A categorification of the one-variable Kamada–Miyazawa polynomial

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[January 23, 2008]

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Khovanov homology for classical links [Khovanov]

L: an ori. classical link, D: a diagram of L

$(C(D), d)$: a chain complex of graded $\mathbb{Z}$-modules

$$C(D) = \oplus_i C^i(D), \qquad d : C^i(D) \to C^{i+1}(D)$$
$$C^i(D) = \oplus_j C^{i,j}(D), \quad d : C^{i,j}(D) \to C^{i+1,j}(D)$$

Each $H^{i,j}(D) = H^i(C^{*,j}(D), d)$ is an invariant of L.

The graded Euler characteristic of $H(L)$

is the Jones polynomial $J(L)$ of L.

$$\sum_i (-1)^i \, q^j \, \mathrm{rank}(H^{i,j}(L)) = J(L) \ \left(\in \mathbb{Z}[q^{\pm 1}]\right)$$

The aim of this talk

To extend Khovanov homology

from classical links to virtual links.

Note

Other extensions were defined in

[Manturov] and [Turaev–Turner].

Plan of my talk

- Reviews of **Khovanov homology**

- Review of **Virtual links**

- Difficulty of extending Khovanov homology to virtual links
  $\longrightarrow$ the existence of **Möbius cobordisms**

- **Miyazawa polynomial** (one-variable version)
  $\nwarrow$ this is defined by using pole diagrams.

- Construction of our homology:
  Degree shifts
  Maps for Möbius and usual saddle cobordisms

- Example of computations

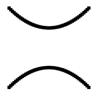
Jones polynomial and Khovanov complex

L: an ori. classical link, D: a diagram of L

($n_+ := \#$  of D, $n_- := \#$  of D)

- Jones polynomial $\left(\langle D \rangle, J(D) \in \mathbb{Z}[q^{\pm 1}]\right)$

$$\begin{cases} \langle\ \text{[crossing]}\ \rangle = \langle\ )\ (\ \rangle - q \langle\ \text{[smoothing]}\ \rangle \\ \langle \bigcirc \cup D \rangle = (q + q^{-1}) \langle D \rangle \end{cases}$$

(crossing: , smoothing: )

$$J(D) := (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle \in \mathbb{Z}[q^{\pm 1}]$$

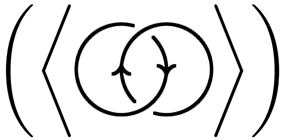
- Khovanov complex

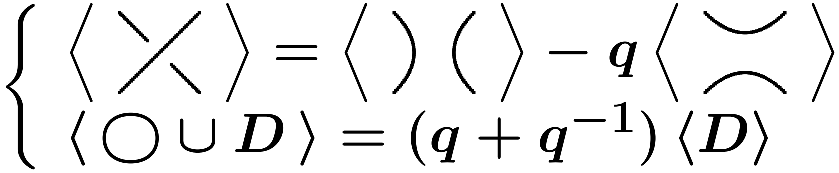
$$\begin{cases} \overline{C}(\ \text{[crossing]}\ ) = \mathbf{Cone}\left(\overline{C}(\ )\ (\ ) \to \overline{C}(\ \text{[smoothing]}\ )\{1\}\right) \\ \overline{C}(\bigcirc \cup D) = V \otimes \overline{C}(D) \quad (\mathrm{qdim}(V) = q + q^{-1}) \end{cases}$$

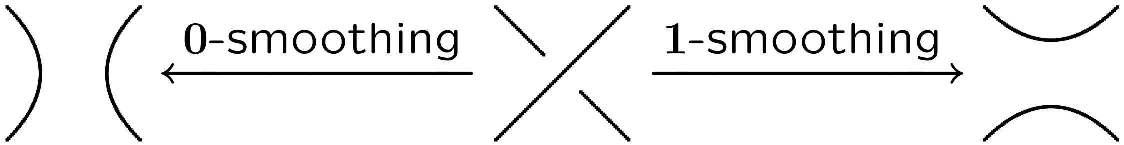
(crossing: , smoothing: )

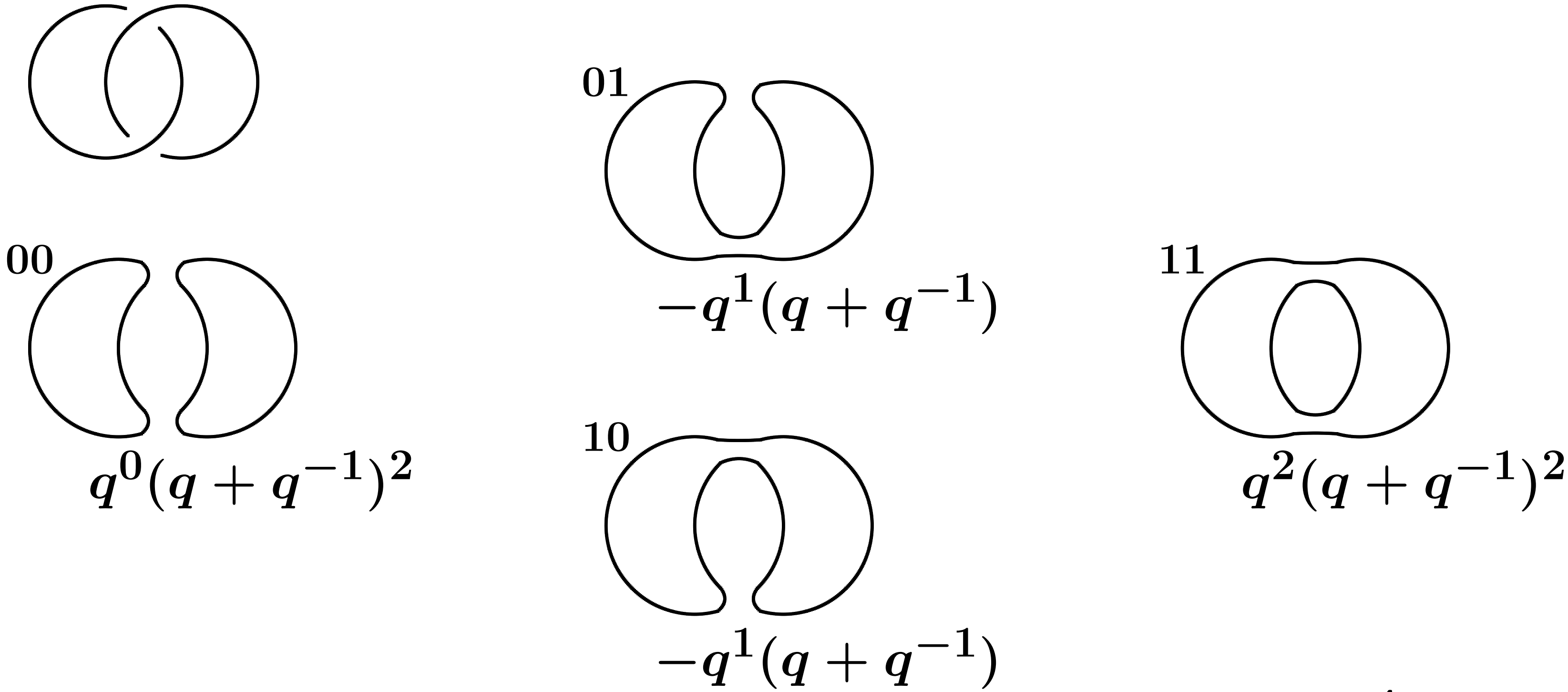
$$C(D) := \overline{C}(D)[-n_-]\{n_+ - 2n_-\},$$

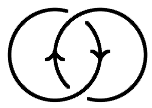
where $(\overline{C}[k]\{l\})^{i,j} = \overline{C}^{i-k,j-l}$.

