Wirtinger presentations and Link diagrams

Zhiqing Yang

January 17, 2008

A (tame) knot is smooth (PL) embedding $f : S^1 \to S^3$. A link is a union of knots with disjoint images. Given a tame link, one can project it to a standard $S^2 \subset S^3$ to get a regular diagram. We assume any link diagram in this talk is reasonably good (being a regular link diagram).

Given a link diagram, there is a standard algorithm to get a Wirtinger presentation of its fundamental group.

Theorem 1. *Let K_i , $i = 1, 2$, be an oriented tame knot in the oriented 3-sphere, (G_i, μ_i, λ_i) be the peripheral system and M_i the knot manifold of K_i. Then:*
(1) K_1 is isotopic to K_2 if and only if there is an isomorphism of triples $(G_1, \mu_1, \lambda_1) \to (G_2, \mu_2, \lambda_2)$
(2) K_1 and K_2 are equivalent (resp. isotopic) if and only if M_1 is homeomorphic to M_2 (resp. by an orientation preserving homeomorphism).

Theorem 2. *(1)(The Whitten rigidity theo rem) Prime knots in the 3-sphere with isomor phic groups have homeomorphic knot mani folds, i.e. they are equivalent.*
(2) If two knots have isomorphic groups, the either both knots are prime or both are com posite.

W. Whitten,
Rigidity among prime-knot complements, Bul Amer. Math. Soc. 14 (1986),299-300.
Knot complements and groups, Topology 2 (1987), 41-44.

However, fundamental group can't distinguisl
1. Any knot and its mirror image.
2. Some different composite knots, like $3_1\#3$ and $3_1\#\overline{3_1}$
3. Sometimes two different links might hav homeomorphic complements.

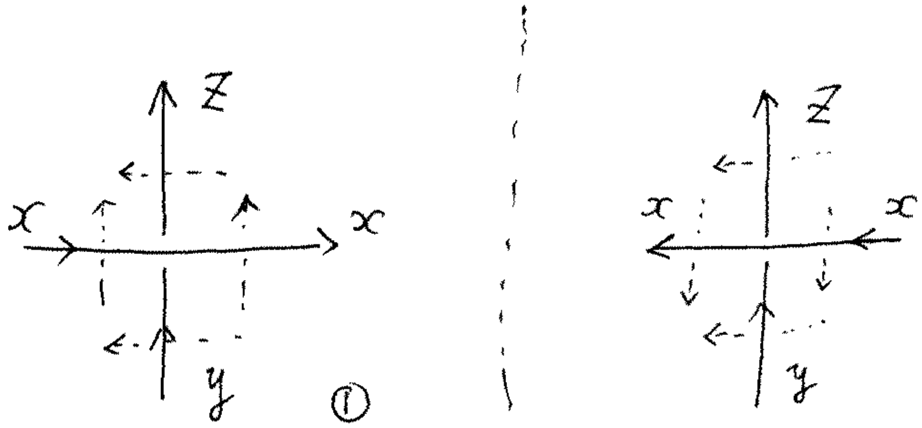
On the other hand, a Wirtinger presentation tells the orientation of knot, and can distinguish:

1. knot and its mirror image.
2. different composite knots with same fundamental group, like $3_1 \# 3_1$ and $3_1 \# \overline{3_1}$
3. different links with homeomorphic complements (stronger than the peripheral system).

Theorem 3. *If two link diagrams have same Wirtinger presentation, then they are the diagrams of the same link.*

Given a Wirtinger presentation, we shall describe an algorithm to draw its diagram.

Hence, if one has a "Wirtinger type/like presentation", we can determine whether or not it is a real Wirtinger presentation for some link.



