Incompressible Surfaces in Graph Link Exteriors

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Problems

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F : an incompressible surface in $E(L)$

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What are the possible types of L?
What are the possible $\chi(F)$?

Preliminaries

Definition 1

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L : **a graph link** $\Leftrightarrow$
$E(L)$ is splitted into Seifert manifold pieces

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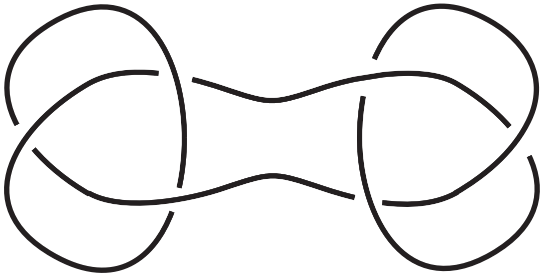
Examples

(1) the granny knot



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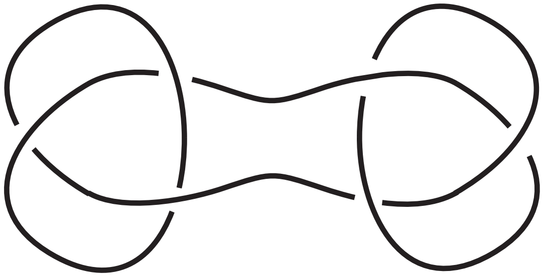

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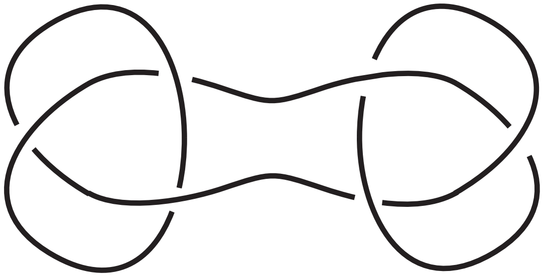

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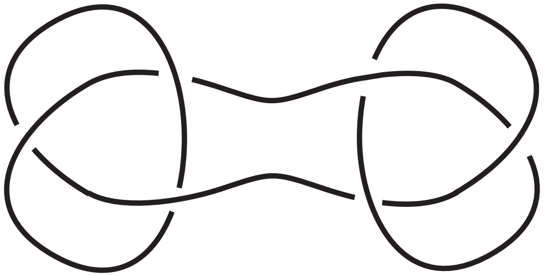

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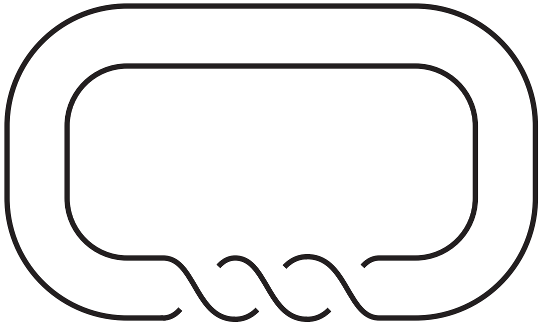

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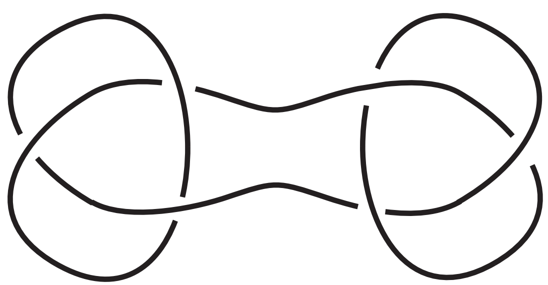

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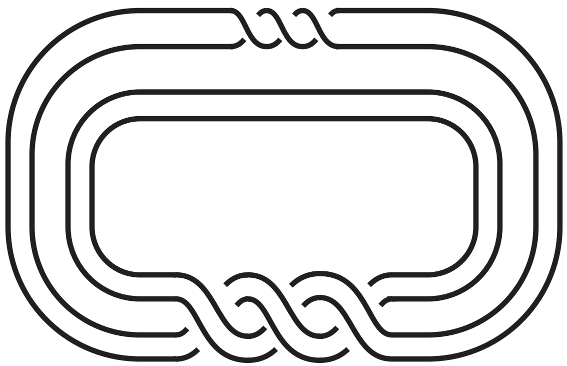