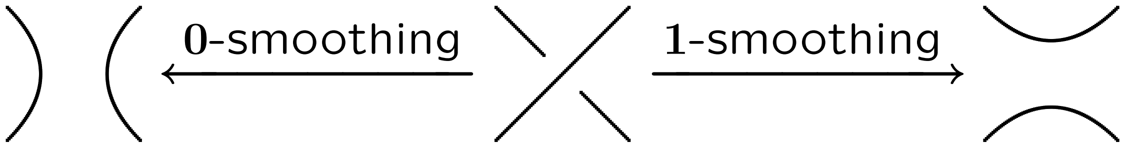
Example 

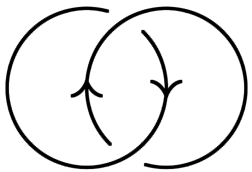

$\langle \bigcirc \cup D \rangle = (q + q^{-1}) \langle D \rangle$


0-smoothing
1-smoothing


00
$q^0(q + q^{-1})^2$
01
$-q^1(q + q^{-1})$
10
$-q^1(q + q^{-1})$
11
$q^2(q + q^{-1})^2$

Example $\Big(J\Big($$\Big)\Big)$ $\qquad J(D) := (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle$





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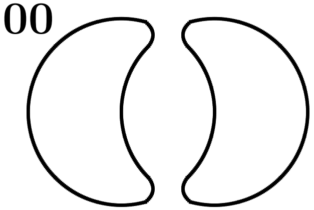
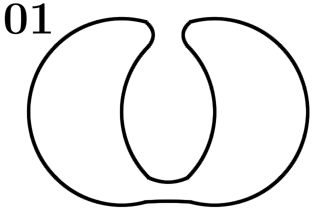


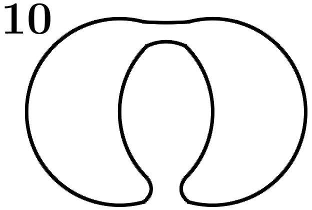
$q^{-4}(q+q^{-1})^2$

01



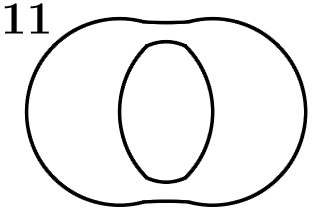
$-q^{-3}(q+q^{-1})$

10



$-q^{-3}(q+q^{-1})$

11



$q^{-2}(q+q^{-1})^2$

Frobenius algebra V and $(1+1)$-TQFT $\mathcal{F}$

- Frobenius algebra (the graded $\mathbb{Z}$-module V)

$$V = \mathbb{Z}[X]/(X^2) \cong \mathbb{Z}\langle 1\rangle \oplus \mathbb{Z}\langle X\rangle$$
$$(\deg(1) = 1,\ \deg(X) = -1) \rightsquigarrow \mathrm{qdim}(V) = q + q^{-1}$$

We give V a Frobenius algebra structure with

$$m : V \otimes V \longrightarrow V \qquad \begin{cases} m(1\otimes 1) = 1,\ m(X\otimes X) = 0 \\ m(1\otimes X) = m(X\otimes 1) = X \end{cases}$$

$$\Delta : V \longrightarrow V \otimes V \qquad \begin{cases} \Delta(1) = 1\otimes X + X\otimes 1 \\ \Delta(X) = X\otimes X \end{cases}$$

$$\iota : \mathbb{Z} \longrightarrow V \qquad \iota(1) = 1$$

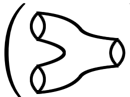
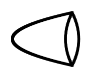
$$\epsilon : V \longrightarrow \mathbb{Z} \qquad \epsilon(1) = 0, \quad \epsilon(X) = 1.$$

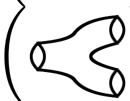
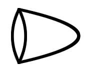
- $(1+1)$-TQFT $\mathcal{F}$

For objects,

$\mathcal{F}(\emptyset) = \mathbb{Z}$, $\mathcal{F}(\bigcirc) = V$, $\mathcal{F}(\bigcirc\,\bigcirc) = V\otimes V, \ldots$ etc.

For morphisms,

$\mathcal{F}\Bigg($  $\Bigg) = m : V\otimes V \longrightarrow V$, $\mathcal{F}\Bigg($  $\Bigg) = \iota : \mathbb{Z} \longrightarrow V$,

$\mathcal{F}\Bigg($  $\Bigg) = \Delta : V \longrightarrow V\otimes V$, $\mathcal{F}\Bigg($  $\Bigg) = \epsilon : V \longrightarrow \mathbb{Z}$.

Jones polynomial and Khovanov complex

L: an ori. classical link, D: a diagram of L

$(n_+ := \#\times$ of D, $n_- := \#\times$ of $D)$

- Jones polynomial $\left(\langle D \rangle, J(D) \in \mathbb{Z}[q^{\pm 1}]\right)$

$$\begin{cases} \langle \times \rangle = \langle)\,(\rangle - q \langle \asymp \rangle \\ \langle \bigcirc \cup D \rangle = (q + q^{-1}) \langle D \rangle \end{cases}$$

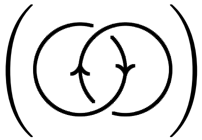
$$J(D) := (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle \in \mathbb{Z}[q^{\pm 1}]$$

- Khovanov complex

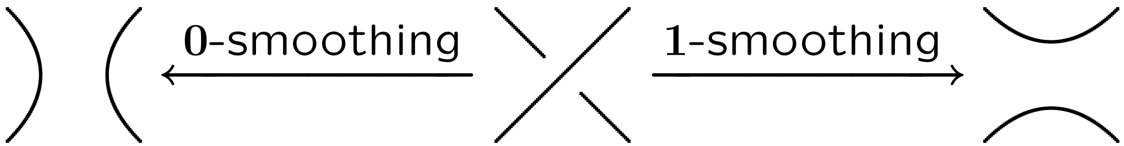
$$\begin{cases} \overline{C}(\times) = \mathrm{Cone}\left(\overline{C}(\,)\,(\,) \to \overline{C}(\asymp)\{1\}\right) \\ \overline{C}(\bigcirc \cup D) = V \otimes \overline{C}(D) \quad (\mathrm{qdim}(V) = q + q^{-1}) \end{cases}$$

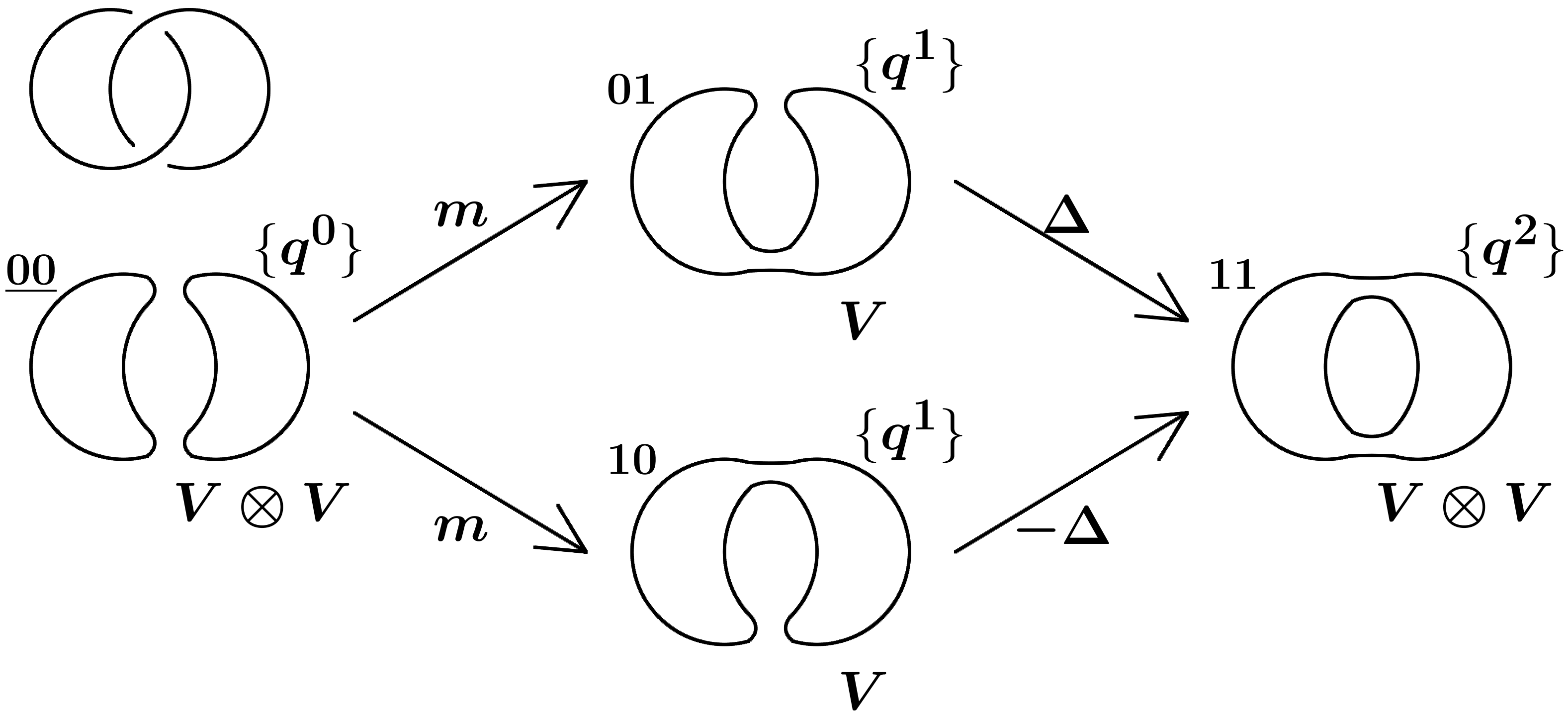
$$C(D) := \overline{C}(D)[-n_-]\{n_+ - 2n_-\},$$