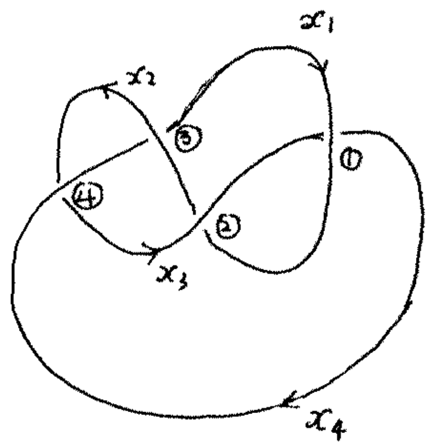
$yx = xZ$ $\qquad$ $Zx = xy$

$y = xZx^{-1}$ $\qquad$ $y = x^{-1}Zx$

mirror images destroy Wirtinger relations.

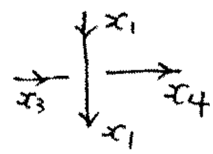
Note: using Wirtinger relators, one can say in figure ①
y is on the right side, Z is on the left side of x.

P10

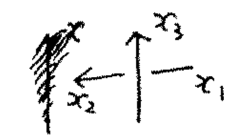


$\langle x_1, x_2, x_3, x_4 \mid$

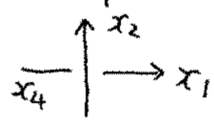
① $x_1 x_4 = x_3 x_1 \Rightarrow$



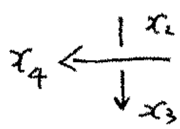
② $x_3 x_2 = x_1 x_3 \Rightarrow$



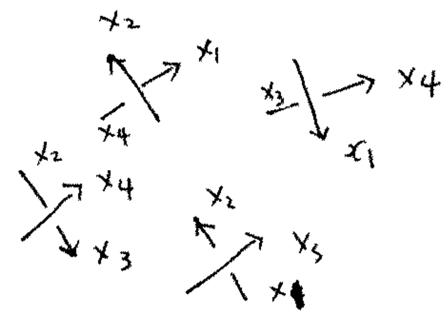
③ $x_2 x_4 = x_1 x_2 \Rightarrow$

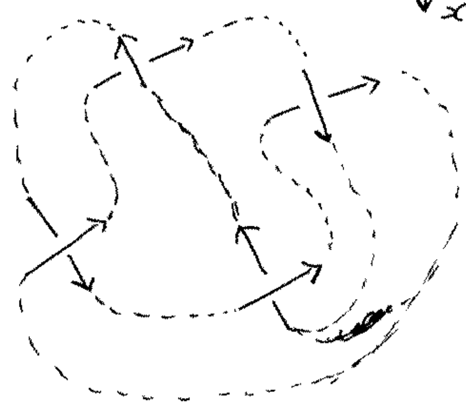


④ $x_4 x_3 = x_2 x_4 \Rightarrow$



$\Rightarrow$





Let's consider the relation:

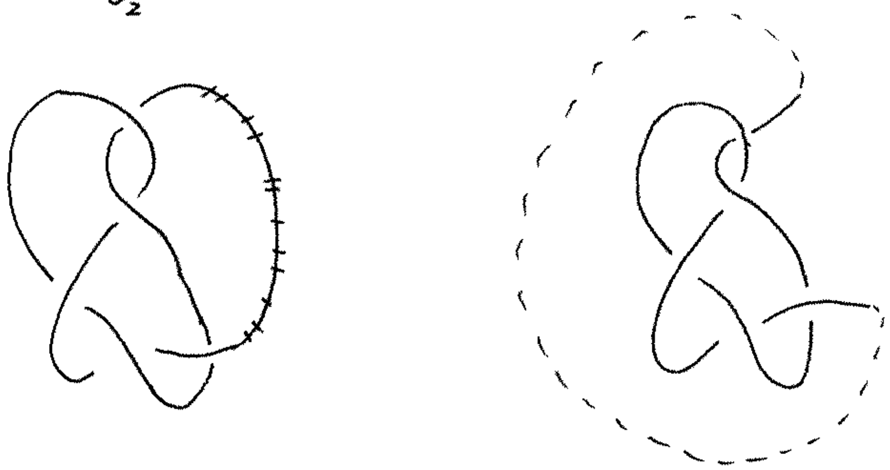
link diagram $\rightleftharpoons$ Wirtinger presentation

First of all, a link diagram does not gives a unique Wirtinger presentation of its fundamental group. One has to mark it with a fixed starting point and a direction on each component.

