváth-Szabó's inequality

Main Theorem vs

trapezoidal property & Ozsváth-Szabó's inequality



Main Theorem vs

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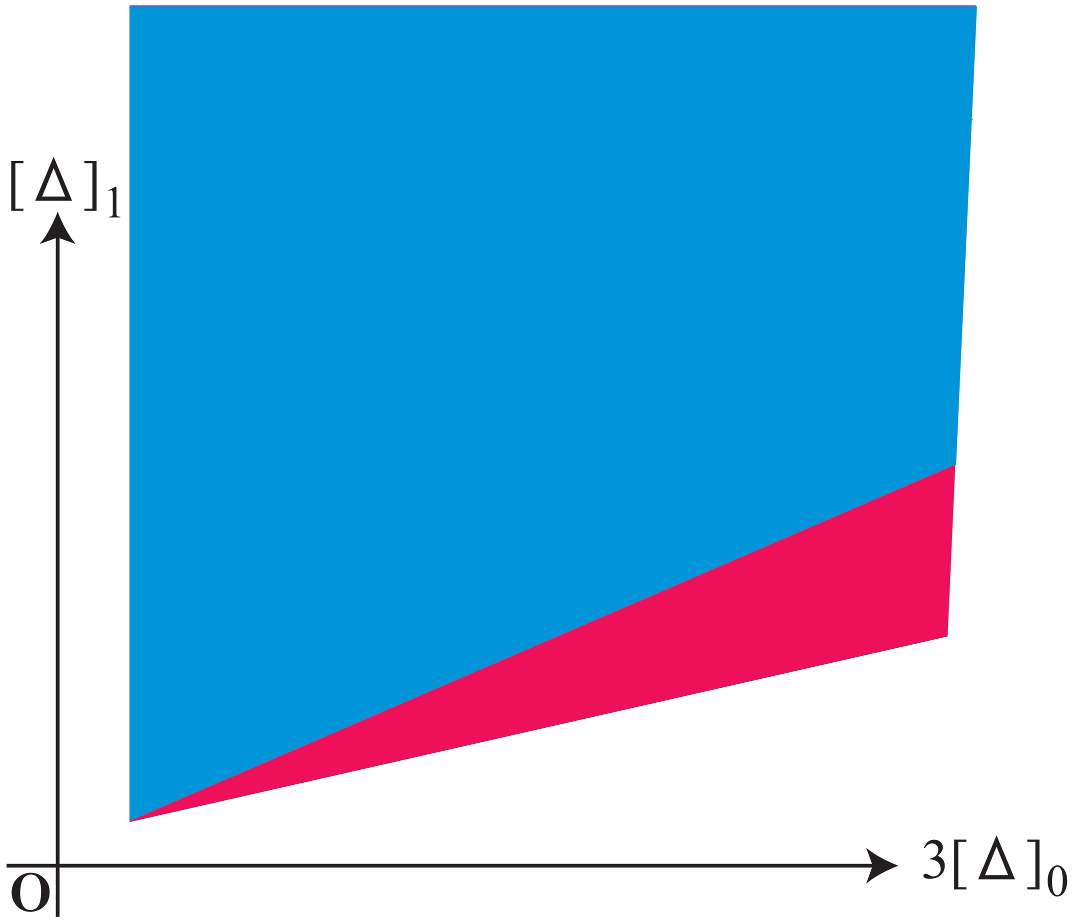
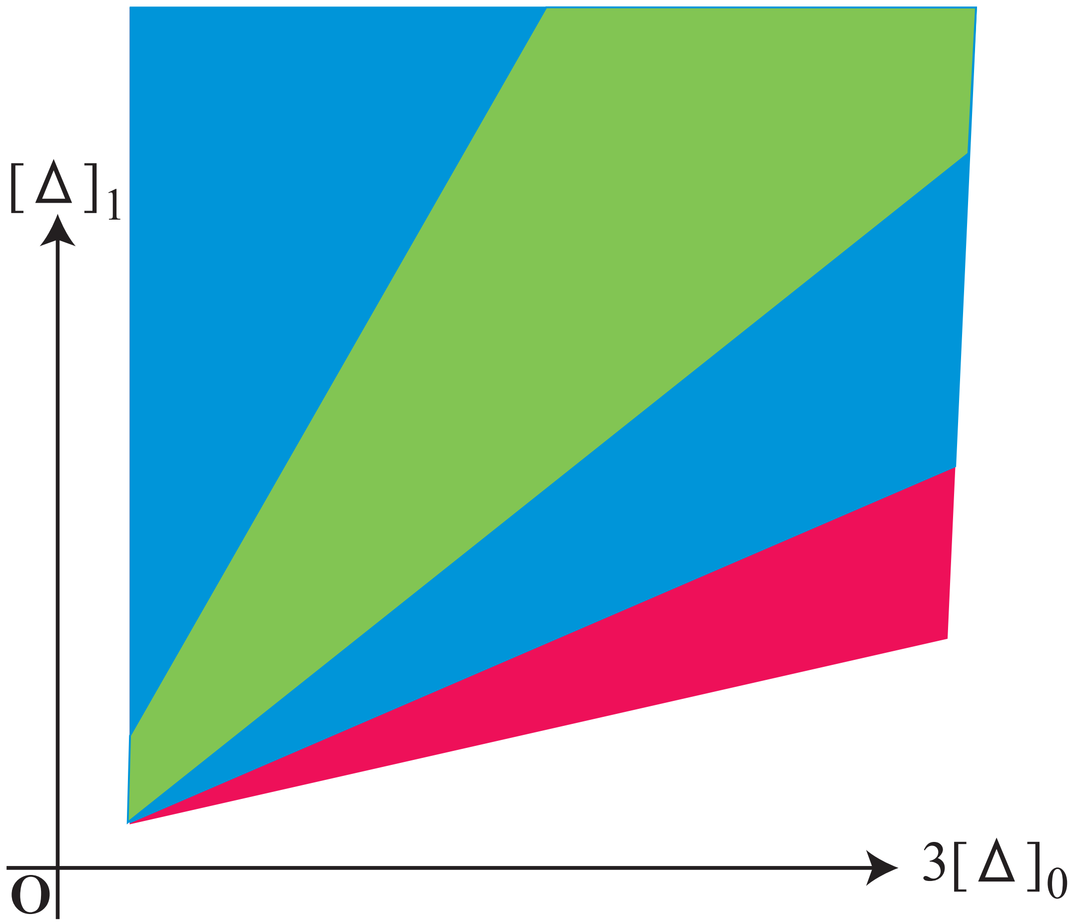


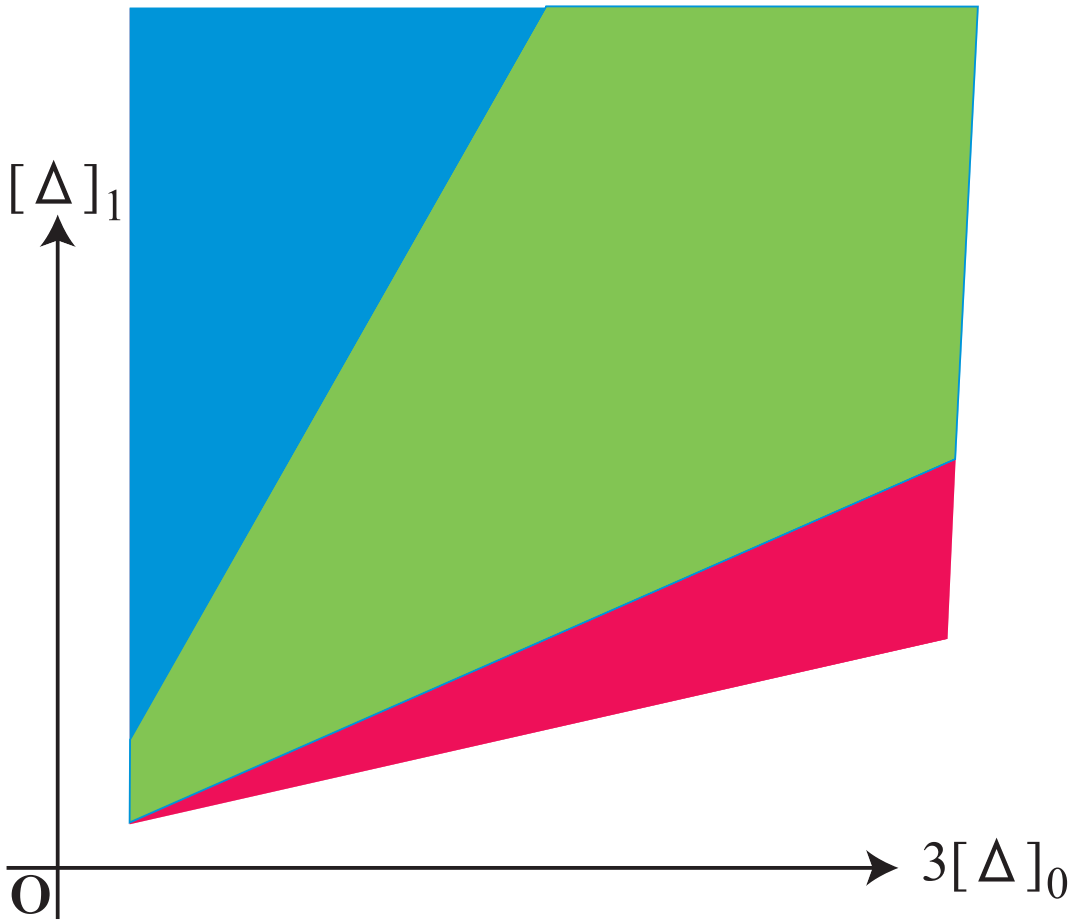
Main Theorem vs ($\sigma = 0$)

trapezoidal property & Ozsváth-Szabó's inequality



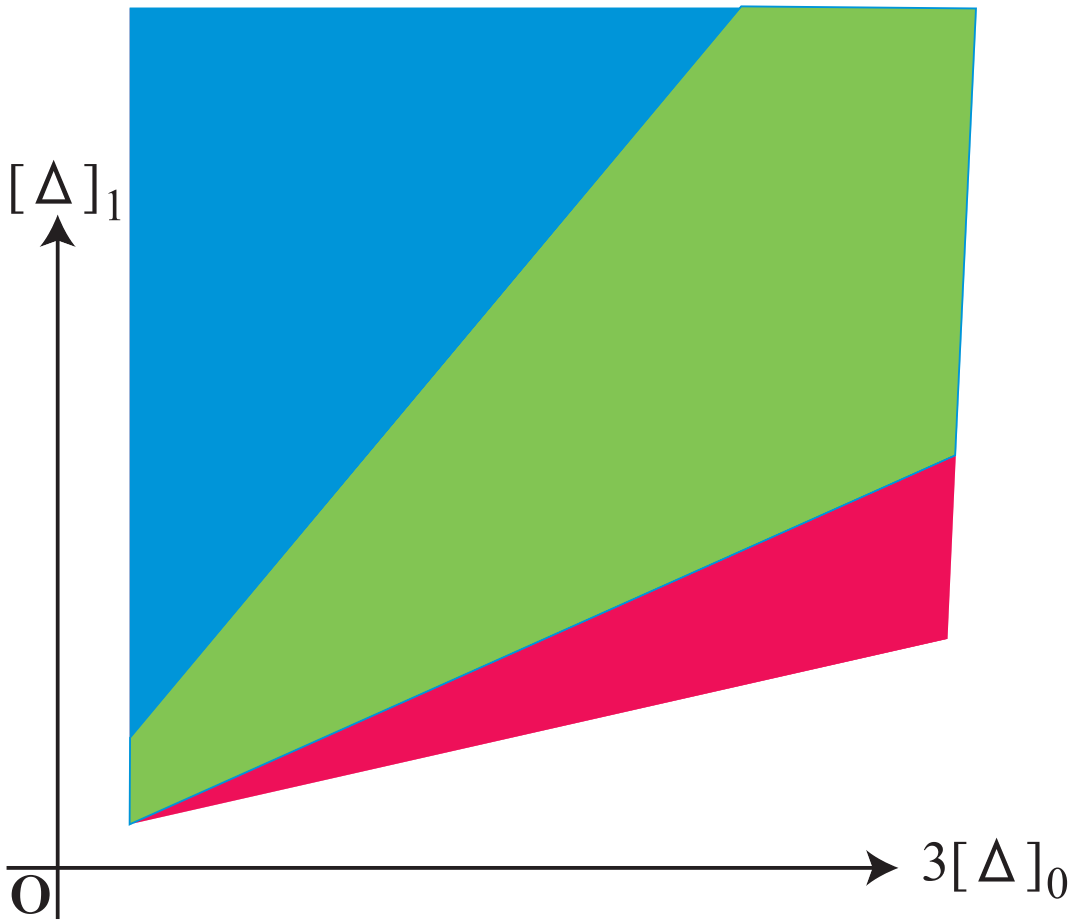
Main Theorem vs ($|\sigma| = 2$)