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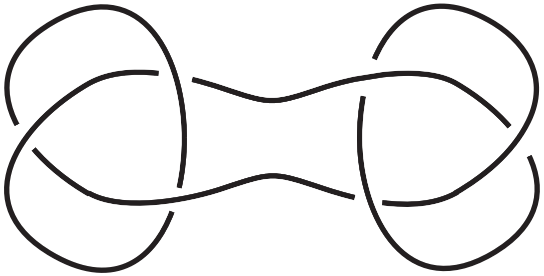

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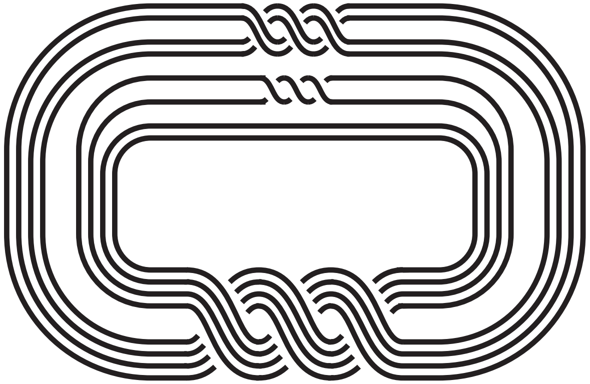

Definition 2

(1) P : a Seifert manifold piece in $E(L)$

$T \subset \partial P$

V : a solid torus s.t. $\partial V = T$

T : an **outer torus** of P $\Leftrightarrow$ $P \subset V$

an **inner torus** $\Leftrightarrow$ otherwise

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longitudinal $\Leftrightarrow$ type (n, nk)
preferred longitudinal $\Leftrightarrow$ type $(n, 0)$

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(3) M : a Seifert manifold
$F \subset M$: a proper surface
F : vertical $\Leftrightarrow F = \bigcup$ fibers
horizontal $\Leftrightarrow$ a fiber of a surface bundle over S^1

2-sided case

Table 1

	n-fold composing space	cable space of type (p,q)	torus knot space of type (p,q)
inner torus	$\left(\lambda, \sum_{i=1}^{n} \mu_i\right)$	$(\lambda, p\mu)$	$(\lambda, 0)$
outer tori	(λ, μ_1), $\dots$, (λ, μ_n)	$(p\lambda, \mu)$	—
remark	$\lambda \neq 0$	$\mu \neq q\lambda$	$\lambda \neq 0$

Boundaries of 2-sided horizontal surfaces

Theorem 1

K : a graph knot
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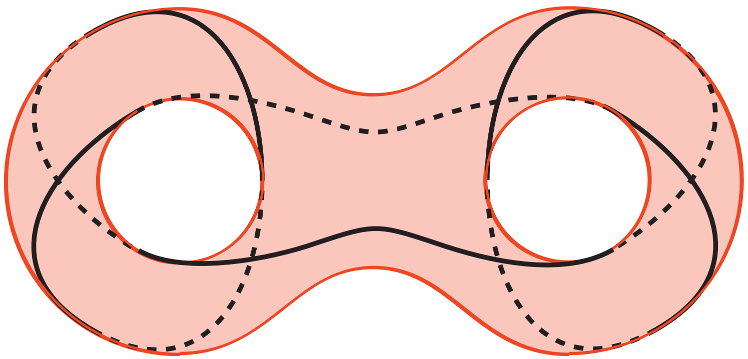

Theorem 2

K : a graph knot
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$F =$ (essential annuli in composing spaces)

$\cup$ (horizontal surfaces in cable spaces)