Then, we can get a unique Wirtinger presentation of the fundamental group. In such a presentation, we denote the generators on corresponding to a component k_i to be $x_1^i, x_2^i, \cdots, x_{n_i}^i$, according to the fixed orientation of this component. If a link diagram has n crossing points, k components, the Wirtinger presentation has n relations, but only $n-1$ of them are necessary. When we write a Wirtinger presentation, we write all the n relations in the presentation. Hence any marked link diagram gives rise to a unique Wirtinger presentation.

One the other hand, different diagrams might share same Wirtinger presentation. For example: the following two diagrams has the same Wirtinger presentation.

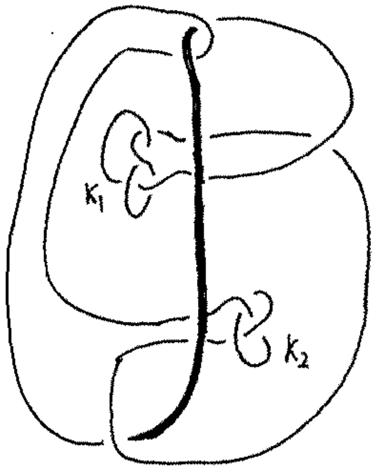
5_2

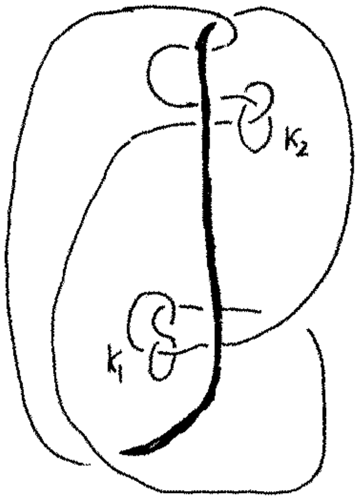


However, the difference is superficial. When we view the two diagrams in S^2, they are isotopic. So the question is when does a Wirtinger presentation determine a unique diagram in S^2?

Examples that different diagrams with same Wirtinger presentation.

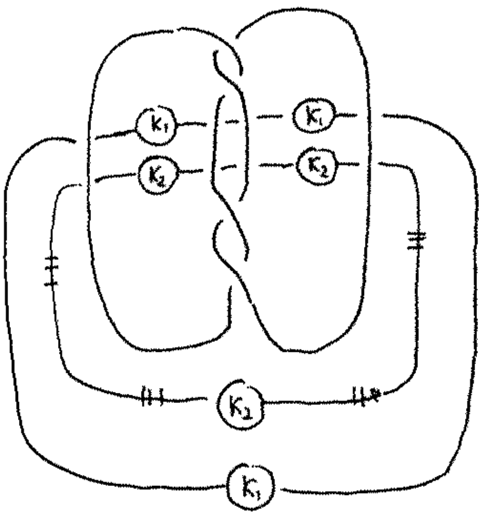
1. knot diagrams

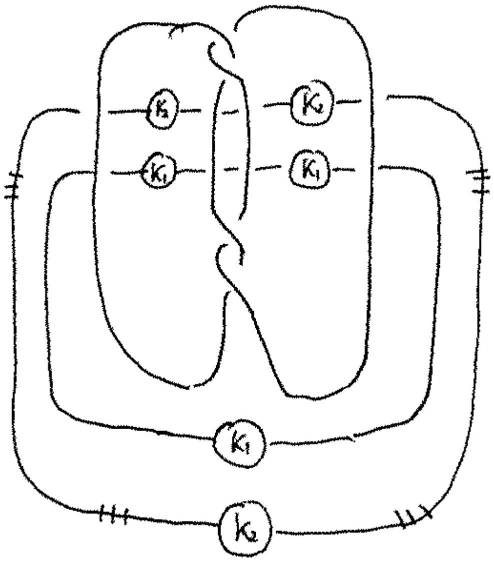




K1, K2.
swith.