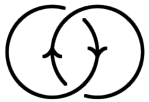
where $(\overline{C}[k]\{l\})^{i,j} = \overline{C}^{i-k,j-l}$.

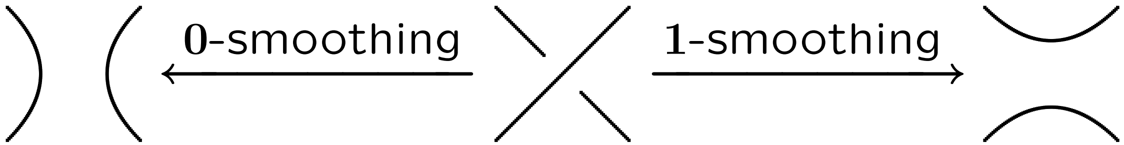
Example $\left(\overline{C}\right.$  $\left.\right)$

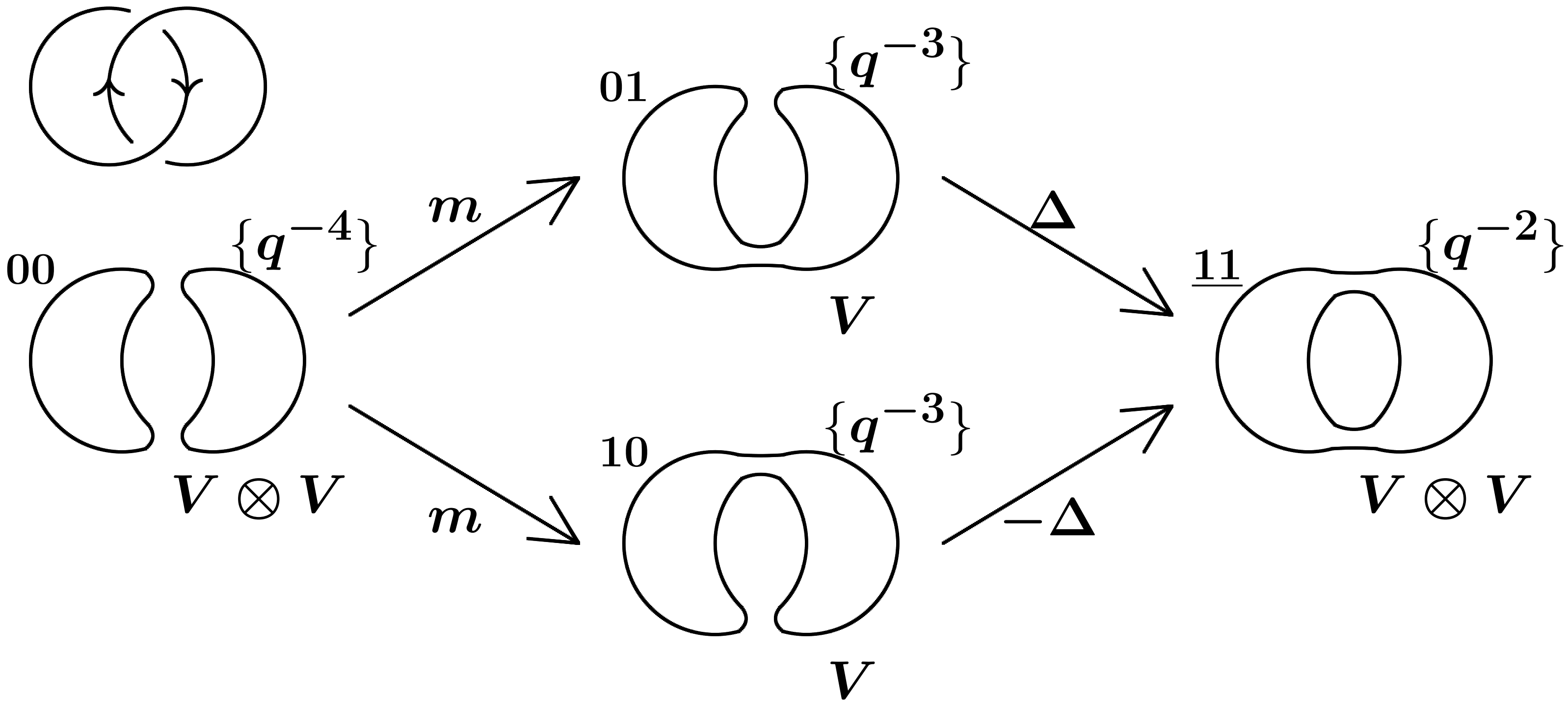
$$\begin{cases} \overline{C}\,(\times) = \mathrm{Cone}\left(\overline{C}\,(\,)\,(\,) \to \overline{C}\,(\asymp)\,\{1\}\right) \\ \overline{C}\,(\bigcirc \cup D) = V \otimes \overline{C}(D) \end{cases}$$





Example $C\Big($  $\Big)$ $\quad C(D) := \overline{C}(D)[-n_-]\{n_+ - 2n_-\}$

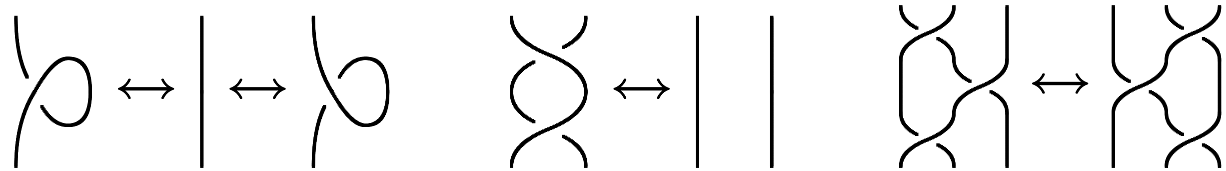




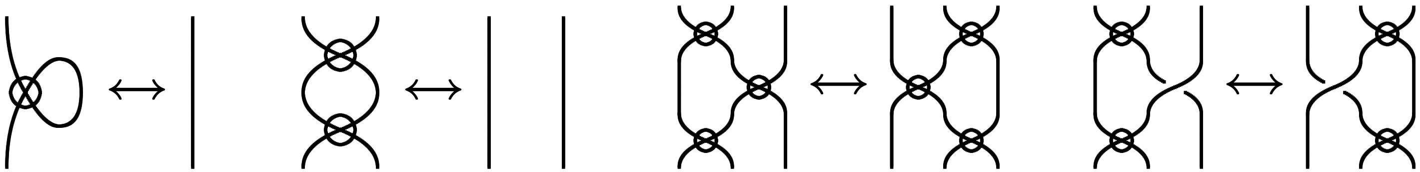
Virtual links [Kauffman]