trapezoidal property & Ozsváth-Szabó's inequality



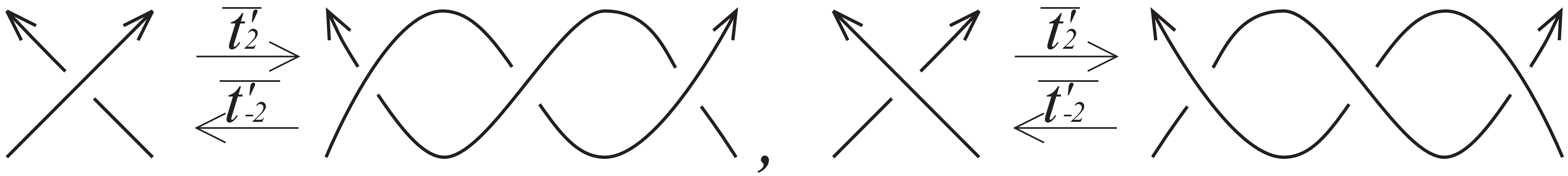
Main Theorem vs ($|\sigma| = 4$)

trapezoidal property & Ozsváth-Szabó's inequality



Generators for genus 2 knots

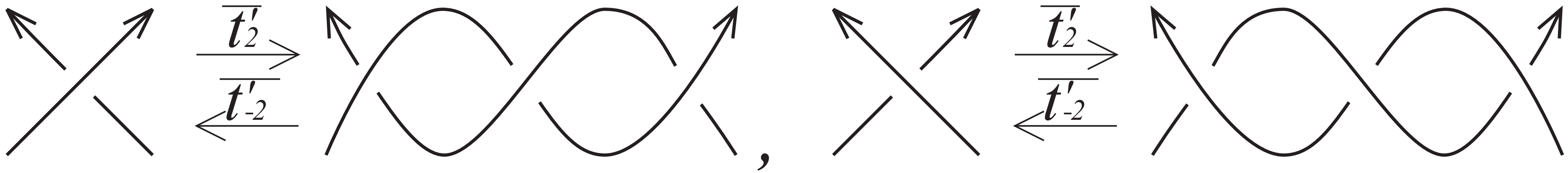
Definition ($\overline{t'_{\pm 2}}$ *move*)



§2. Proof of Main Theorem

Generators for genus 2 knots

Definition ($\overline{t'_{\pm 2}}$ *move*)



Lemma 2.1. ['05 A. Stoimenow]

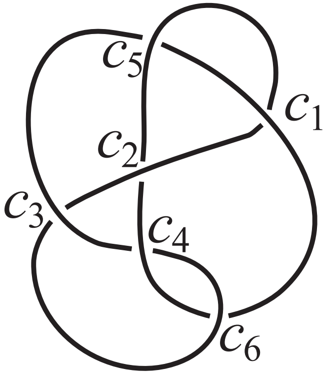
$\{$reduced alternating knot diagrams of genus 2$\}$

$/\overline{t'_{\pm 2}}$ move, mirror image, flype

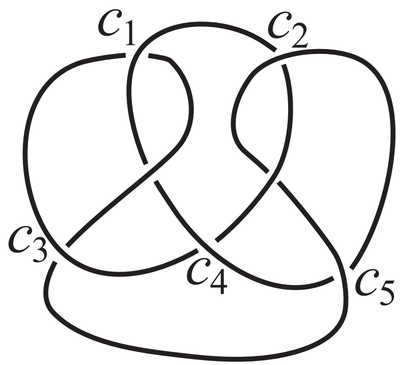
$=\{5_1, 6_2, 6_3, 7_5, 7_6, 7_7, 8_{12}, 8_{14}, 8_{15}, 9_{23}, 9_{25}, 9_{38}, 9_{39}, 9_{41}, 10_{58},$
$10_{97}, 10_{101}, 10_{120}, 11_{123}, 11_{148}, 11_{329}, 12_{1097}, 12_{1202},$
$13_{4233}, 3_1\#3_1, 3_1\#4_1, 3_1\#3_1^*, 4_1\#4_1\} =: G_2$

We name crossings of the diagrams in G_2 as follows:

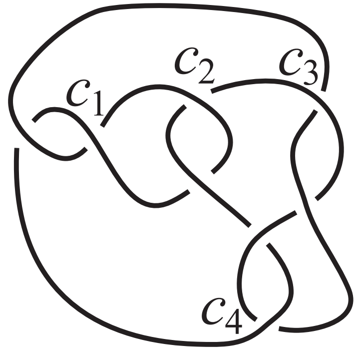
$\underline{\sigma = 0}$



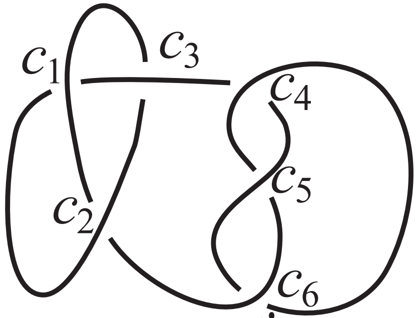
6_3



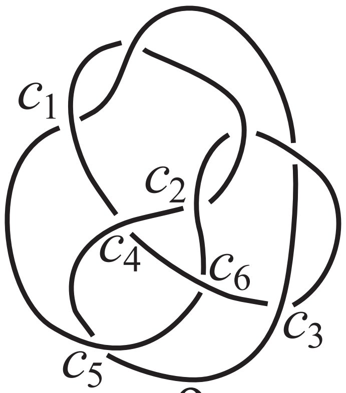
7_7



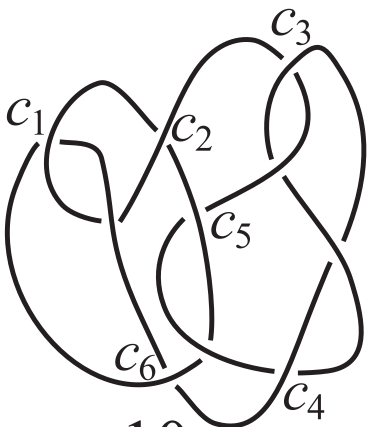
8_{12}



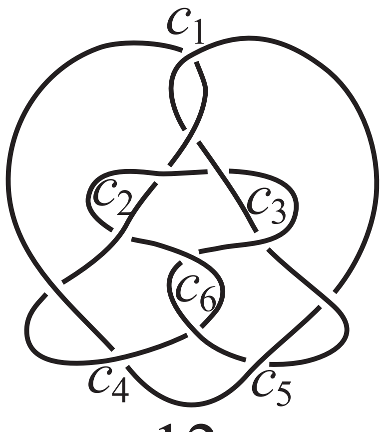
$3_1 \# 3_1^*$



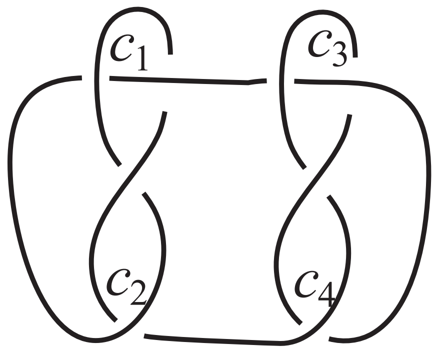
9_{41}



10_{58}

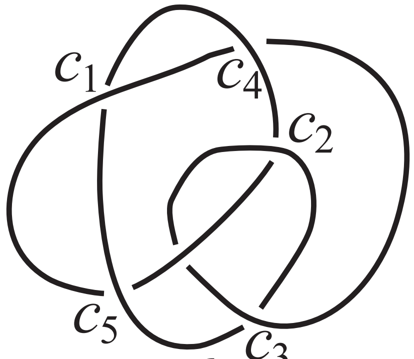


12_{1202}

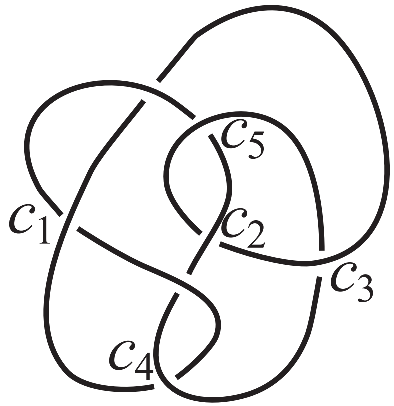


$4_1 \# 4_1$

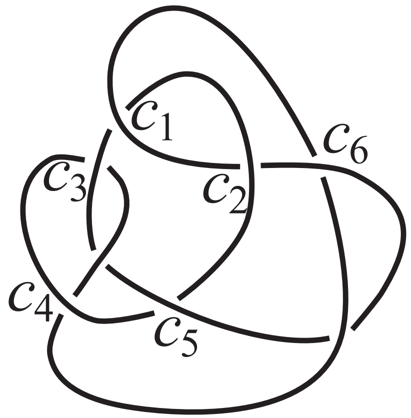
$|\sigma| = 2$



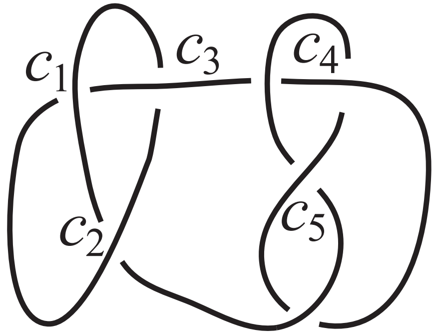
6_2



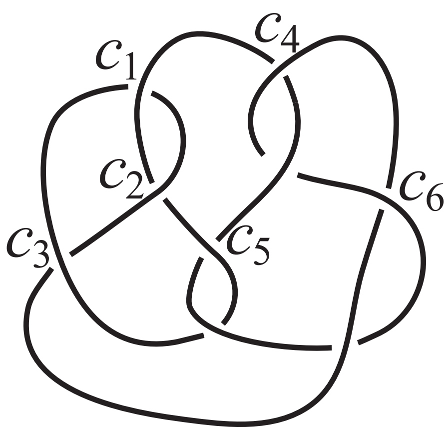
7_6



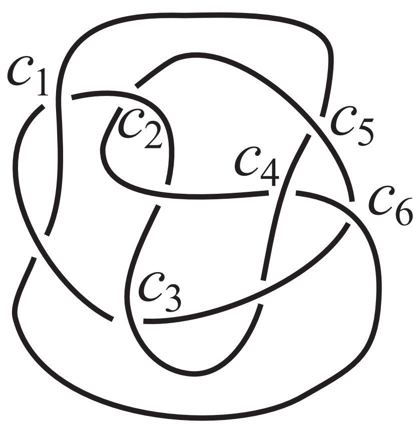
8_{14}



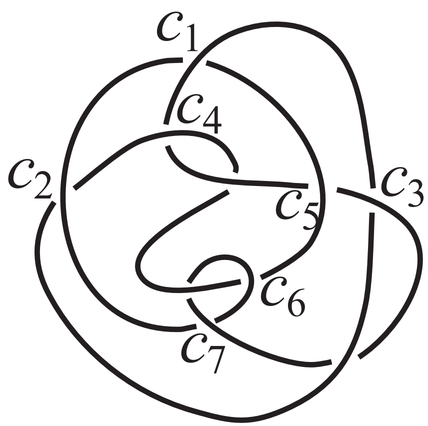
$3_1 \# 4_1$



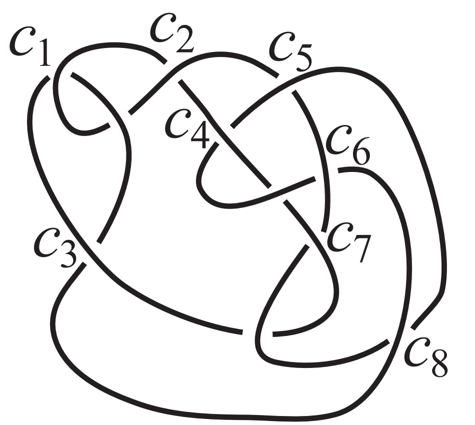
9_{25}



9_{39}

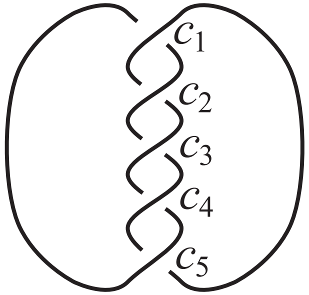


10_{97}

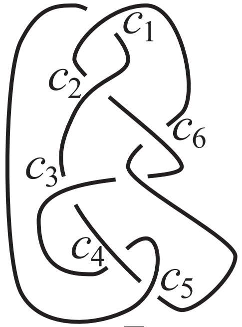


11_{148}

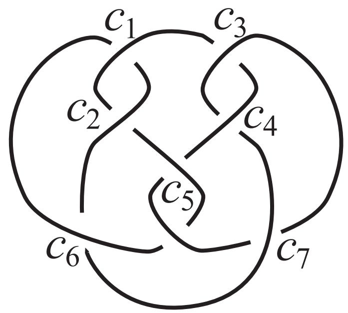
$|\sigma| = 4$



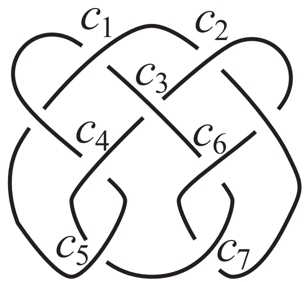
5_1



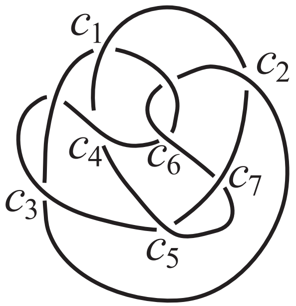
7_5



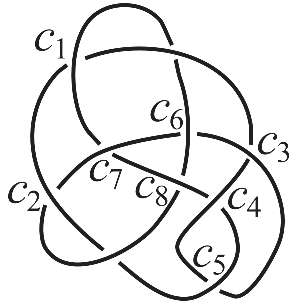
8_{15}



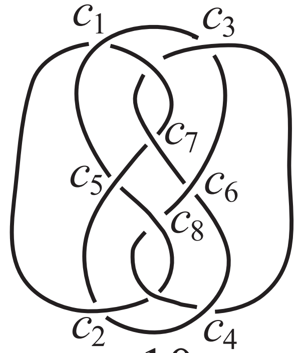
9_{23}



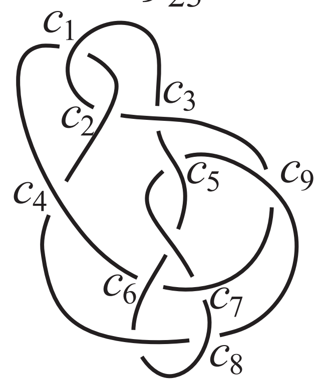
9_{38}



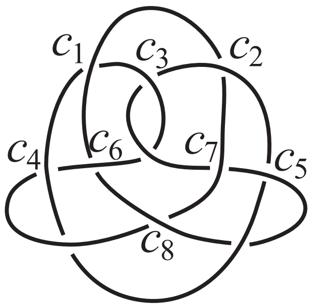
10_{101}



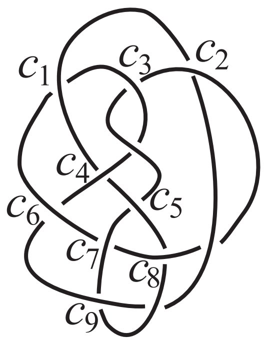
10_{120}



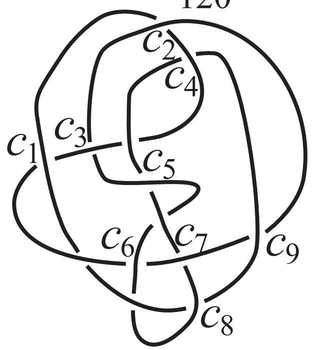
11_{123}



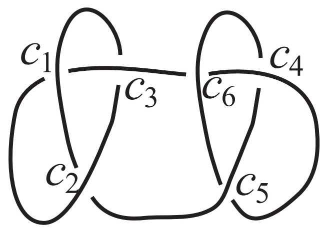
11_{329}



12_{1097}



13_{4233}



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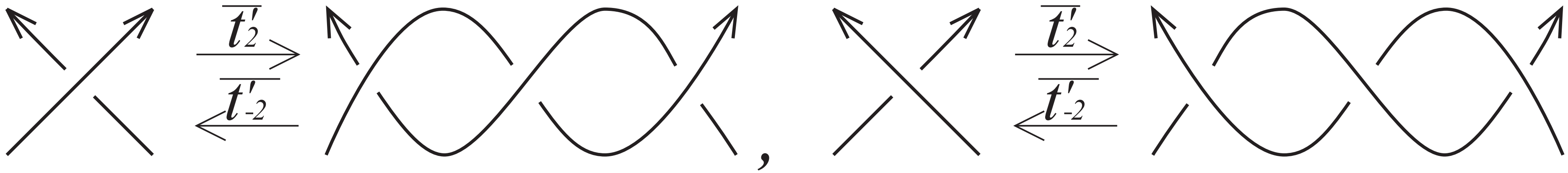
k_i-times $\overline{t'_2}$ moves at c_i for $i = 1, 2, \dots, m$



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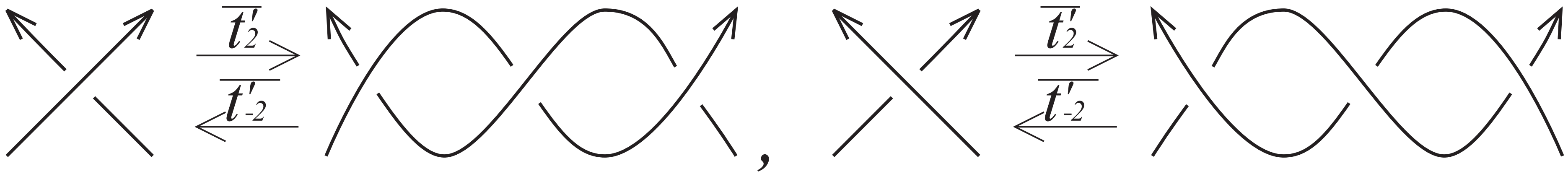
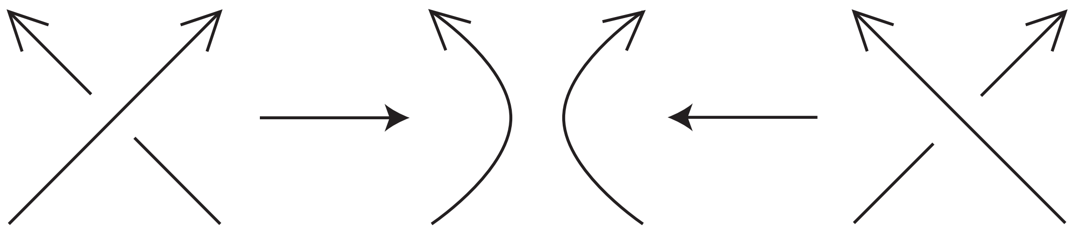
k_i-times $\overline{t_2'}$ moves at c_i for $i = 1, 2, \ldots, m$



$D/c_1 \cdots c_m$: the diagram obtained by smoothing $c_1, \ldots, c_m$



Main Theorem

Let K be an alternating knot of genus 2.

Then the following inequalities hold ($[\Delta]_0 \geq 1$):

$$3[\Delta_K]_0 - 1 \leq [\Delta_K]_1 \leq 6[\Delta_K]_0 + 1 \text{ if } \sigma(K) = 0,$$

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Moreover, any other linear inequality on $[\Delta]_0$ and $[\Delta]_1$ for all alternating knots of genus 2 is a consequence of our inequalities. (Completeness)

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c: a crossing of D. Then

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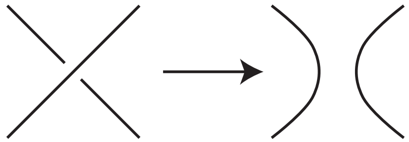
Lemma 2.3. ['05, E. S. Lee]

D: a reduced alternating diagram.

$p(D) = \#\{$positive crossings of $D\}$

$o(D) = \#\{$circles obtained by splicing all crossings as

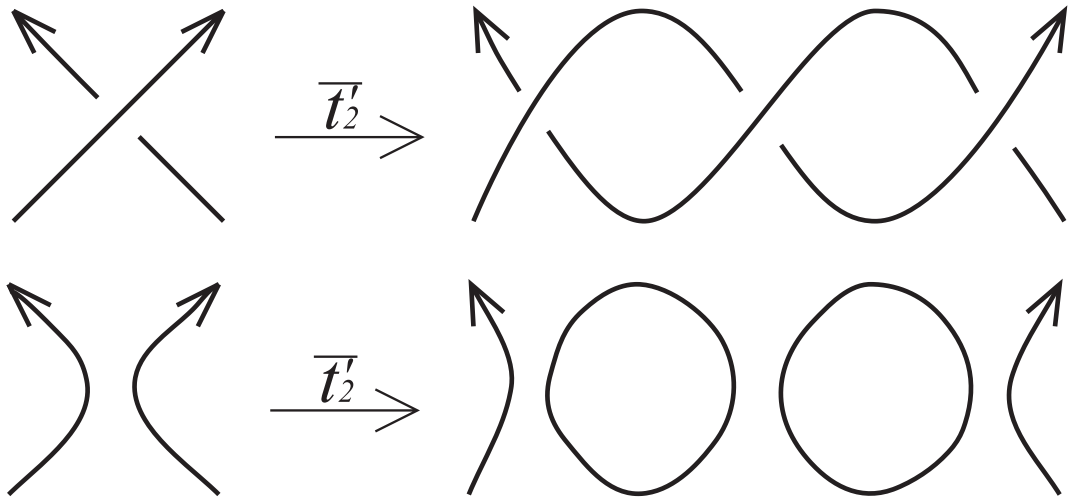
 $\}$

Then

$$\sigma(D) = o(D) - p(D) - 1.$$

Proof of Proposition 2.2.

- c is positive.



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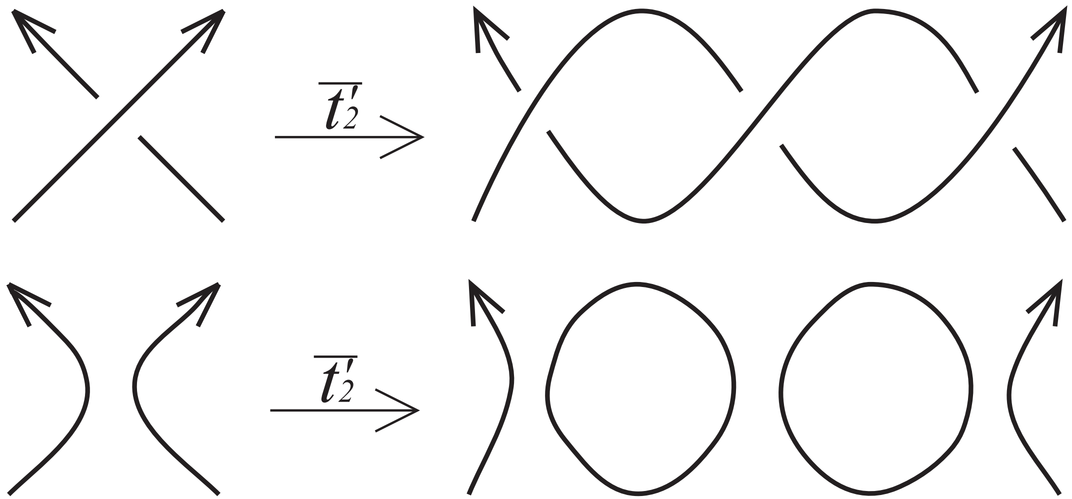

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\begin{aligned}
\sigma\left(D(c)\right) &= o\left(D(c)\right) - p\left(D(c)\right) - 1 \\
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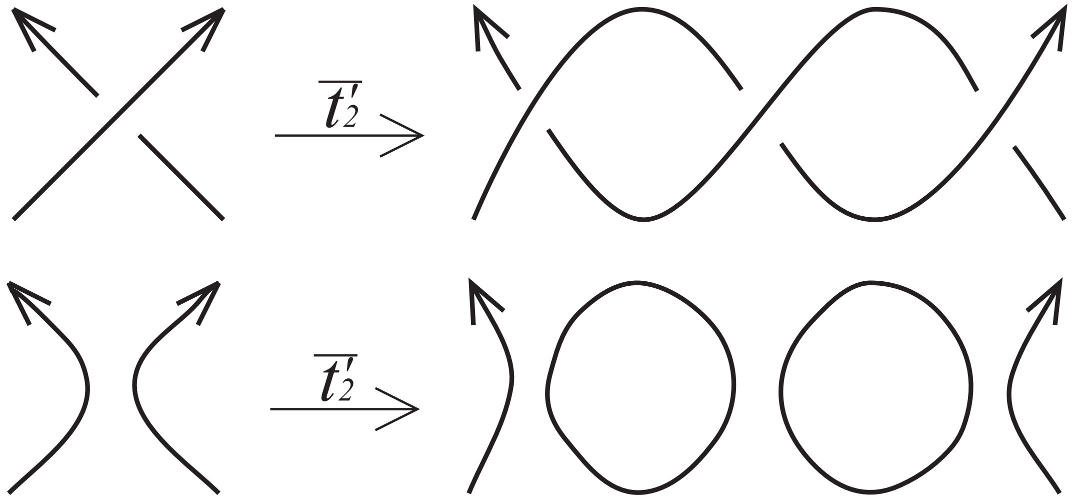
- c is positive. (c: negative $\Rightarrow$ take the mirror image)



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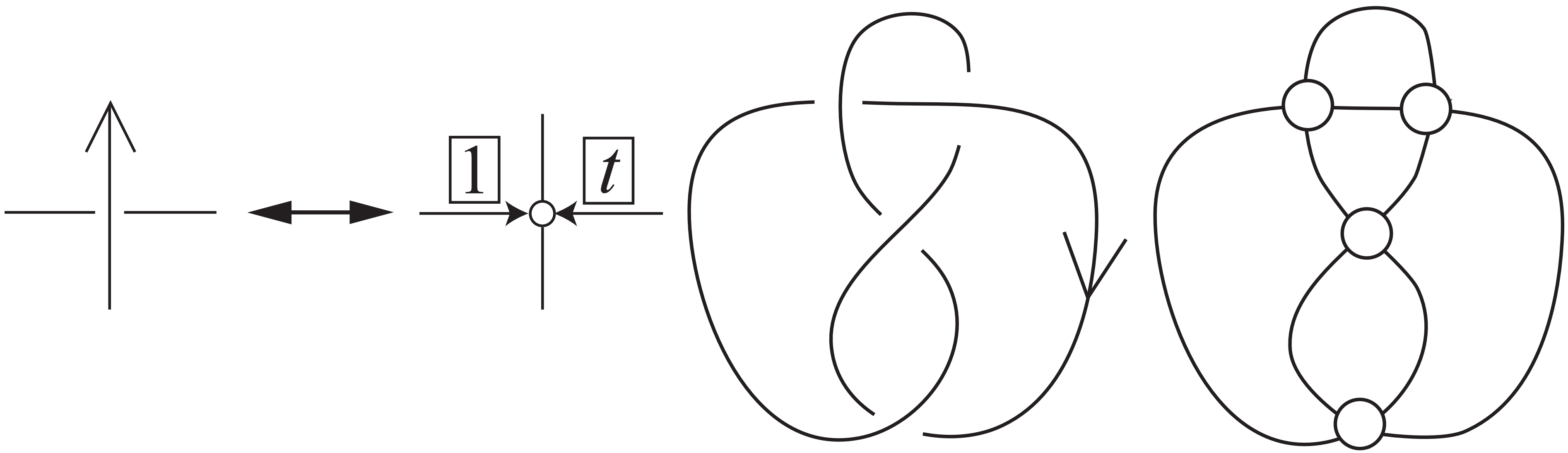
Method for calculating $\Delta(-t)$ of an alternating link

D : an alternating diagram

Step 1 : Constructing an oriented graph with a weight map from the alternating diagram D.