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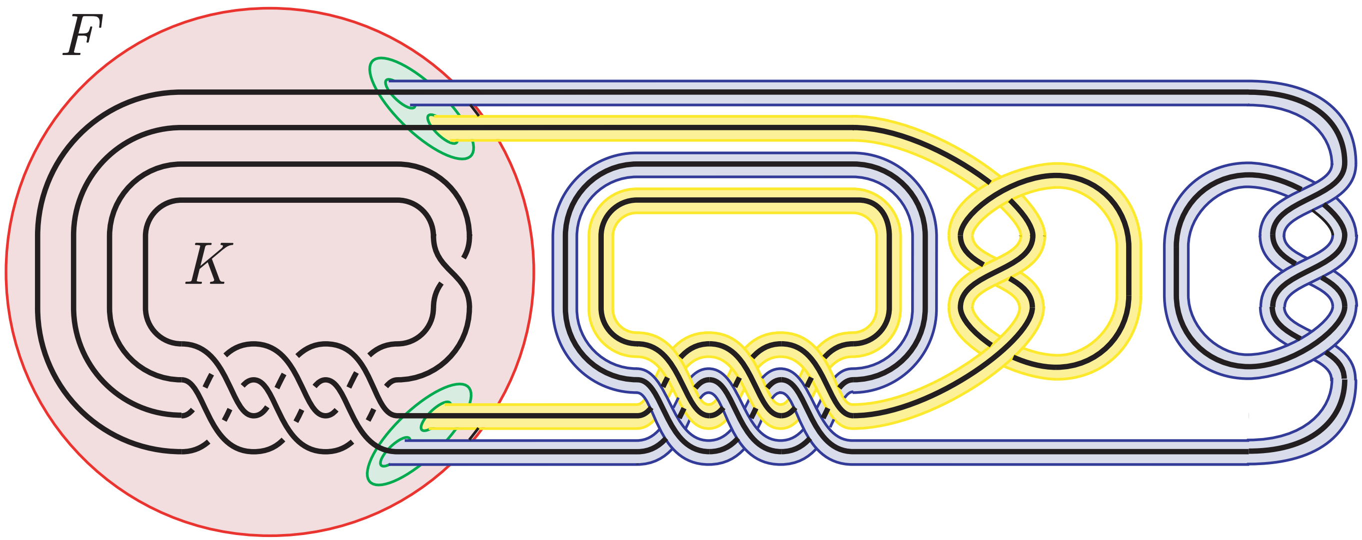
Corollary

Any iterated torus knot exterior contains no closed essential surface F of $\chi(F) < 0$.

Example



1-sided case

Known results

Lemma (Frohman and Rannard)

M : a Seifert manifold
$\varepsilon_1, \ldots, \varepsilon_n$: exceptional fibers
$F \subset M$: an incompressible surface

Then F can be isotoped so that

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(2) $\forall l \subset \partial V$: a loop of type $(2p, 2q+1)$,
$\exists F \subset V$: 1-sided incompressible surface s.t. $\partial F = l$

Table 2

	cable space of type (p, q)	torus knot space of type (p, q)
inner torus	(λ, μ)	$(\lambda, 2\mu)$
outer torus	$\left(p\lambda + 2\lambda', \dfrac{\mu + 2q\lambda'}{p}\right)$	—
remark	$\dfrac{\mu + 2q\lambda'}{p} \in \mathbb{Z}$, $\mu \neq pq\lambda$	$2\mu \neq pq\lambda$

Boundaries of possibly 1-sided horizontal surfaces

Theorem 3

K : a graph knot
$F \subset E(K)$: a bounded incompressible surface

Then

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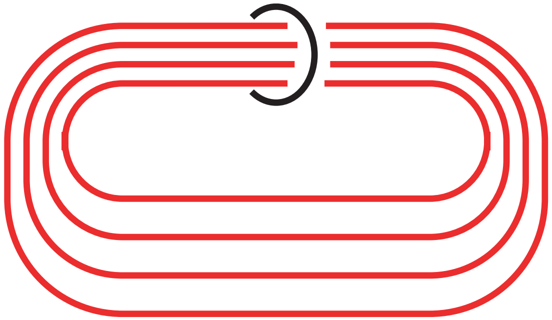
K : a graph knot
$F \subset E(K)$: a bounded incompressible surface