$\{\text{virtual link}\} := \{\text{virtual link diagram}\}/\text{R1–3, V1–4}$

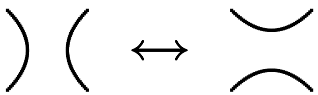
R1-move R2-move R3-move

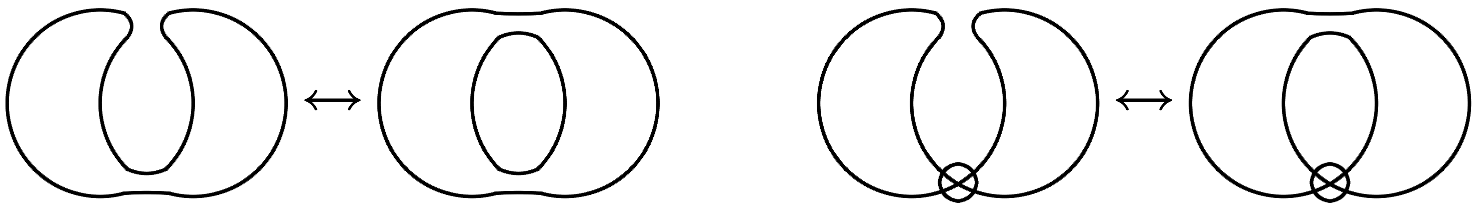


V1-move V2-move V3-move V4-move



Remark

The move  may Not change the number of circles on the plane if they have virtual intersections.



What is the difficulty? ($\Rightarrow$ Möbius cobordisms)

- Classical link diagram

  bifurcations of type $\mathbf{2} \longrightarrow \mathbf{1}$ or of type $\mathbf{1} \longrightarrow \mathbf{2}$
- Virtual link diagram

  bifurcations of type $\mathbf{2} \longrightarrow \mathbf{1}$ or of type $\mathbf{1} \longrightarrow \mathbf{2}$

  &

  It may appear bifurcations of type $\mathbf{1} \longrightarrow \mathbf{1}$

We call a bifurcation of type $\mathbf{1} \longrightarrow \mathbf{1}$ a Möbius cobordism.

for type $\mathbf{2} \longrightarrow \mathbf{1}$

$\boldsymbol{m : V \otimes V \longrightarrow V\{1\}}$ is a degree-preserving map.

$\mathbf{-2, 0, 2} \qquad \mathbf{0, 2}$ (degrees)

for type $\mathbf{1} \longrightarrow \mathbf{2}$

$\boldsymbol{\Delta : V \longrightarrow V \otimes V\{1\}}$ is a degree-preserving map.

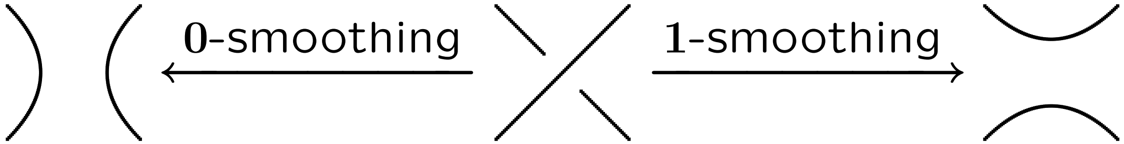
$\mathbf{-1, 1} \qquad \mathbf{-3, -1, 1}$ (degrees)

for type $\mathbf{1} \longrightarrow \mathbf{1}$

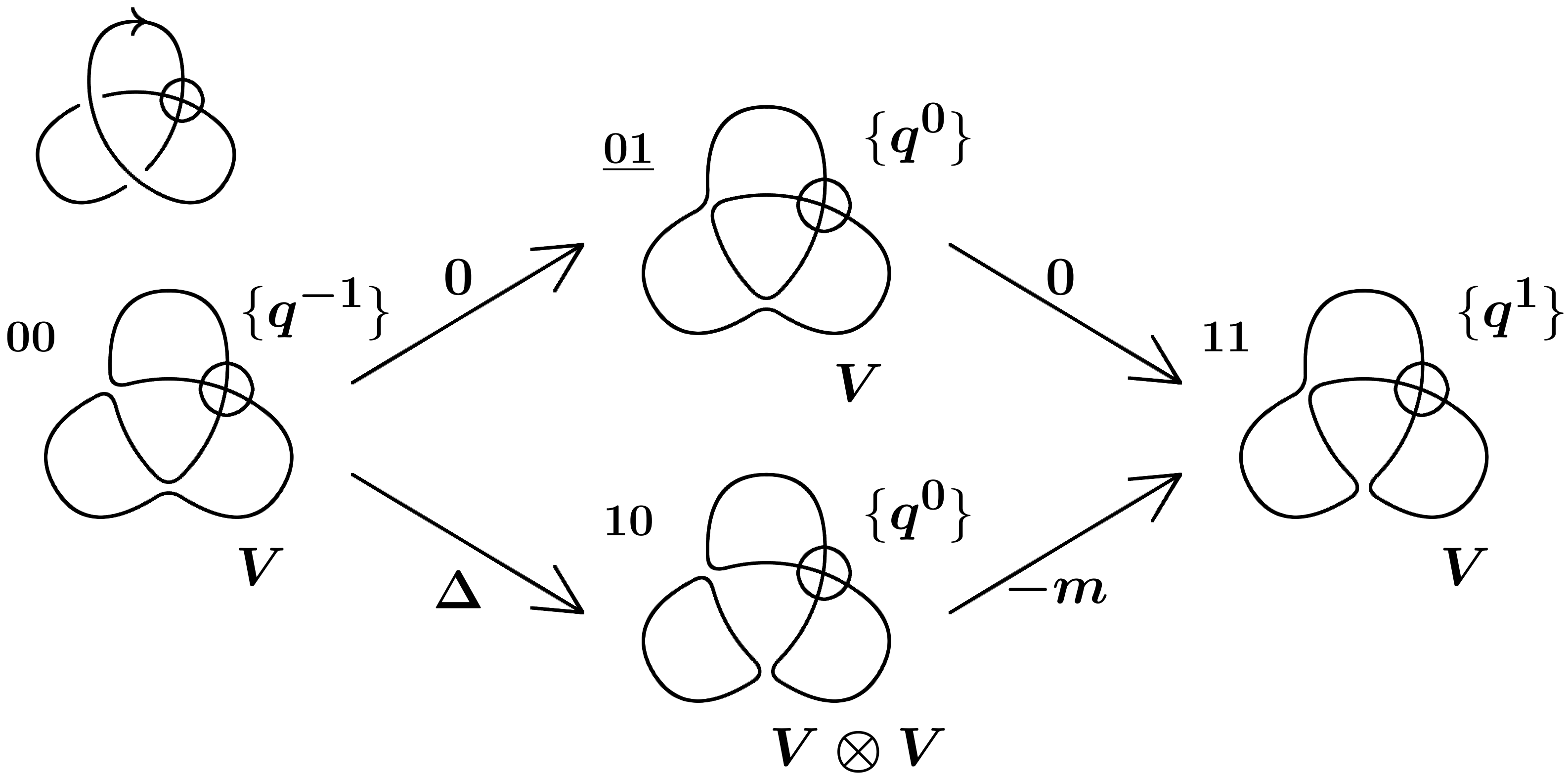
"?" $\boldsymbol{: V \longrightarrow V\{1\}}$ is a degree-preserving map??

$\mathbf{-1, 1} \qquad \mathbf{0, 2}$ (degrees) $\rightsquigarrow$ "?" should be the $\mathbf{0}$-map.

However...



0-smoothing
1-smoothing



00
$\{q^{-1}\}$
V
0
Δ
01
$\{q^0\}$
V
10
$\{q^0\}$
$V \otimes V$
0
$-m$
11
$\{q^1\}$
V

Maps for Möbius cobordisms (How to overcome?)

- [Manturov] used $\mathbf{0}$-maps for Möbius cobordisms.
He changed (the sign of) the basis of $\boldsymbol{V}$ while passing from one crossing to another, and used exterior product instead of the symmetric product.

- [Turaev–Turner] used an unoriented $(\mathbf{1}+\mathbf{1})$-TQFT.
Over $\mathbb{Q}$, their theories are all singly graded.
(Over $\mathbb{Z}_{\mathbf{2}}$, their theories contain bigraded theories.)