- Orientation : terminal points = undercrossings.
- Weight : the weight of the edges which are on the left (resp. right) of the crossings = 1 (resp. t).



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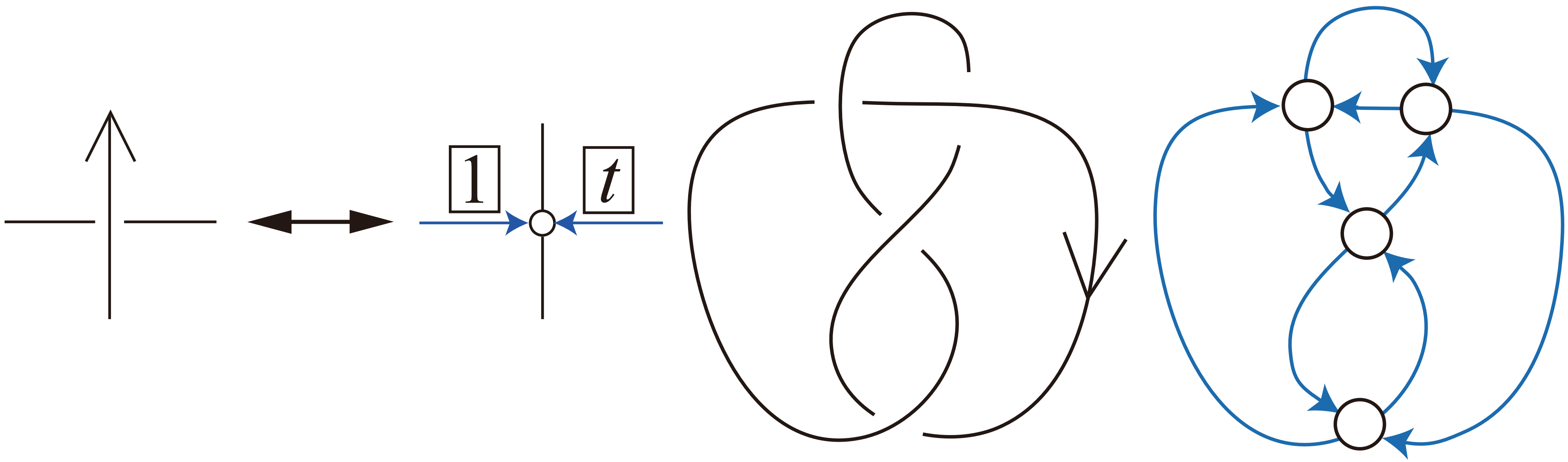

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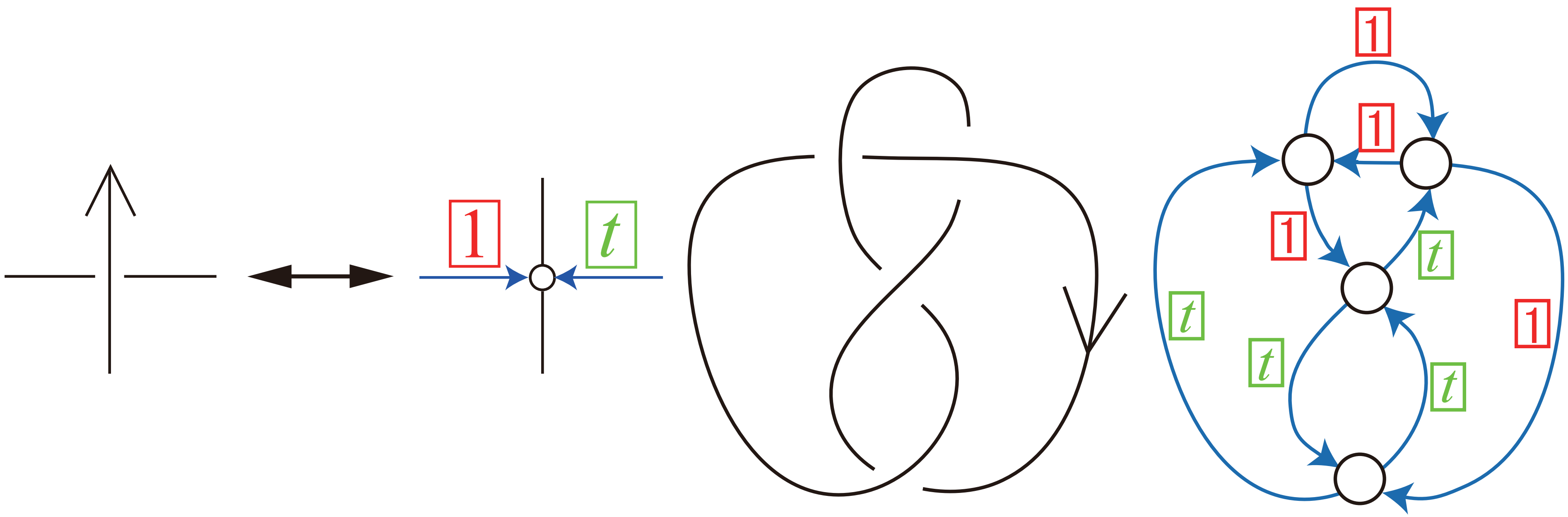
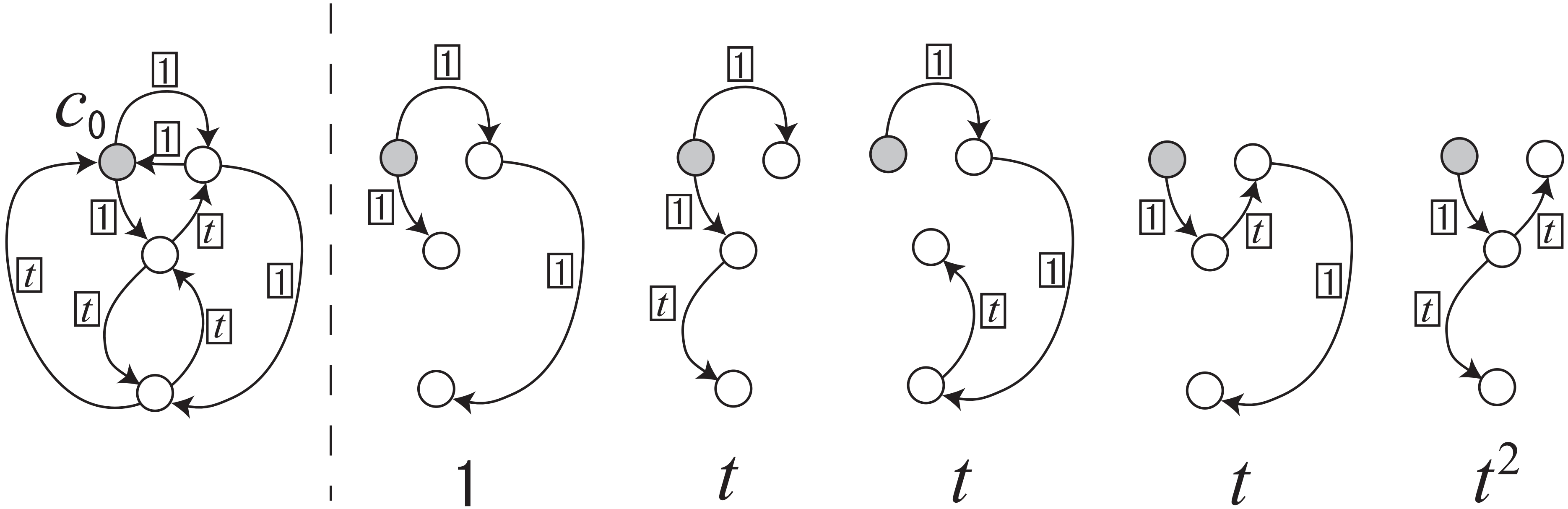


Step 2 : Fixing a root vertex c_0 and enumerating the maximal rooted trees with the root vertex c_0.



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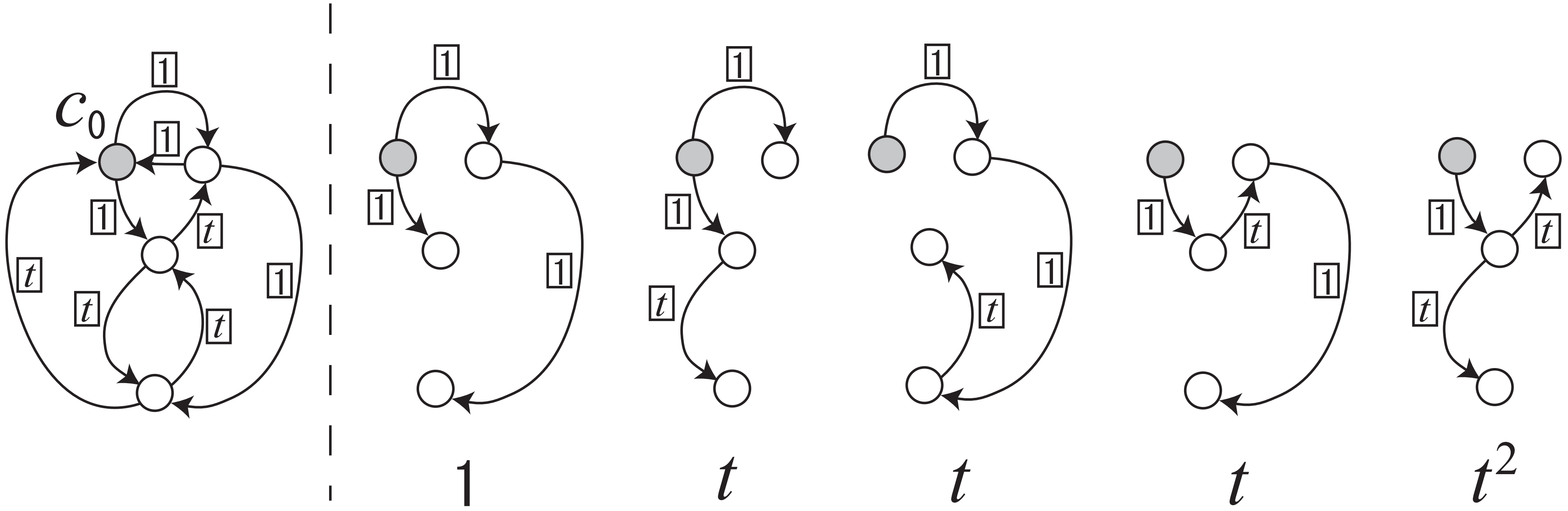