Then

∂F : not of type $(2p, 2q+1)$, where $p, q \in \mathbb{Z}$

Graph link case

Composing spaces

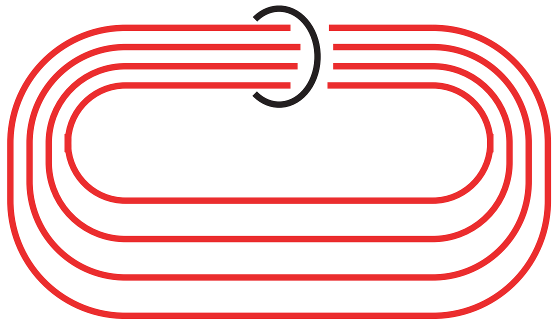


Type I

Red : outer tori

Black or Blue : inner tori

Composing spaces



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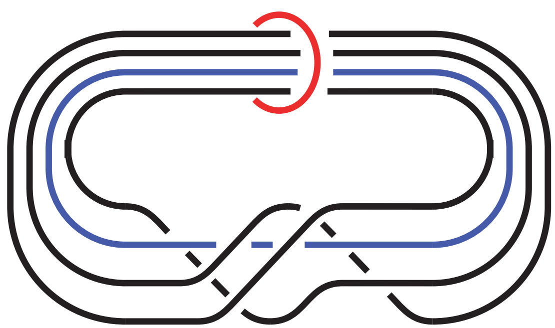
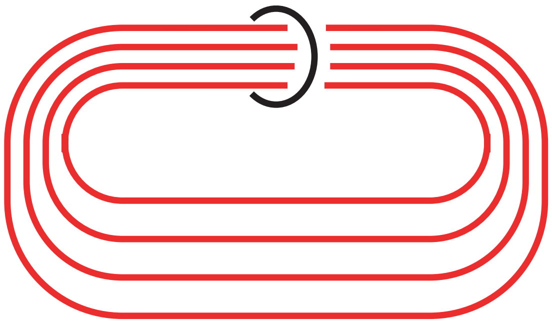


Type II

Red : outer tori

Black or Blue : inner tori

Composing spaces



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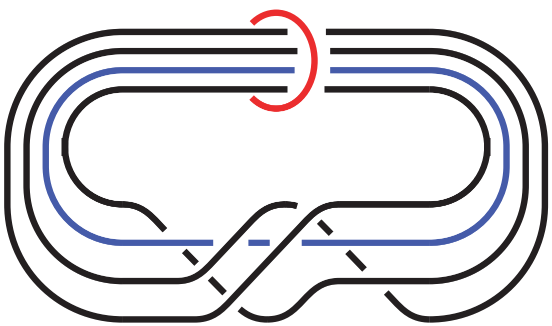

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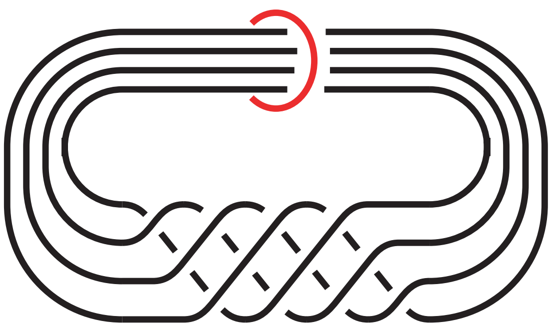

Type III

Table 3-1

	n-fold composing space		
type	I	II $-\ ((n-1)p, (n-1)q)$	
inner tori	$\left(\lambda, \sum_{i=1}^{n} \mu_i\right)$	$\left(\lambda_1, \frac{\mu}{\lvert p\rvert} + \frac{q\lambda_1}{p}\right)$	$(\lambda_2, \mu + pq\lambda_2)$, $\cdots$, $(\lambda_n, \mu + pq\lambda_n)$
outer tori	(λ, μ_1), $\cdots$, (λ, μ_n)	$\left(\overline{\lambda}, \frac{\mu}{\lvert p\rvert} + \frac{q\overline{\lambda}}{p}\right)$	
remark	$\lambda \neq 0$	$\overline{\lambda} = \lambda_1 + \lvert p\rvert \sum_{i=2}^{n} \lambda_i$	
	$\gcd(p, q) = 1, \quad \lvert p\rvert > 1, \quad \mu \neq 0$		