2. link diagrams



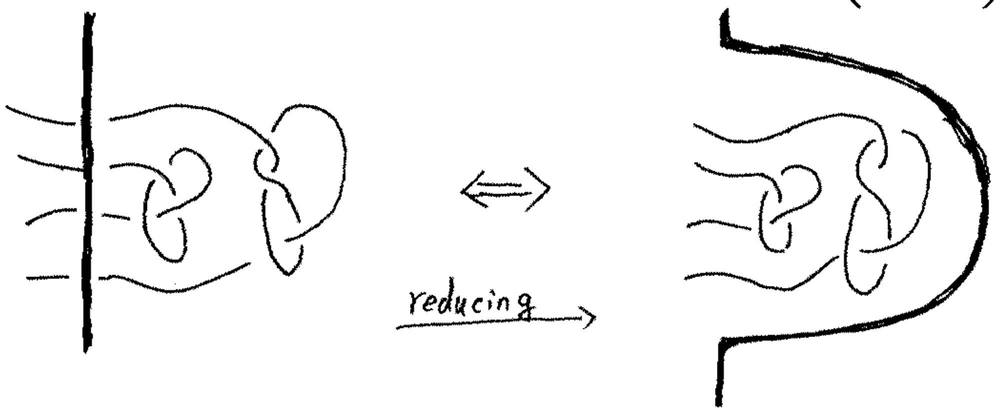


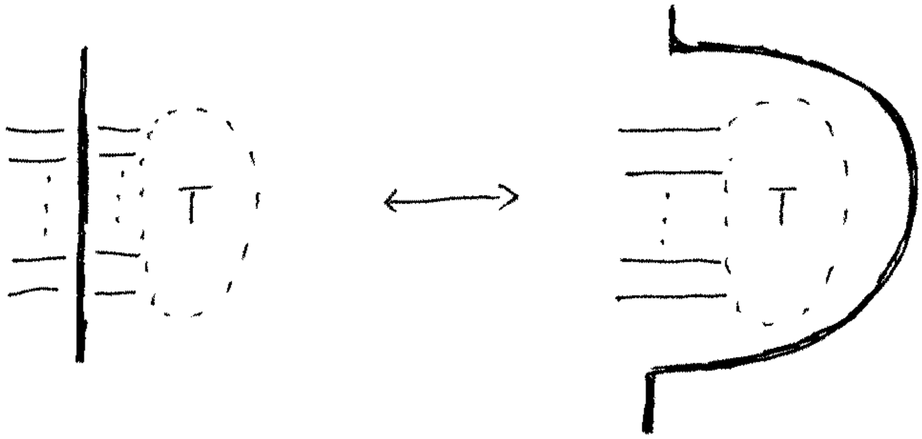
inner
outer

parallel swith.

They are the only cases that cause problems!

Generalized Reidemeister move-II (GR2)





If a diagram can't be isotopic to another diagram with less crossing number by applying GR2 once, we call it a GR2-reduced diagram.

Theorem 4. *A GR2-reduced knot diagram is uniquely determined by its Wirtinger presentation.*

Cor: A minimal Crossing knot diagram is uniquely determined by its Wirtinger presentation.

To understand the proof, let's first warm up by a baby case:

Theorem 5. *An marked alternating(or almost alternating) link diagram is uniquely determined by its Wirtinger presentation.*

Note: Not only the diagram, but also the marking is also unique. However, in the case of a general link diagram, there might be different ways to mark the link diagram and get the same Wirtinger presentation.