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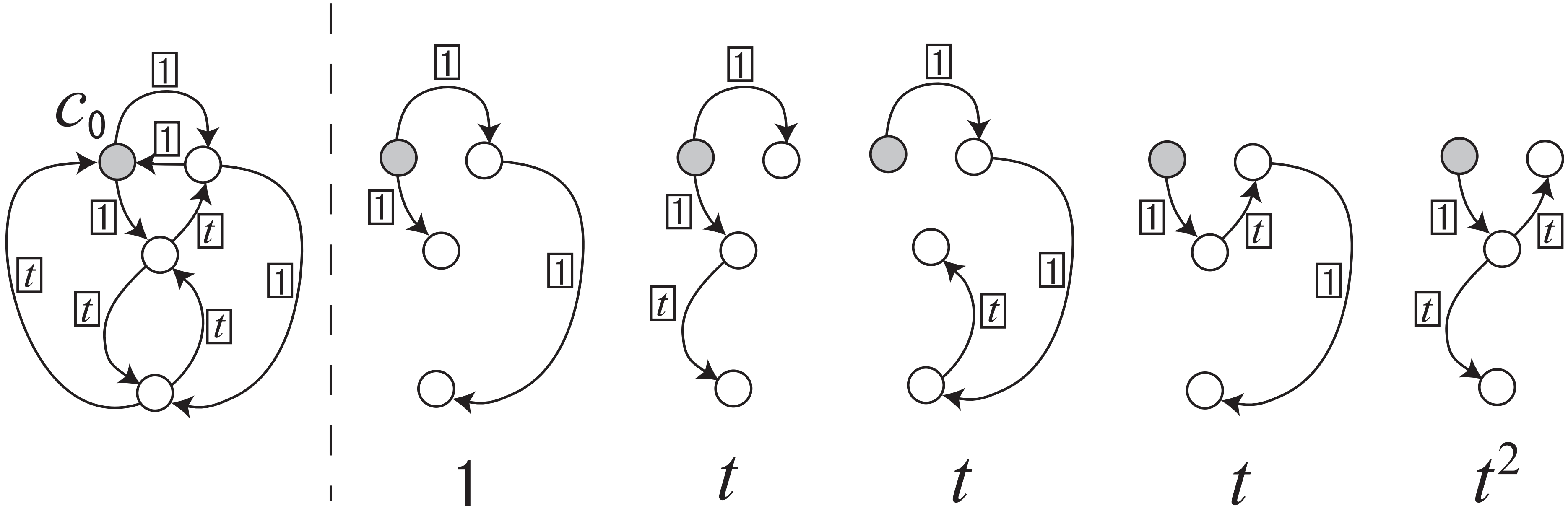

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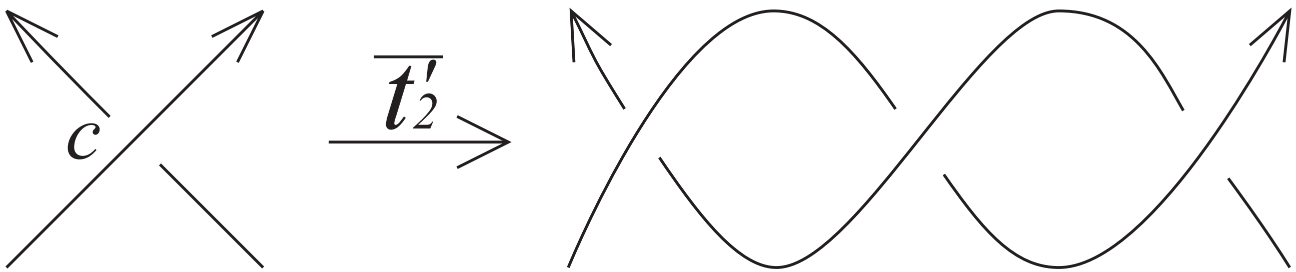
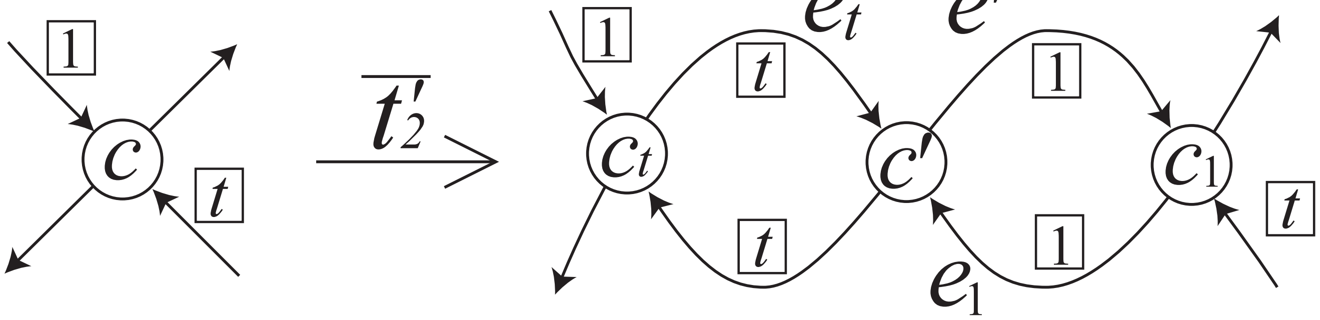
※ (ambiguity of choices of c_0) "=" (ambiguity of $\times t^l$).

Lemma 2.4.

c : a crossing of an alternating diagram D

$\Rightarrow \Delta_{D(c)} = \Delta_D + (1+t)\Delta_{D/c}$





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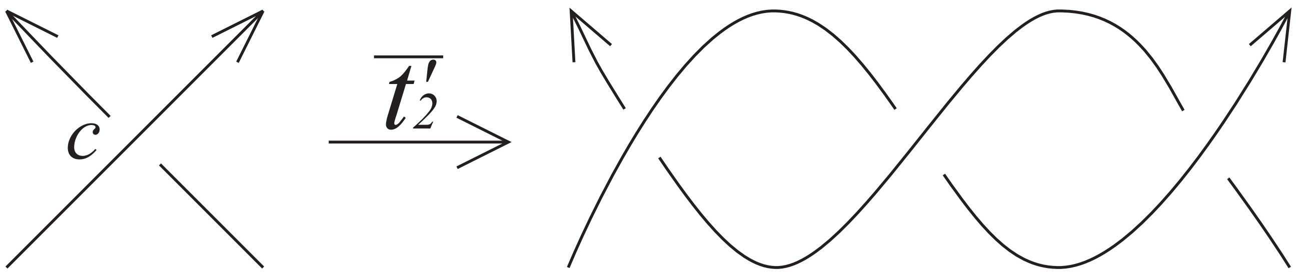

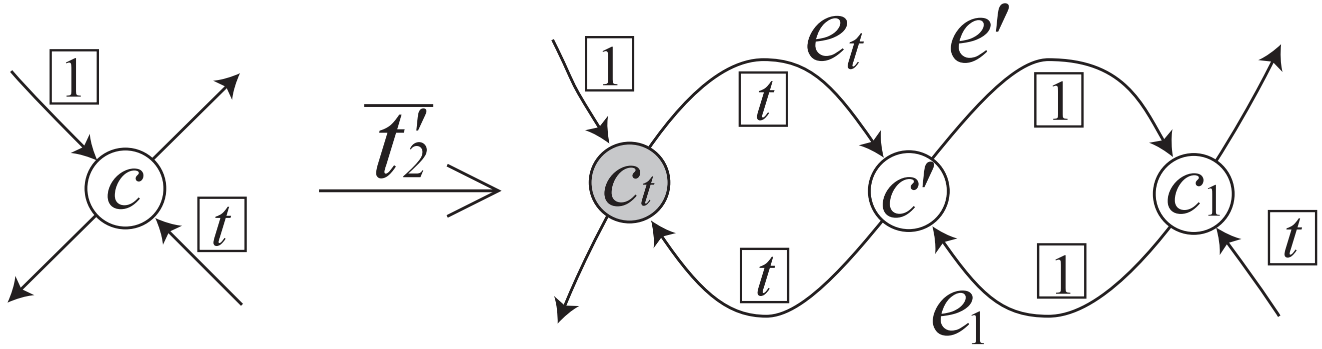

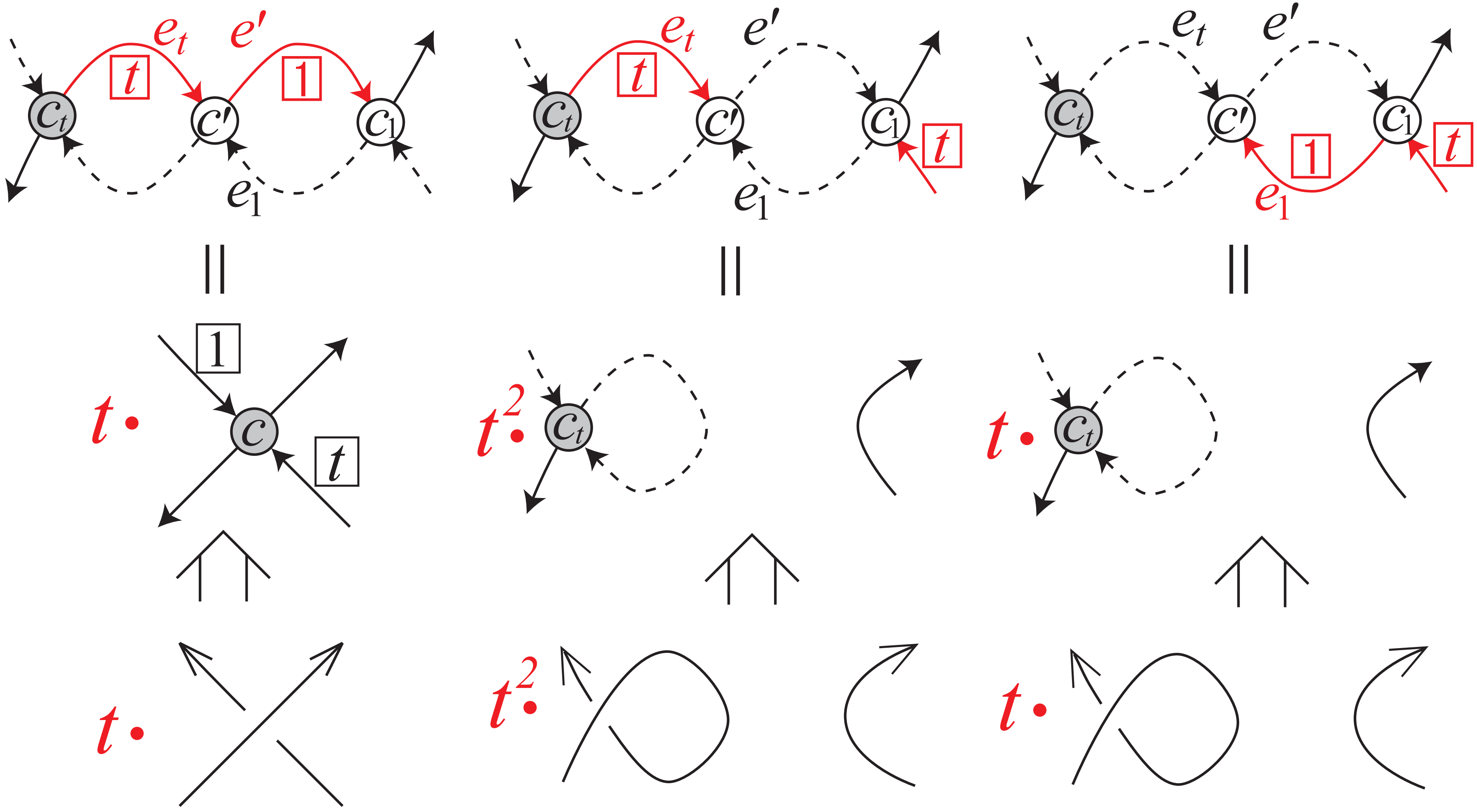

$$t\Delta_D \quad + \quad t(1+t)\Delta_{D/c}$$

$$\doteq \Delta_D + (1+t)\Delta_{D/c}$$

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Fact

c : a crossing of a reduced alternating diagram D

$\Rightarrow \deg \Delta_{D/c} = \deg \Delta_D - 1$

Lemma 2.5.