- [Ishii–Tanaka] use non-zero maps for Möbius cobordisms.
Our key ideas are
to take a suitable grading shift for each states,
to assign one of two non-zero maps
to each of the usual saddle cobordisms, and
to assign one of two non-zero maps
to each of the Möbius cobordisms.

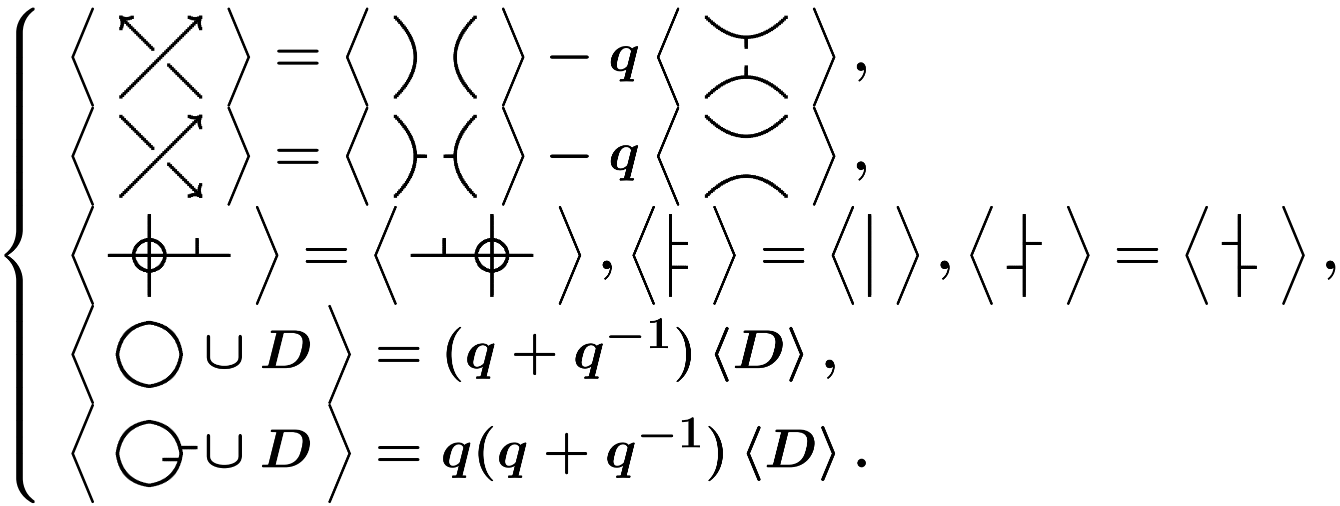
Details are explained later...

One-variable Miyazawa polynomial [Miyazawa] (cf. [Ishii])

L: an ori. virtual link, D: a diagram of L

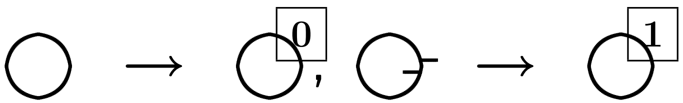
($n_+ :=$ #  of D, $n_- :=$ #  of D)

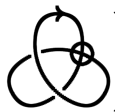
- The bracket polynomial $\left(\langle D \rangle \in \mathbb{Z}[q^{\pm 1}]\right)$

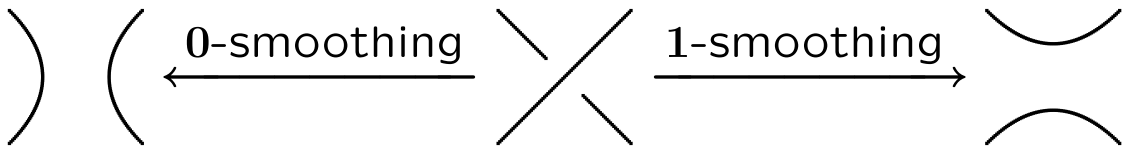


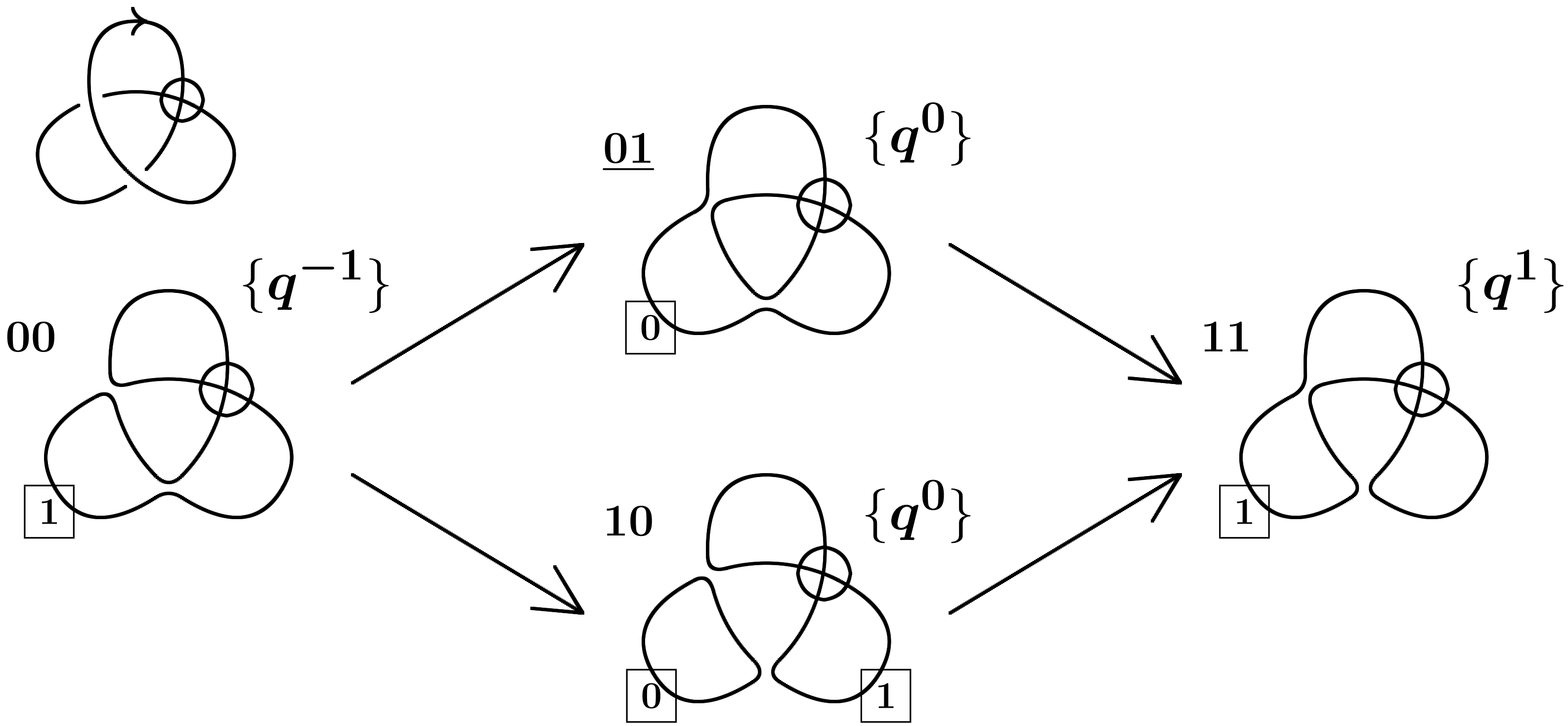
- The one-variable Miyazawa polynomial

$$\widetilde{J}(D) := (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle \quad (\in \mathbb{Z}[q^{\pm 1}])$$