Boundaries of 2-sided horizontal surfaces

Table 3-2

	n-fold composing space	cable space	torus link space
type	III – (np, nq)	(np, nq)	(np, nq)
inner tori	$(\lambda_1, \mu + q\lambda_1), \dots, (\lambda_n, \mu + q\lambda_n)$	$(\lambda_1, \mu + pq\lambda_1), \dots, (\lambda_n, \mu + pq\lambda_n)$	$(\lambda_1, \mu + pq\lambda_1), \dots, (\lambda_n, \mu + pq\lambda_n)$
outer tori	$(\overline{\lambda}, \mu + q\overline{\lambda})$	$\left(\overline{\lambda}, \dfrac{\mu}{\lvert p\rvert} + \dfrac{q\overline{\lambda}}{p}\right)$	—
remark	$\overline{\lambda} = \sum_{i=1}^{n} \lambda_i$	$\overline{\lambda} = \lvert p\rvert \sum_{i=1}^{n} \lambda_i$	$\mu = -pq \sum_{i=1}^{n} \lambda_i$
	$\gcd(p, q) = 1, \quad \lvert p\rvert > 1, \quad \mu \neq 0$		

Boundaries of 2-sided horizontal surfaces

Theorem 4

L : a non-splittable graph link
$E(L)$: N Seifert manifold pieces
$\exists F \subset E(L)$: a closed essential surface of $\chi(F) < 0$

Then

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$N = 2 \quad \Rightarrow \quad L$: an $((p, q), (r, 0))$-iterated torus link

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Then

$N = 2 \quad \Rightarrow \quad L$: an $((p,q),(r,0))$-iterated torus link
$N = 3 \quad \Rightarrow \quad L$: (1) an iterated torus link of type
$((p,q),(r,0),(t,u))$, $((p,q),(r,s),(t,0))$,
$(\underline{(0,q)},(r,s),(0,u))$, $((0,q),(r,s),(n,0))$,
$((p,q),(r,s),(n,npqr^2))$ or $(\underline{(p,q)},(r,s),(t,u))$,

(2) an iterated cable of the Hopf link of type
$((p,q),(r,s),(n,npqr^2))$ or $(\underline{(p,q)},(r,s),(t,u))$

(3) a $(2p,2q)$-torus link with both components cabled

(red : special cable, blue : cable or special cable,
underline : exceptional component for iterated cabling)

Table 4

	(np, nq)-cable space	(np, nq)-torus link space
χ	$(k, n, \|p\|)$	$(k, n, \|p\|, \|q\|)$
-1	$(2,1,2)$	$(6,1,2,3), (6,1,3,2)$
-2	$(3,1,3), (4,1,2)$	$(12,1,2,3), (12,1,3,2)$
-3	$(6,1,2), (4,1,4),$ $(2,2,2)$	$(18,1,2,3), (18,1,3,2),$ $(10,1,2,5), (10,1,5,2)$
-4	$(8,1,2), (6,1,3),$ $(5,1,5)$	$(24,1,2,3), (24,1,3,2)$

Horizontal surfaces of $\chi \geq -4$
(k-fold branched covers of the orbit-manifolds)

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χ	$(k, n, \|p\|)$	$(k, n, \|p\|, \|q\|)$
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-2	$(3, 1, 3), (4, 1, 2)$	$(12, 1, 2, 3), (12, 1, 3, 2)$
-3	$(6, 1, 2), (4, 1, 4),$ $(2, 2, 2)$	$(18, 1, 2, 3), (18, 1, 3, 2),$ $(10, 1, 2, 5), (10, 1, 5, 2)$
-4	$(8, 1, 2), (6, 1, 3),$ $(5, 1, 5)$	$(24, 1, 2, 3), (24, 1, 3, 2)$

Horizontal surfaces of $\chi \geq -4$
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$E(L)$: no composing space
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(1) $\chi(F) \geq 0$ or $\chi(F) \leq -6$

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(2) no cable space of type $(4, 4r + 2)$

$$\Rightarrow \chi(F) \neq -6,\ -8$$

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$$\Rightarrow \chi(F) \neq -6, \ -8$$

(3) no cable space of type $(2, 2r+1)$

$$\Rightarrow \chi(F) \neq -8$$