$D \in G_2$,

$c_1, \ldots, c_m$: crossings of D,

$D' = D(c_1^{k_1}, c_2^{k_2}, \ldots, c_m^{k_m})$, $(k_1, \ldots, k_m \in \mathbb{Z}_{\geq 0})$. Then

$$\begin{aligned}\Delta_{D'} = \Delta_D &+ \sum_{1 \leq i \leq m} k_i(1+t)\Delta_{D/c_i} \\ &+ \sum_{1 \leq i < j \leq m} k_i k_j (1+t)^2 \Delta_{D/c_i c_j} \\ &+ \sum_{1 \leq i < j < l \leq m} k_i k_j k_l (1+t)^3 \Delta_{D/c_i c_j c_l} \\ &+ \sum_{1 \leq i < j < l < p \leq m} k_i k_j k_l k_p (1+t)^4 \Delta_{D/c_i c_j c_l c_p}.\end{aligned}$$

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Remark

$[(1+t)^l \Delta_{D/c_{i_1}\cdots c_{i_l}}]_0 = [\Delta_{D/c_{i_1}\cdots c_{i_l}}]_0$,
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To estimate the ratios

$$\frac{[(1+t)^l \Delta_{D/c_{i_1}\cdots c_{i_l}}]_1}{[(1+t)^l \Delta_{D/c_{i_1}\cdots c_{i_l}}]_0} = \frac{[\Delta_{D/c_{i_1}\cdots c_{i_l}}]_1}{[\Delta_{D/c_{i_1}\cdots c_{i_l}}]_0} + l,$$

we define $m(D)$ and $M(D)$ by

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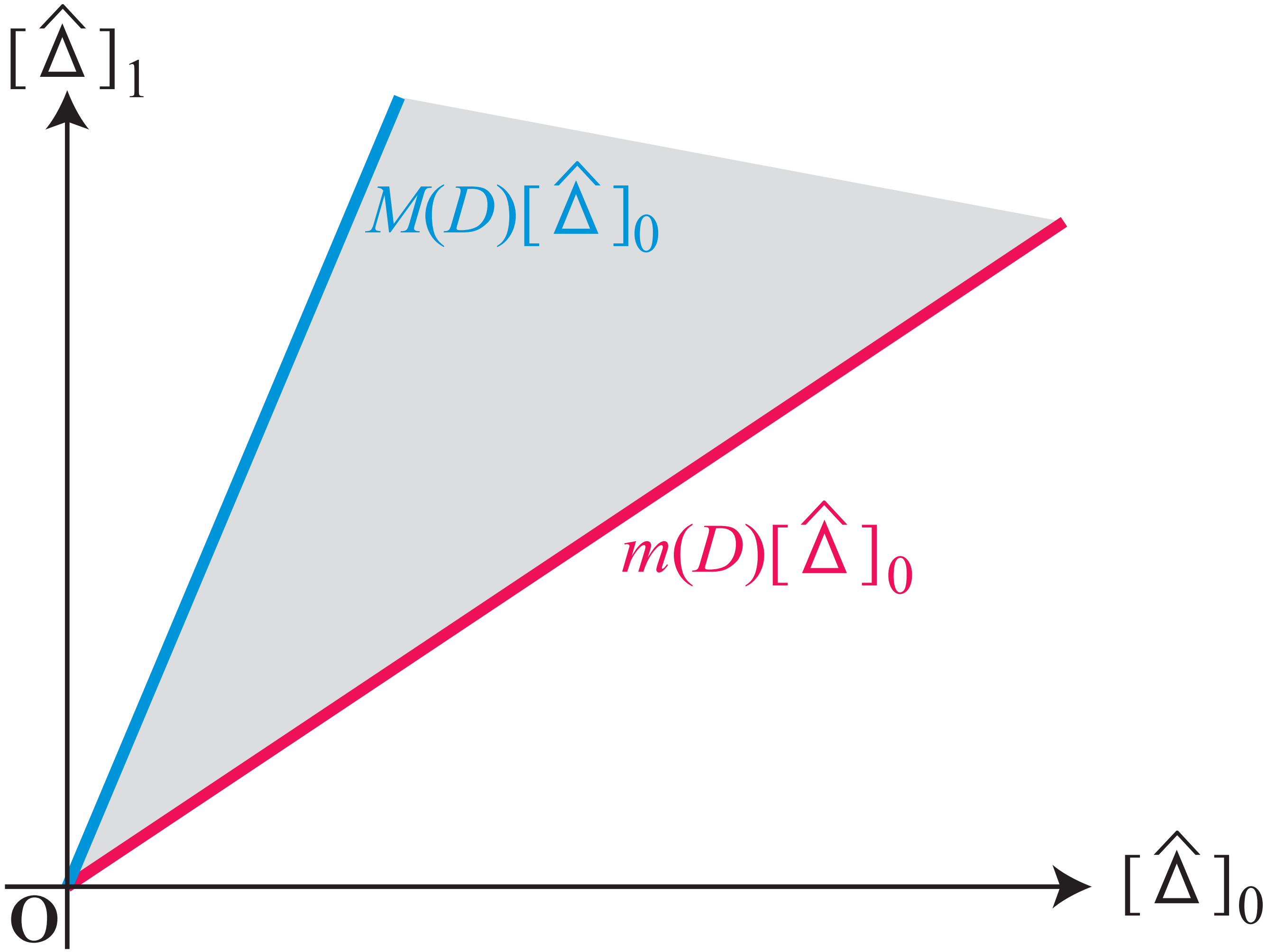

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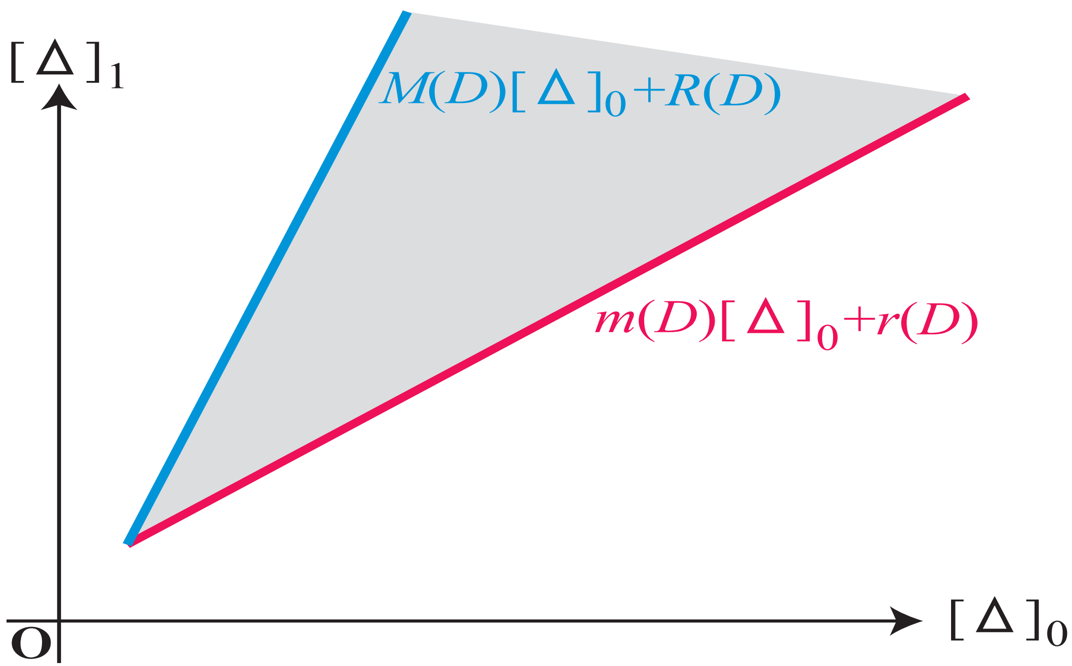
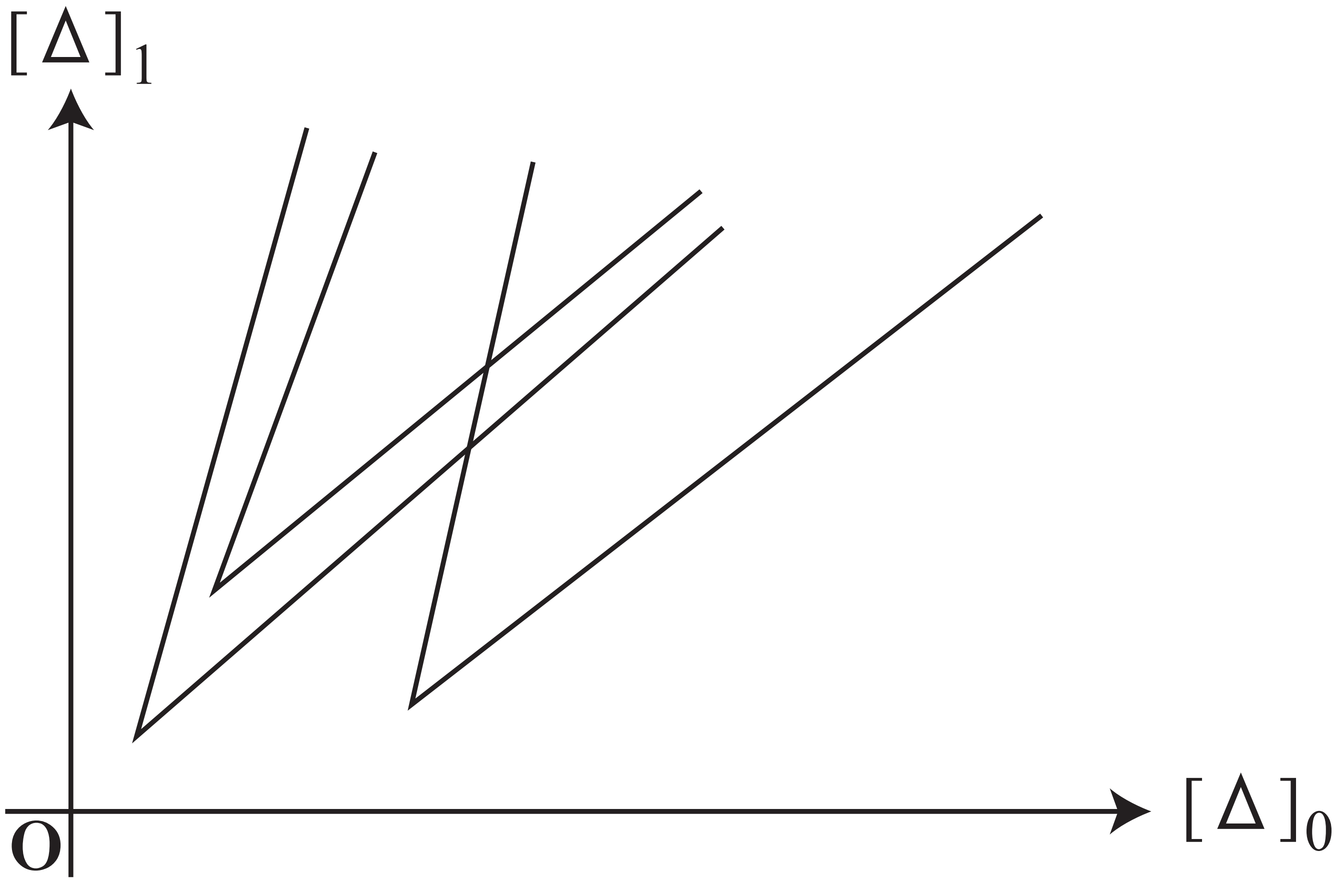
Set $r(D) = [\Delta_D]_1 - m(D)[\Delta_D]_0$ and

$R(D) = [\Delta_D]_1 - M(D)[\Delta_D]_0$. Then we obtain

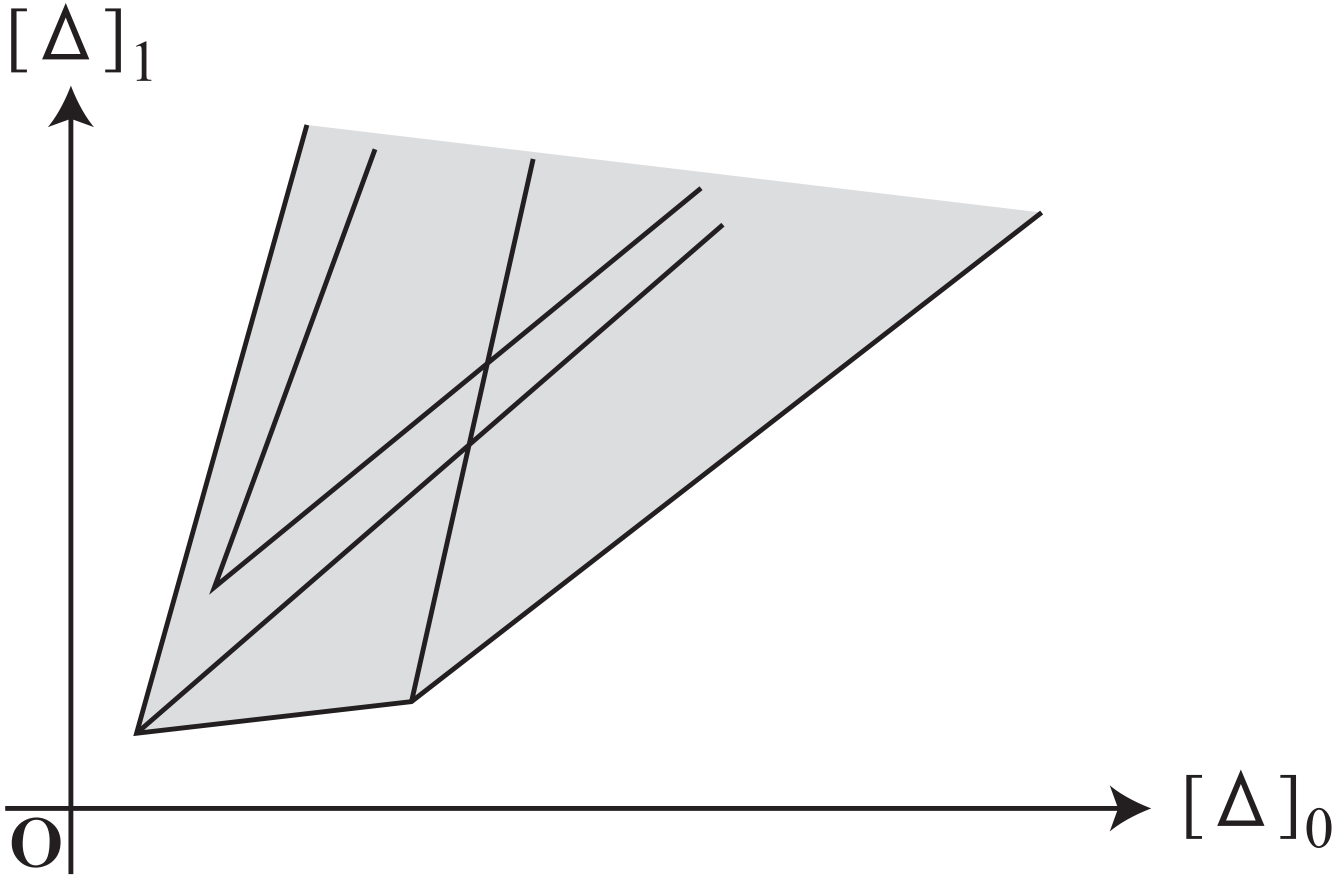
$$m(D)[\Delta_{D'}]_0 + r(D) \le [\Delta_{D'}]_1 \le M(D)[\Delta_{D'}]_0 + R(D).$$



By calculating $m(D), M(D), r(D)$, and $R(D)$ of the 27 generators, we obtain 27 inequalities.