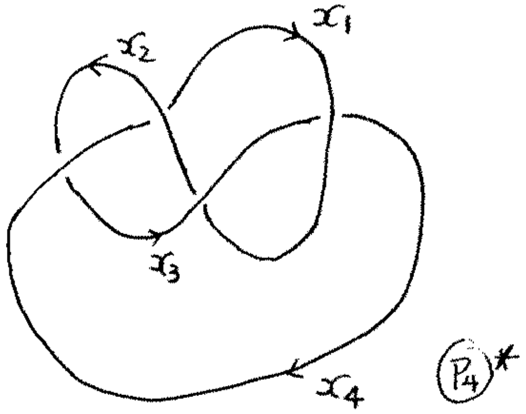
The idea of the proof:

Step 1. From the Wirtinger presentation, one can get a unique oriented graph.

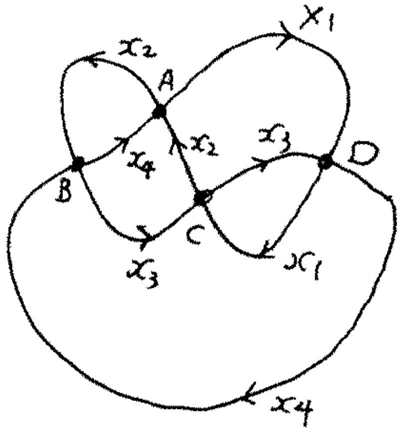
At each vertex, one can get a orientation from the Wirtinger relation.

Step 2. There is a unique way to embed the graph in S^2 and preserve the orientation at each vertex.

4_1

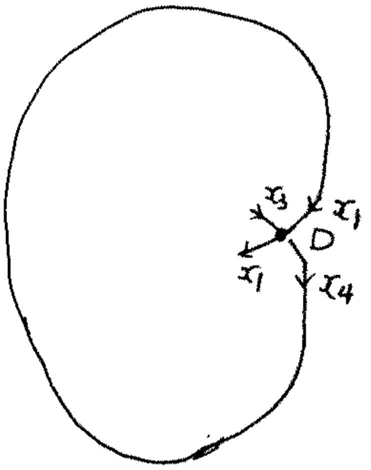


knot diagram

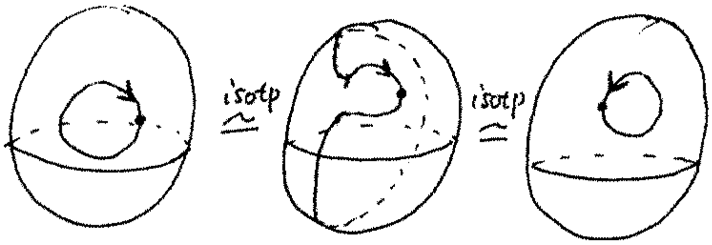


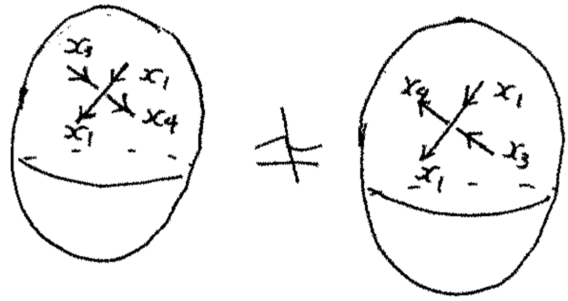
marked oriented graph

Alternating
⇓
unique way
to connect.