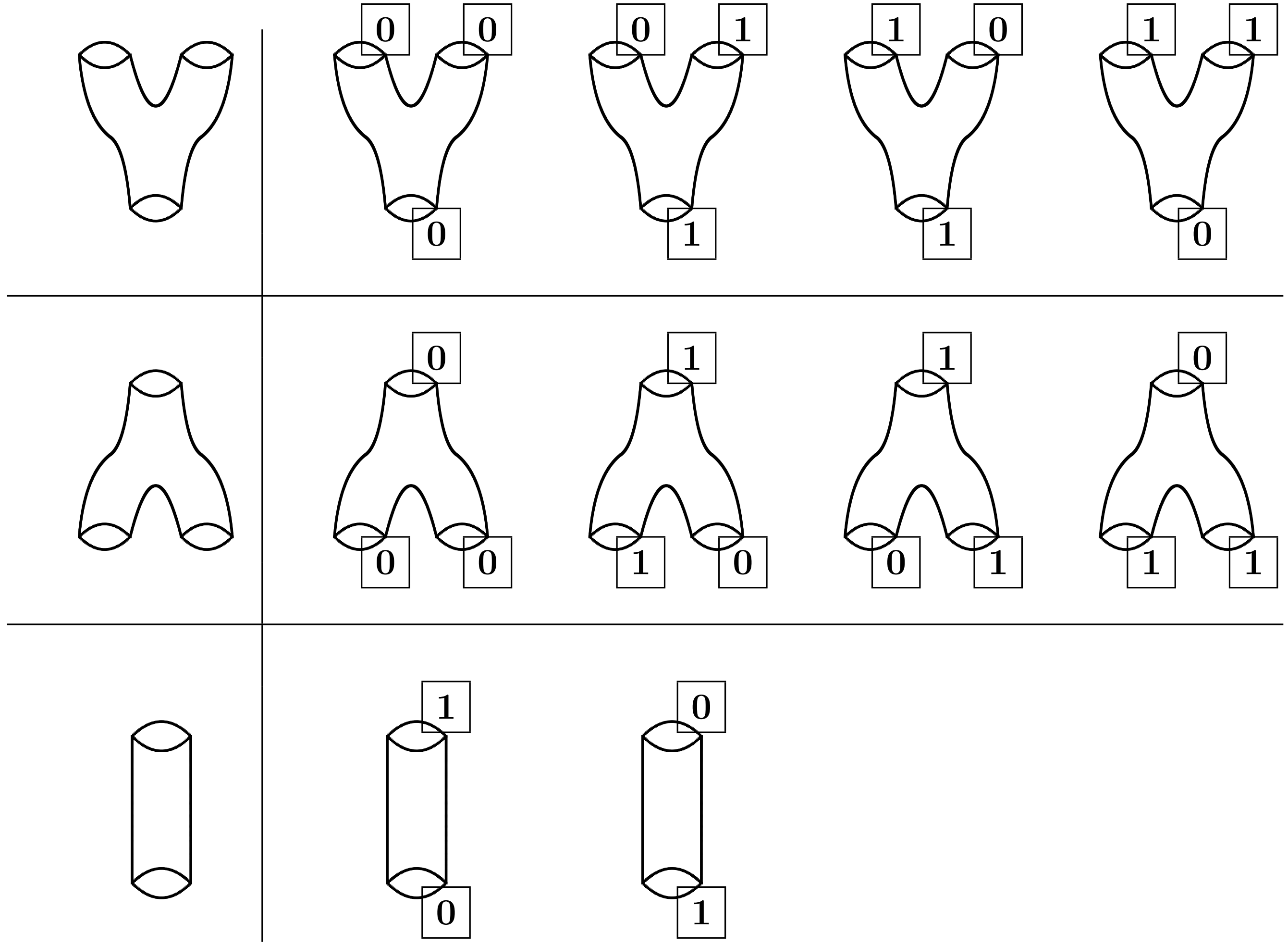
- Notation: 

Example (state of the diagram )





Types of bifurcations



Our homology for virtual links ($H_+(\cdot)$ and $H_-(\cdot)$)

L: an ori. virtual link, D: a diagram of L

$(C_+(D), d)$: a chain complex of graded $\mathbb{Q}[t^{\pm 1}]$-modules
$(C_-(D), d)$: a chain complex of graded $\mathbb{Q}[t^{\pm 1}]$-modules

Each graded $\mathbb{Q}[t^{\pm 1}]$-module

$$H^i_+(D) = H^i(C^*_+(D), d),\ H^i_-(D) = H^i(C^*_-(D), d)$$

is an invariant of L.

If L is a classical link, we have

$$H^i_+(L) \cong H^i_-(L) \cong H^i(L; \mathcal{F}')$$

as graded $\mathbb{Q}[t^{\pm 1}]$-modules. ↖ is NOT the same as $H^i(L)(= H^i(L; \mathcal{F}))$.

Frobenius algebra V' and $(1+1)$-TQFT $\mathcal{F}'$

- Frobenius algebra (the graded $\mathbb{Q}[t^{\pm 1}]$-module V')

$$V' = \mathbb{Q}[t^{\pm 1}, X]/(X^2 - t) \cong \mathbb{Q}[t^{\pm 1}]\langle 1 \rangle \oplus \mathbb{Q}[t^{\pm 1}]\langle X \rangle$$

$$(\deg(1) = 1,\ \deg(X) = -1,\ \deg(t) = -4)$$

We give V' a Frobenius algebra structure with

$$m : V' \otimes V' \to V' \qquad \begin{cases} m(1 \otimes 1) = 1,\ m(X \otimes X) = \underline{t \cdot 1} \\ m(1 \otimes X) = m(X \otimes 1) = X \end{cases}$$

$$\Delta : V' \to V' \otimes V' \qquad \begin{cases} \Delta(1) = 1 \otimes X + X \otimes 1 \\ \Delta(X) = X \otimes X + \underline{t \cdot 1 \otimes 1} \end{cases}$$

$$\iota : \mathbb{Q}[t^{\pm 1}] \to V' \qquad \iota(1) = 1$$

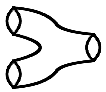
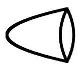
$$\epsilon : V' \to \mathbb{Q}[t^{\pm 1}] \qquad \epsilon(1) = 0, \quad \epsilon(X) = 1.$$

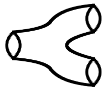
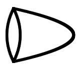
- $(1+1)$-TQFT $\mathcal{F}'$

For objects,

$\mathcal{F}'(\emptyset) = \mathbb{Q}[t^{\pm 1}]$, $\mathcal{F}'(\bigcirc) = V'$, $\mathcal{F}'(\bigcirc\,\bigcirc) = V' \otimes V', \ldots$ etc.

For morphisms,

$\mathcal{F}'\left(\right.$  $\left.\right) = m : V' \otimes V' \to V'$, $\mathcal{F}'\left(\right.$  $\left.\right) = \iota : \mathbb{Q}[t^{\pm 1}] \to V'$,

$\mathcal{F}'\left(\right.$  $\left.\right) = \Delta : V' \to V' \otimes V'$, $\mathcal{F}'\left(\right.$  $\left.\right) = \epsilon : V' \to \mathbb{Q}[t^{\pm 1}]$.

Our maps for Möbius and usual saddle cobordisms

- $C_+(\,\cdot\,)$ case (degree shift: $\bigcirc^{\boxed{1}}\{q^n\} \rightsquigarrow V'\{q^{n+1}\}$)

From $\bigcirc^{\boxed{0}}\{q^n\}$ to $\bigcirc^{\boxed{1}}\{q^{n+1}\}$, $\swarrow$ degree -2

assign $m \circ \Delta : V'\{q^n\} \longrightarrow V'\{q^{n+2}\}$.

From $\bigcirc^{\boxed{1}}\{q^n\}$ to $\bigcirc^{\boxed{0}}\{q^{n+1}\}$, $\swarrow$ degree 0

assign $\mathbf{id} : V'\{q^{n+1}\} \longrightarrow V'\{q^{n+1}\}$.

From $\bigcirc^{\boxed{1}}\bigcirc^{\boxed{1}}\{q^n\}$ to $\bigcirc^{\boxed{0}}\{q^{n+1}\}$, $\swarrow$ degree 1

assign $\frac{1}{4t}(m \circ \Delta \circ m) : V' \otimes V'\{q^{n+2}\} \longrightarrow V'\{q^{n+1}\}$.

From $\bigcirc^{\boxed{0}}\{q^n\}$ to $\bigcirc^{\boxed{1}}\bigcirc^{\boxed{1}}\{q^{n+1}\}$, $\swarrow$ degree -3

assign $\Delta \circ m \circ \Delta : V'\{q^n\} \longrightarrow V' \otimes V'\{q^{n+3}\}$.