By calculating $m(D), M(D), r(D)$, and $R(D)$ of the 27 generators, we obtain 27 inequalities.
By taking the convex hull for each σ, we obtain Main Theorem.



The inequalities which decide the boundary of the convex hull are the following red ones.

$\underline{\sigma = 0}$

G_2	m	M	r	R	$([\Delta]_0, [\Delta]_1, [\Delta]_2)$
6_3	3	5	0	-2	$(1, 3, 5)$
7_7	3	6	2	-1	$(1, 5, 9)$
8_{12}	4	6	3	1	$(1, 7, 13)$
9_{41}	3	$\frac{14}{3}$	3	-2	$(3, 12, 19)$
10_{58}	$\frac{10}{3}$	$\frac{14}{3}$	6	2	$(3, 16, 27)$
12_{1202}	4	$\frac{13}{3}$	6	3	$(9, 42, 67)$
$3_1\#3_1^*$	3	4	-1	-2	$(1, 2, 3)$

$|\sigma| = 2$

G_2	m	M	r	R	$([\Delta]_0, [\Delta]_1, [\Delta]_2)$
6_2	2	5	1	-2	$(1, 3, 3)$
7_6	3	6	2	-1	$(1, 5, 7)$
8_{14}	3	5	2	-2	$(2, 8, 11)$
9_{25}	$\frac{10}{3}$	$\frac{14}{3}$	2	-2	$(3, 12, 17)$
9_{39}	$\frac{10}{3}$	$\frac{11}{2}$	4	$-\frac{5}{2}$	$(3, 14, 21)$
10_{97}	$\frac{18}{5}$	$\frac{14}{3}$	4	$-\frac{4}{3}$	$(5, 22, 33)$
11_{148}	$\frac{25}{7}$	$\frac{23}{5}$	4	$-\frac{16}{5}$	$(7, 29, 43)$
$3_1\#4_1$	3	5	-1	1	$(1, 4, 5)$

$|\sigma| = 4$

G_2	m	M	r	R	$([\Delta]_0, [\Delta]_1, [\Delta]_2)$
5_1	2	4	-1	-3	$(1, 1, 1)$
7_5	$\frac{5}{2}$	4	-1	-4	$(2, 4, 5)$
8_{15}	3	4	-1	-4	$(3, 8, 11)$
9_{23}	$\frac{13}{4}$	4	-2	-5	$(4, 11, 15)$
9_{38}	3	4	-1	-6	$(5, 14, 19)$
10_{101}	$\frac{13}{4}$	4	$-\frac{7}{4}$	-7	$(7, 21, 29)$
10_{120}	$\frac{25}{7}$	4	$-\frac{18}{7}$	-6	$(8, 26, 37)$
11_{123}	$\frac{24}{7}$	4	$-\frac{13}{7}$	-7	$(9, 29, 41)$
11_{329}	$\frac{7}{2}$	4	$-\frac{5}{2}$	-8	$(11, 36, 51)$
12_{1097}	$\frac{7}{2}$	4	-2	-10	$(16, 54, 77)$
13_{4233}	$\frac{11}{3}$	4	-3	-10	$(21, 74, 107)$
$3_1 \# 3_1$	3	4	-1	-2	$(1, 2, 3)$

Main Theorem

Let K be an alternating knot of genus 2.

Then the following inequalities hold ($[\Delta]_0 \geq 1$):

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Moreover, any other linear inequality on $[\Delta]_0$ and $[\Delta]_1$ for all alternating knots of genus 2 is a consequence of our inequalities. (Completeness)

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Then the following inequalities hold ($[\Delta]_0 \geq 1$):

$$3[\Delta_K]_0 - 1 \leq [\Delta_K]_1 \leq 6[\Delta_K]_0 + 1 \text{ if } \sigma(K) = 0,$$

$$2[\Delta_K]_0 + 1 \leq [\Delta_K]_1 \leq 6[\Delta_K]_0 - 1 \text{ if } |\sigma(K)| = 2,$$

$$2[\Delta_K]_0 - 1 \leq [\Delta_K]_1 \leq 4[\Delta_K]_0 - 2 \text{ if } |\sigma(K)| = 4.$$

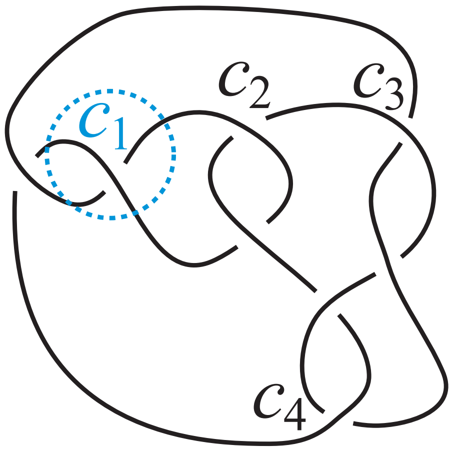
Moreover, any other linear inequality on $[\Delta]_0$ and $[\Delta]_1$ for all alternating knots of genus 2 is a consequence of our inequalities. (**Completeness**)

Proof of the completeness for $\sigma = 0$ (upper bound)

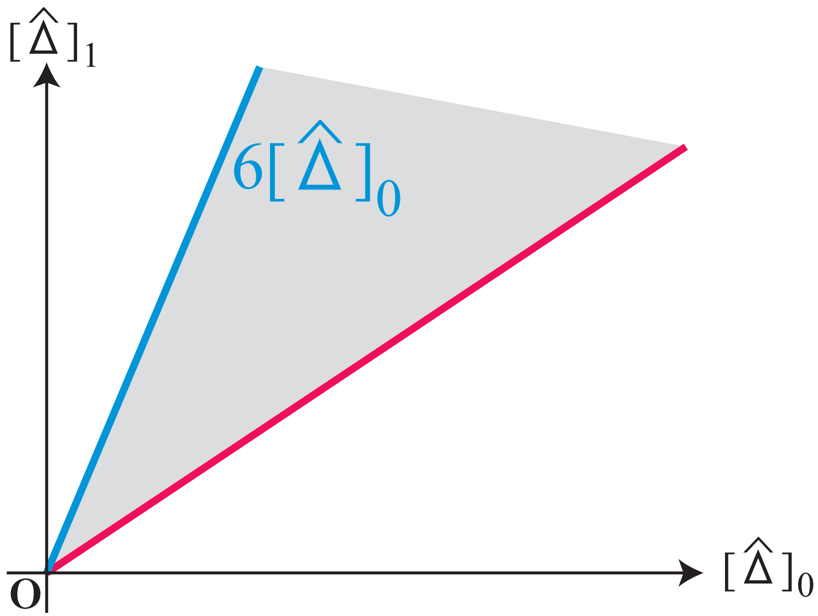
Let $D = 8_{12}$.

Then $[\widehat{\Delta}_{D(c_1^n)}]_0 = n$, $[\widehat{\Delta}_{D(c_1^n)}]_1 = 6n$, we have

$$\frac{[\widehat{\Delta}_{D(c_1^n)}]_1}{[\widehat{\Delta}_{D(c_1^n)}]_0} = 6.$$



8_{12}



§3. Characterization of the alternating knots of genus two with $[\Delta]_0 \leq 3$

Corollary 3.1.

D: a reduced alternating diagram

$\Rightarrow [\Delta_{D(c)}]_0 > [\Delta_D]_0$.

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Let L be an alternating link. L is fibered $\Leftrightarrow [\Delta]_0 = 1$.

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Let L be an alternating link. L is fibered $\Leftrightarrow [\Delta]_0 = 1$.

Fact

The fibered knots of genus one are just 3_1, 3_1^* and 4_1.

The alternating fibered knots of genus 2 ($[\Delta]_0 = 1$)

The knots in G_2 with $[\Delta]_0 = 1$ are just 5_1, 6_2, 6_3, 7_6, 7_7, 8_{12}, $3_1\#3_1$, $3_1\#3_1^*$, $3_1\#4_1$, and $4_1\#4_1$ up to $*$.

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Theorem 3.4.

The alternating fibered knots of genus 2 are just the following knots: 5_1, 5_1^*, 6_2, 6_2^*, 6_3, 7_6, 7_6^*, 7_7, 7_7^*, 8_{12}, $3_1\#3_1$, $3_1\#3_1^*$, $3_1^*\#3_1^*$, $3_1\#4_1$, $3_1^*\#4_1$, and $4_1\#4_1$.

The alternating fibered knots of genus 2 ($[\Delta]_0 = 1$)

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We denote the set of these knot diagrams by AF_2.

Corollary 3.5.

The Alexander polynomials which have the trapezoidal property

$1 - n_1 t + (2n_1 - 1)t^2 - n_1 t^3 + t^4$ for $n_1 = 4$ or $n_1 \geq 8$,

$1 - n_2 t + (2n_2 - 3)t^2 - n_2 t^3 + t^4$ for $n_2 \geq 6$

are never realized by an alternating knot.

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Remark

$\Delta_K(t) = 1 - 4t + 7t^2 - 4t^3 + t^4$

The knot with this Δ satisfies the inequality in Main Theorem.

However, this polynomial is never realized by an alternating knot.

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Incidentally, $\Delta_{9_{44}}(t) = 1 - 4t + 7t^2 - 4t^3 + t^4$.

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α_1, α_2, $\overline{\alpha_1}$, $\overline{\alpha_2} \in \mathbb{C}$: zeros of $\Delta_K(t) \Rightarrow |\alpha_1|, |\alpha_2| \neq 1$.

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$\therefore \sigma(K) = 0$. Then

The ineq. in Main Thm $\Leftrightarrow 3[\Delta]_0 - 1 \leq [\Delta]_1 \leq 6[\Delta]_0 + 1$.

(Ozsváth-Szabó's ineq. $\Leftrightarrow 2[\Delta]_0 \leq [\Delta]_1$.)

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$\therefore$The knot with $\Delta_K(t) = 1 - 4t + 7t^2 - 4t^3 + t^4$ satisfies the inequality in Main Theorem. (Trapezoidal property and Ozsváth-Szabó's inequality are also satisfied.)

However, a knot with this Δ is non-alternating knot.

The alternating knots of genus 2 with $[\Delta]_0 = 2$

The knots in G_2 with $[\Delta]_0 = 2$ are just 7_5, 8_{14} up to $*$.

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Other alternating knots of genus 2 with $[\Delta]_0 = 2$ are obtained by applying once $\overline{t'_2}$ move at a crossing of a diagram in AF_2: $5_1(c_1) = 7_3$, $6_2(c_1) = 8_{11}$, $6_2(c_2) = 8_4$, $6_2(c_3) = 8_6$, $6_3(c_1) = 8_{13}$, $6_3(c_2) = 8_8$, $6_3(c_3) = 8_8$, $6_3(c_4) = 8_{13}$, $7_6(c_1) = 9_8$, $7_6(c_2) = 9_{21}$, $7_6(c_3) = 9_{15}$, $7_6(c_4) = 9_{12}$, $7_7(c_1) = 9_{14}$, $7_7(c_2) = 9_{14}$, $7_7(c_3) = 9_{19}$, $7_7(c_4) = 9_{37}$, $7_7(c_5) = 9_{19}$, $8_{12}(c_1) = 10_{13}$, $8_{12}(c_2) = 10_{35}$, $8_{12}(c_3) = 10_{13}$, $8_{12}(c_4) = 10_{35}$.
The composite alternating knots of genus 2 with $[\Delta]_0 = 2$ are just $3_1\#5_2$, $3_1\#6_1$, $4_1\#5_2$, and $4_1\#6_1$ up to $*$ for each factor.

Theorem 3.6.

The alternating knots of genus 2 with $[\Delta]_0 = 2$ are just the following knots up to $*$: 7_3, 7_5, 8_4, 8_6, 8_8, 8_{11}, 8_{13}, 8_{14}, 9_8, 9_{12}, 9_{14}, 9_{15}, 9_{19}, 9_{21}, 9_{37}, 10_{13}, 10_{35}, $3_1\#5_2$, $3_1^*\#5_2$, $3_1\#6_1$, $3_1^*\#6_1$, $4_1\#5_2$, and $4_1\#6_1$.