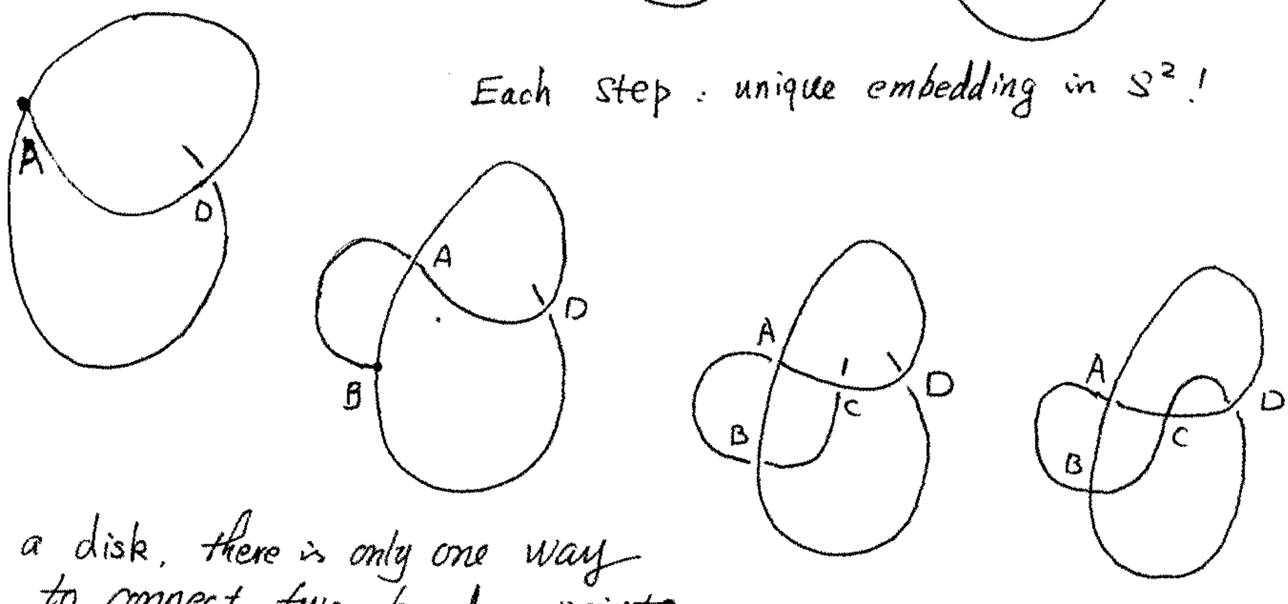


This oriented circle uniquely embeds in S^2





Each step: unique embedding in S^2!



In a disk, there is only one way to connect two boundary points.
Each vertex is of valency 4.

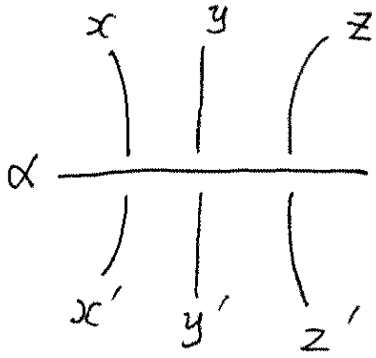
Theorem: A GR2-reduced knot diagram is uniquely determined by its Wirtinger presentation.
What do we know from a Wirtinger presentation $W = < X \mid R >$?

1. One can get a graph D, which is not unique.
2. A trick: contract each overcrossing edges, one get a new graph $C(D)$, which is uniquely determined by W.

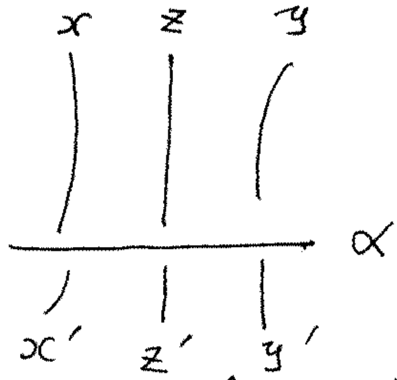
The graph inherits the Wirtinger label (generator x_i on each edge), and orientation from D and W.

Lemma 1. *1. An embedding of $D \to S^2$ as a link diagram gives rise to an unique embedding $C(D) \to S^2$.*
2. An embedding $C(D) \to S^2$ gives rise to an unique embedding $D \to S^2$.

Hence we pass to the problem how to embed the graph $C(D) \to S^2$, with the label and orientation.