- $\underline{C_{-}(\,\cdot\,)\ \text{case}}$ (degree shift: $\bigcirc^{\boxed{1}}\{q^n\} \rightsquigarrow V'\{q^{n-1}\}$)

From $\bigcirc^{\boxed{0}}\{q^n\}$ to $\bigcirc^{\boxed{1}}\{q^{n+1}\}$, $\swarrow$ degree $\mathbf{0}$

assign $\mathbf{id} : V'\{q^n\} \to V'\{q^n\}$.

From $\bigcirc^{\boxed{1}}\{q^n\}$ to $\bigcirc^{\boxed{0}}\{q^{n+1}\}$, $\swarrow$ degree $\mathbf{-2}$

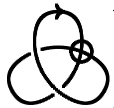
assign $m \circ \Delta : V'\{q^{n-1}\} \to V'\{q^{n+1}\}$.

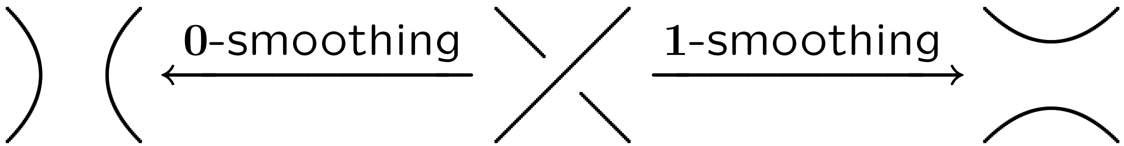
From $\bigcirc^{\boxed{1}}\bigcirc^{\boxed{1}}\{q^n\}$ to $\bigcirc^{\boxed{0}}\{q^{n+1}\}$, $\swarrow$ degree $\mathbf{-3}$

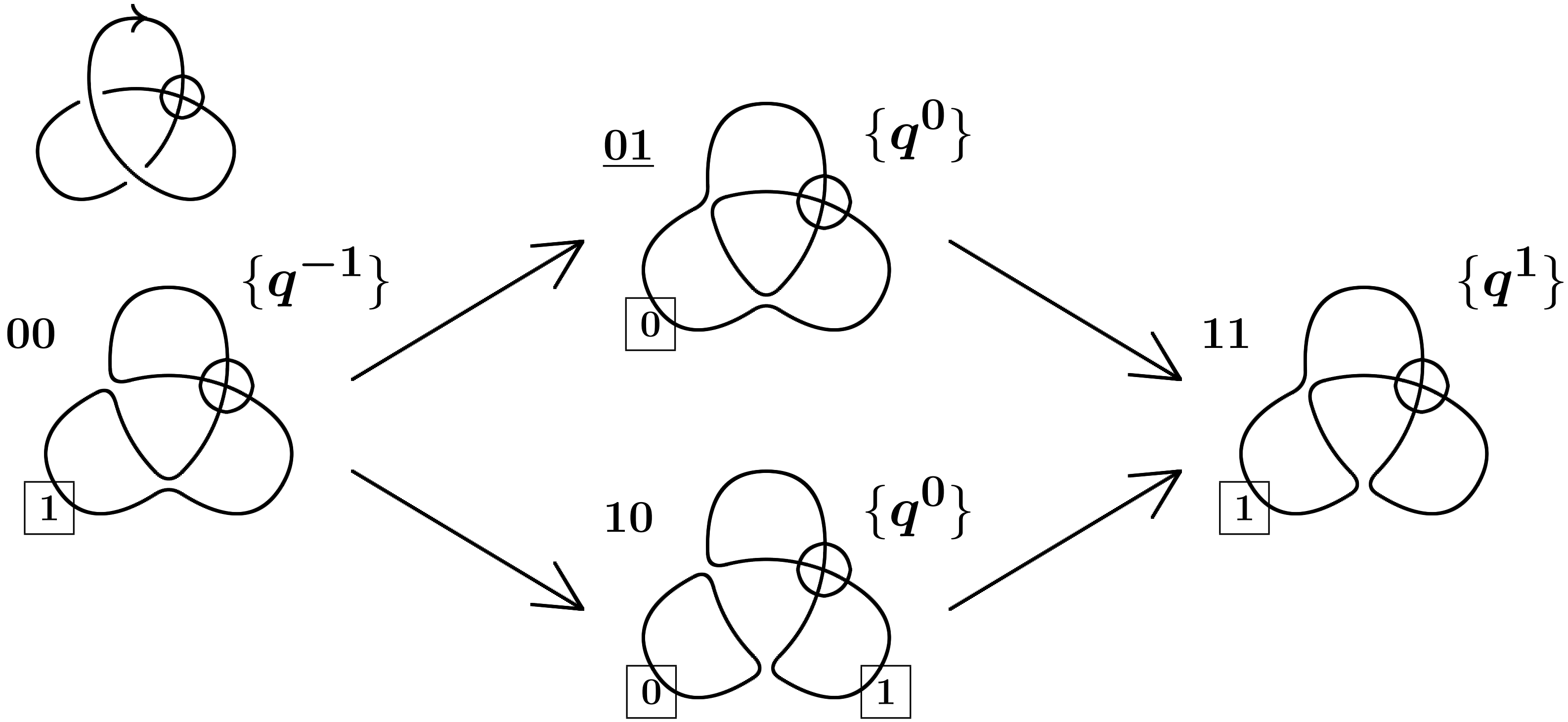
assign $m \circ \Delta \circ m : V' \otimes V'\{q^{n-2}\} \to V'\{q^{n+1}\}$.

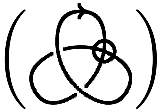
From $\bigcirc^{\boxed{0}}\{q^n\}$ to $\bigcirc^{\boxed{1}}\bigcirc^{\boxed{1}}\{q^{n+1}\}$, $\swarrow$ degree $\mathbf{1}$

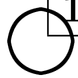
assign $\frac{1}{4t}(\Delta \circ m \circ \Delta) : V'\{q^n\} \to V' \otimes V'\{q^{n-1}\}$.

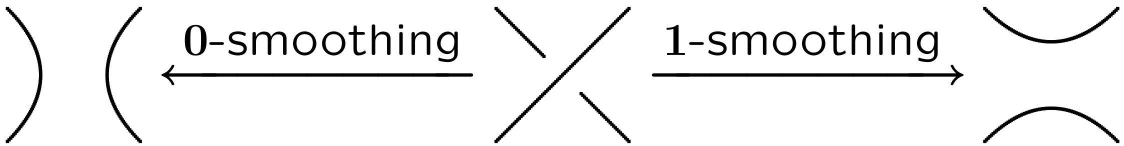
Recall (state of the diagram )