The alternating knots of genus 2 with $[\Delta]_0 = 3$

By the same way (i.e. by applying twice $\overline{t'_2}$ moves on AF_2), we have the following theorem.

Theorem 3.7.

The alternating prime knots of genus 2 with $[\Delta]_0 = 3$ are just the following knots up to $*$: 9_4, 9_7, 10_4, 10_7, 10_{10}, 10_{20}, 10_{34}, 10_{36}, 11_{13}, 11_{59}, 11_{65}, 11_{195}, 11_{211}, 11_{214}, 11_{230}, 12_{197}, 12_{691}, $3_1\#7_2$, $3_1^*\#7_2$, $3_1\#8_1$, $3_1^*\#8_1$, $4_1\#7_2$, and $4_1\#8_1$.

Corollary 3.8.

The Alexander polynomials which have the trapezoidal property

$2 - n_1 t + (2n_1 - 3)t^2 - n_1 t^3 + 2t^4$ for $n_1 = 8$ or $n_1 \geq 14$,

$2 - n_2 t + (2n_2 - 5)t^2 - n_2 t^3 + 2t^4$ for $n_2 \geq 12$

are never realized by an alternating knot.

Corollary 3.9.

The Alexander polynomials which have the trapezoidal property

$3 - n_1 t + (2n_1 - 5)t^2 - n_1 t^3 + 3t^4$ for $n_1 = 6, 10, 14, 18$ or $n_1 \geq 20$,

$3 - n_2 t + (2n_2 - 7)t^2 - n_2 t^3 + 3t^4$ for $n_2 = 8, 16$ or $n_2 \geq 18$

are never realized by an alternating knot.

§4. Non-alternating knots up to 10 crossings

Fact

$[\Delta]_0 \leq 3$ holds for any non-alternating prime knot up to 10 crossings (in Rolfsen's table).

There exists non-alternating knots which satisfy our inequality. (e.g. $\Delta(t) = 1 - 4t + 7t^2 - 4t^3 + t^4$.)

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There exists non-alternating knots which satisfy our inequality. (e.g. $\Delta(t) = 1 - 4t + 7t^2 - 4t^3 + t^4$.) i.e. we have non-alternating knots whose Δ are similar to those of alternating knots.

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$[\Delta]_0 \leq 3$ holds for any non-alternating prime knot up to 10 crossings (in Rolfsen's table).

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We enumerate these non-alternating knots with $\deg \Delta =$ 4 up to 10 crossings.

$[\Delta]_0 = 1$

K	$([\Delta]_0,\ [\Delta]_1,\ [\Delta]_2)$	$\lvert\sigma\rvert$	alt. knot
8_{20}	$(1,2,3)$	0	$3_1\#3_1$
8_{21}	$(1,4,5)$	2	$3_1\#4_1$
9_{44}	$(1,4,7)$	0	$\nexists$
9_{45}	$(1,6,9)$	2	$\nexists$
9_{48}	$(1,7,11)$	2	$\nexists$
10_{132}	$(1,1,1)$	0	5_1
10_{133}	$(1,5,7)$	2	7_6
10_{136}	$(1,4,5)$	2	$3_1\#4_1$
10_{137}	$(1,6,11)$	0	$4_1\#4_1$
10_{140}	$(1,2,3)$	0	$3_1\#3_1$

$[\Delta]_0 = 1$

K	$([\Delta]_0,\ [\Delta]_1,\ [\Delta]_2)$	$\lvert\sigma\rvert$	alt. knot
8_{20}	$(1,2,3)$	0	$3_1\#3_1$
8_{21}	$(1,4,5)$	2	$3_1\#4_1$
9_{44}	$(1,4,7)$	0	$\nexists$
9_{45}	$(1,6,9)$	2	$\nexists$
9_{48}	$(1,7,11)$	2	$\nexists$
10_{132}	$(1,1,1)$	0	5_1
10_{133}	$(1,5,7)$	2	7_6
10_{136}	$(1,4,5)$	2	$3_1\#4_1$
10_{137}	$(1,6,11)$	0	$4_1\#4_1$
10_{140}	$(1,2,3)$	0	$3_1\#3_1$

Corollary 3.5.

$1 - n_1 t + (2n_1 - 1)t^2 - n_1 t^3 + t^4$ for $n_1 = 4$ or $n_1 \geq 8$,

$1 - n_2 t + (2n_2 - 3)t^2 - n_2 t^3 + t^4$ for $n_2 \geq 6$

are never realized by an alternating knot.

$[\Delta]_0 = 2$

K	$([\Delta]_0,\ [\Delta]_1,\ [\Delta]_2)$	$\|\sigma\|$	alt. knot
10_{129}	$(2,6,9)$	0	8_8
10_{130}	$(2,4,5)$	0	7_5
10_{131}	$(2,8,11)$	2	$8_{14}, 9_8$
10_{146}	$(2,8,13)$	0	$\nexists$
10_{147}	$(2,7,9)$	2	8_{11}
10_{166}	$(2,10,15)$	2	9_{15}

$[\Delta]_0 = 3$

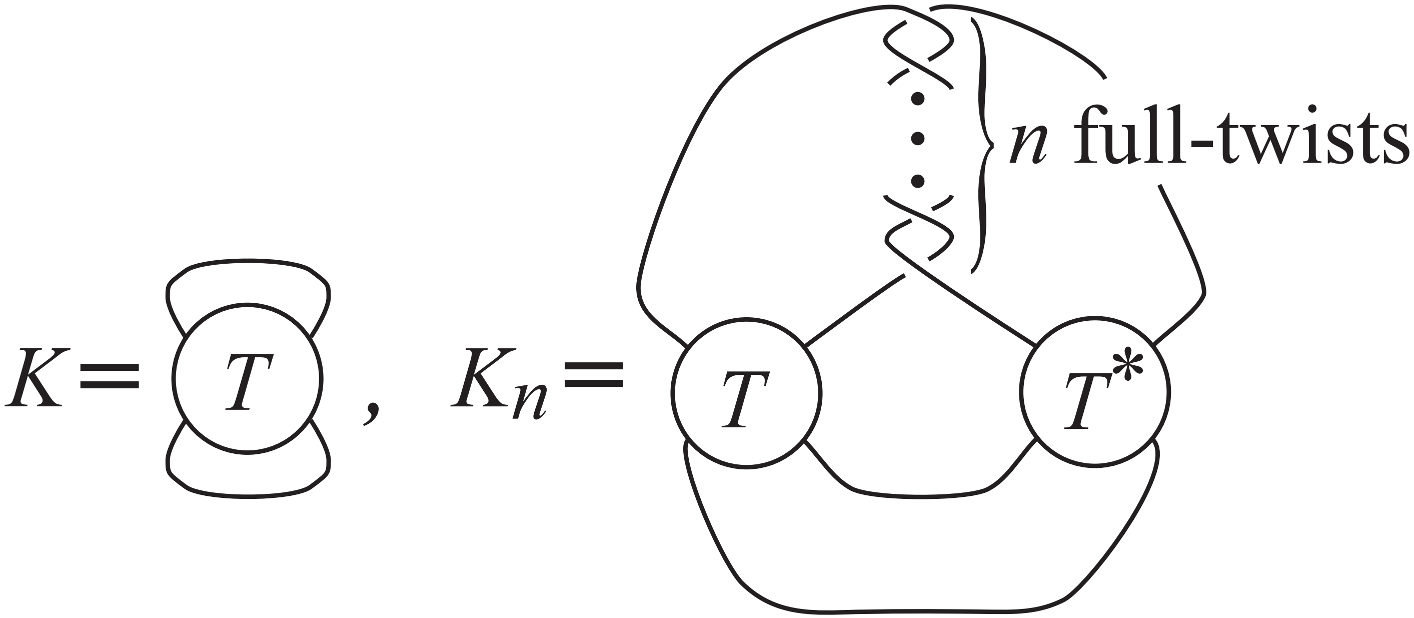
K	$([\Delta]_0,\ [\Delta]_1,\ [\Delta]_2)$	$\|\sigma\|$	alt. knot
9_{49}	$(3,6,7)$	4	$\nexists$
10_{135}	$(3,9,13)$	0	10_{34}
10_{144}	$(3,10,13)$	2	$3_1\#8_1$
10_{163}	$(3,9,11)$	2	10_{20}
10_{165}	$(3,11,17)$	0	10_{10}

Symmetric union

K: a knot represented by a closure of a tangle T.

Definition ['57 S. Kinoshita-H. Terasaka]

The *symmetric unions* of K, denoted by K_n, are the knots represented by the following diagrams.



Proposition 4.1. $\Delta_{K_n} = \Delta_K^2$ for $\forall n \in \mathbb{Z}$.

- $K = 3_1 \Rightarrow K_0 = 3_1 \# 3_1^*,\ K_2 = 8_{20},\ K_4 = 10_{140}$.

- $K = 3_1 \Rightarrow K_0 = 3_1 \# 3_1^*$, $K_2 = 8_{20}$, $K_4 = 10_{140}$.
- $K = 4_1 \Rightarrow K_0 = 4_1 \# 4_1$, $K_2 = 10_{137}$.

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$[\Delta]_0 = 1$

K	$([\Delta]_0, [\Delta]_1, [\Delta]_2)$	$\lvert\sigma\rvert$	alt. knot
8_{20}	$(1,2,3)$	0	$3_1 \# 3_1$
8_{21}	$(1,4,5)$	2	$3_1 \# 4_1$
9_{44}	$(1,4,7)$	0	$\nexists$
9_{45}	$(1,6,9)$	2	$\nexists$
9_{48}	$(1,7,11)$	2	$\nexists$
10_{132}	$(1,1,1)$	0	5_1
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10_{136}	$(1,4,5)$	2	$3_1 \# 4_1$
10_{137}	$(1,6,11)$	0	$4_1 \# 4_1$
10_{140}	$(1,2,3)$	0	$3_1 \# 3_1$

$[\Delta]_0 = 2$

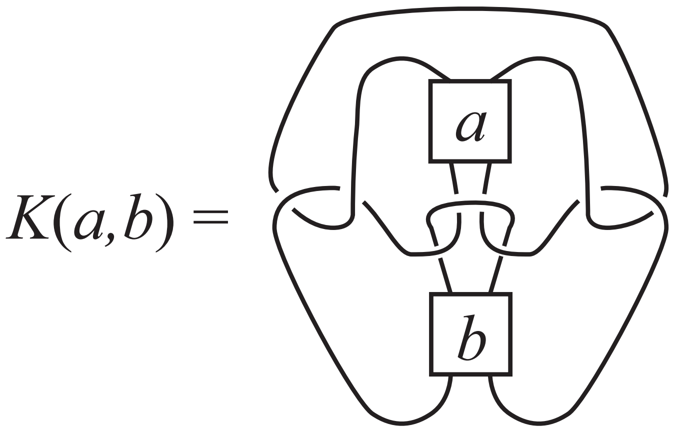
K	$([\Delta]_0,\ [\Delta]_1,\ [\Delta]_2)$	$\lvert\sigma\rvert$	alt. knot
10_{129}	$(2, 6, 9)$	0	8_8
10_{130}	$(2, 4, 5)$	0	7_5
10_{131}	$(2, 8, 11)$	2	$8_{14}, 9_8$
10_{146}	$(2, 8, 13)$	0	$\nexists$
10_{147}	$(2, 7, 9)$	2	8_{11}
10_{166}	$(2, 10, 15)$	2	9_{15}

$[\Delta]_0 = 3$

K	$([\Delta]_0,\ [\Delta]_1,\ [\Delta]_2)$	$\lvert\sigma\rvert$	alt. knot
9_{49}	$(3, 6, 7)$	4	$\nexists$
10_{135}	$(3, 9, 13)$	0	10_{34}
10_{144}	$(3, 10, 13)$	2	$3_1\#8_1$
10_{163}	$(3, 9, 11)$	2	10_{20}
10_{165}	$(3, 11, 17)$	0	10_{10}

Kanenobu's knot family

T. Kanenobu discovered families of knots, denoted by $K(a,b)$.



$K(0,-1) = 8_8$,

$K(2,-1) = 10_{129}$,

$K(0,0) = 4_1 \# 4_1$,

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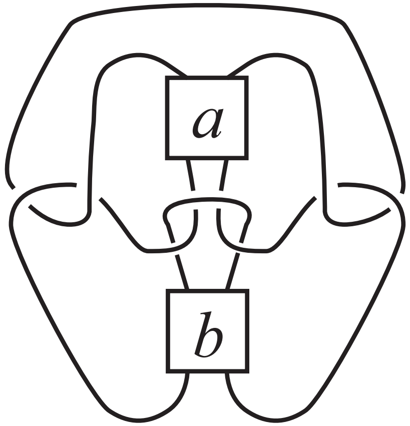
$K(a,b) =$ 

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Proposition 4.2. ['86 T. Kanenobu]

$\Delta_{K(a,b)} = \Delta(\varepsilon,\delta)$ $(\varepsilon \equiv a, \delta \equiv b \mod 2)$.

Here $\Delta(0,0) = (1,6,11)$, $\Delta(0,1) = \Delta(1,0) = (2,6,9)$, $\Delta(1,1) = (1,3,5,7)$.

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