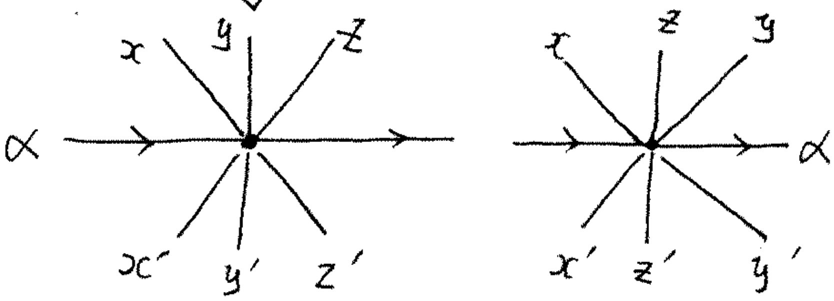


or?



Can't be determined in the Graph.

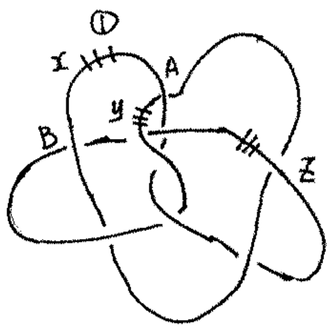
⇓

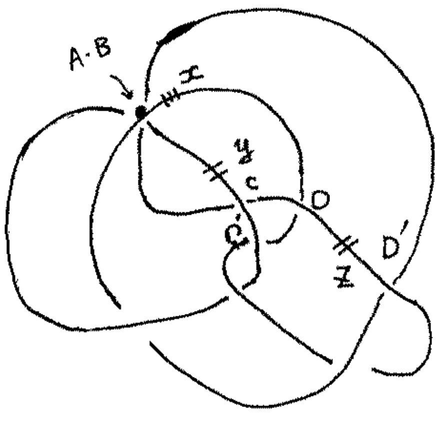


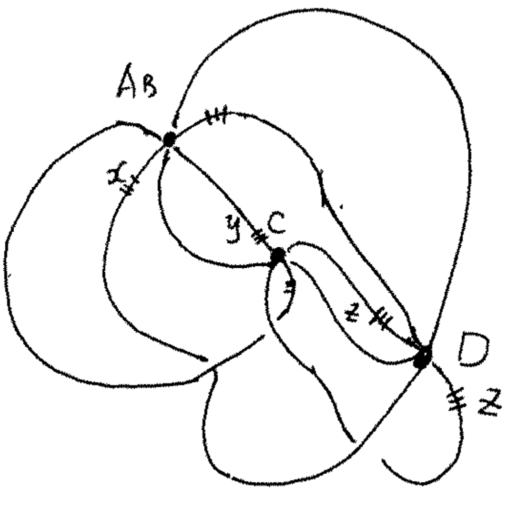
same graph with

1. undercrossing / overcrossing
2. pairing
3. orientation of each component
4. label on each edge.

9_{49}

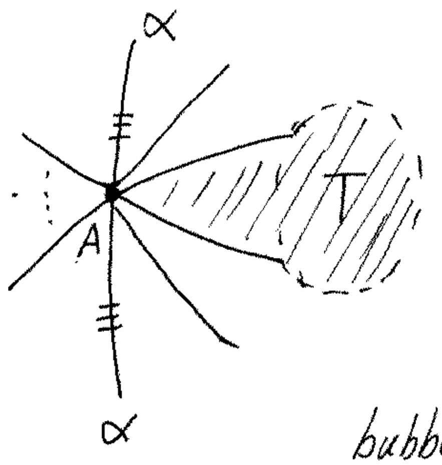






Now some vertex has valency > 4. we need GR2-reduced.

If D is GR2-reduced. it can't have the following case in C(D).



← a bubble connected to a vertex A.

bubble is not occuring.

Theorem 6. *Any GR2 reduced knot diagram is uniquely determined by its Wirtinger presentation.*

Theorem 7. *Any Wirtinger presentation uniquely determines a knot type.*

Theorem 8. *Any Wirtinger presentation uniquely determines a link type.*