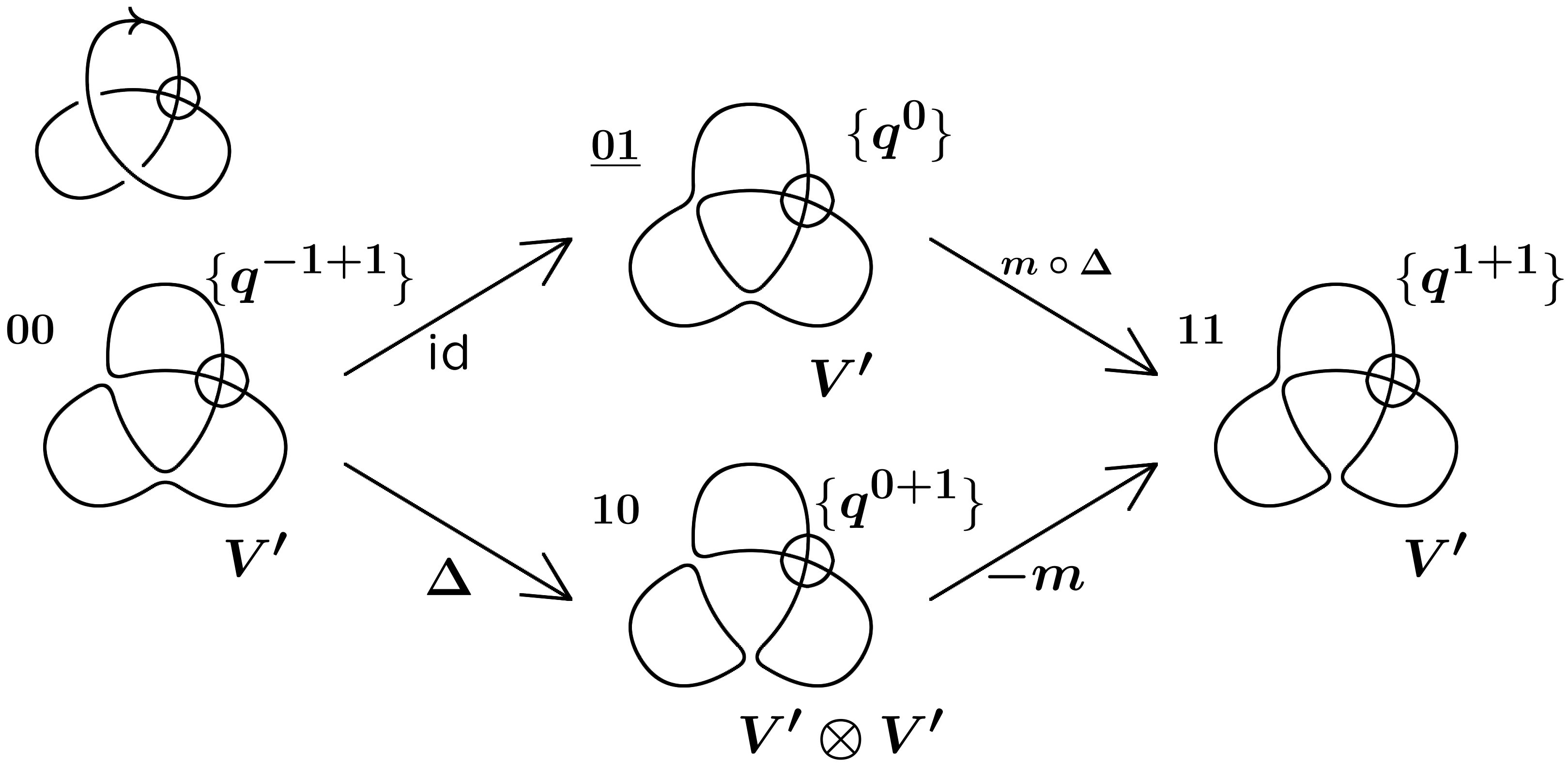


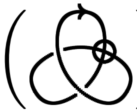


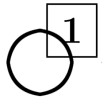
Example $\left(C_+\left(\right.\right.$  $\left.\left.\right)\right)$

 $^{\boxed{1}}\{q^n\} \rightsquigarrow V'\{q^{n+1}\}$

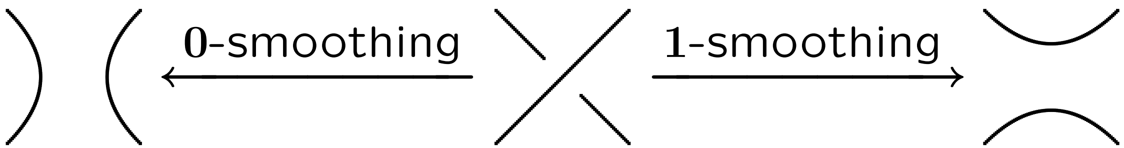


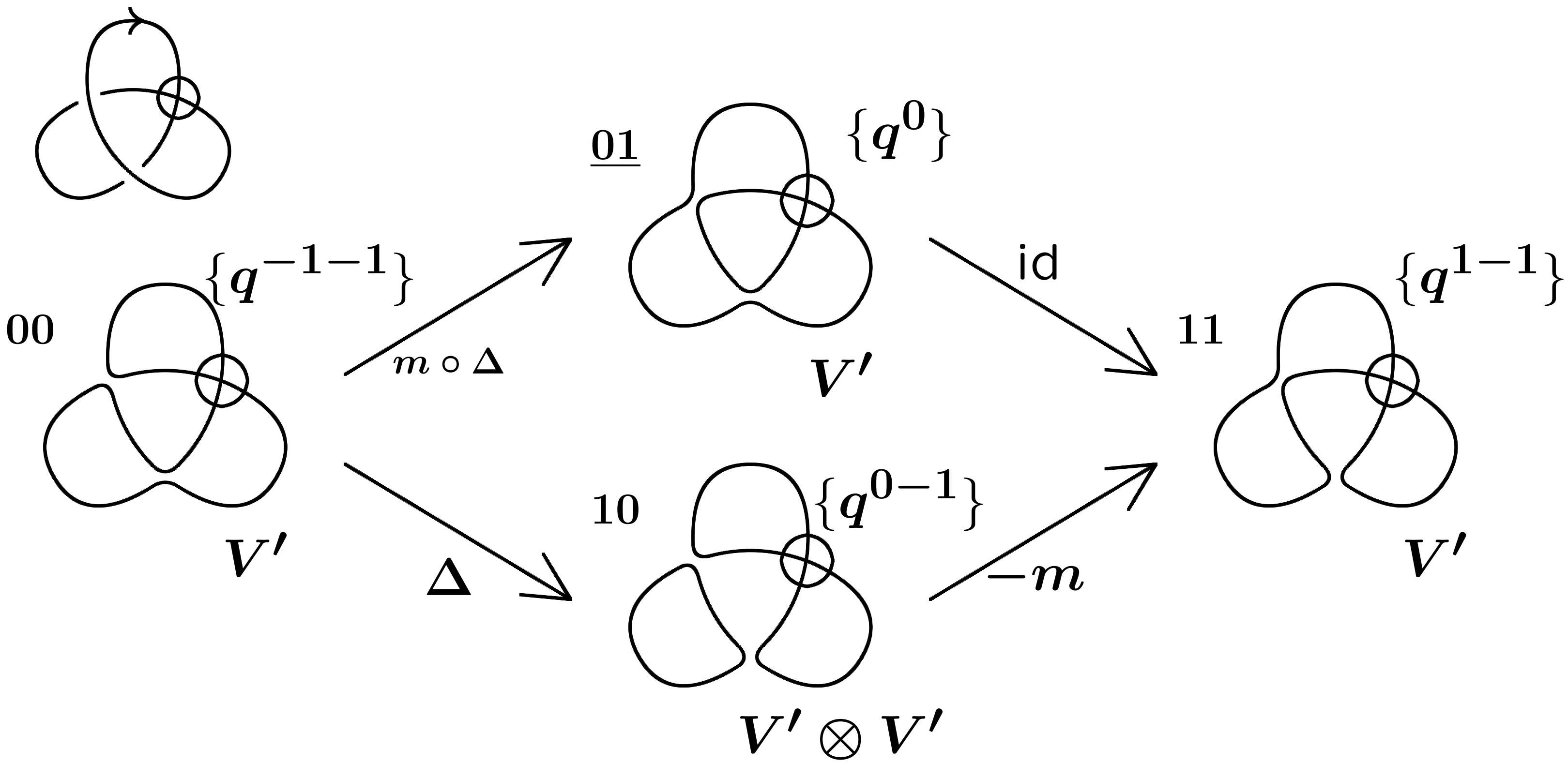


Example $\left(C_{-}\left(\right.\right.$  $\left.\left.\right)\right)$

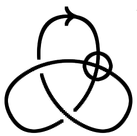


$\{q^n\} \rightsquigarrow V'\{q^{n-1}\}$



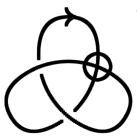


Example (computation of the virtual trefoil)

H_+ () =

$j \diagdown i$	**0**	**1**	**2**
7			$\mathbb{Q}[t^{\pm 1}]$
5			$\mathbb{Q}[t^{\pm 1}]$
3			
1			

$$\chi(H_+) = q^5 + q^7$$

H_- () =

$j \diagdown i$	**0**	**1**	**2**
7			
5			$\mathbb{Q}[t^{\pm 1}]$
3			$\mathbb{Q}[t^{\pm 1}]$
1			

$$\chi(H_-) = q^3 + q^5$$