Wirtinger presentations and Link diagrams

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January 17, 2008
A (tame) knot is smooth (PL) embedding $f : S^1 \to S^3$. A link is a union of knots with disjoint images. Given a tame link, one can project it to a standard $S^2 \subset S^3$ to get a regular diagram. We assume any link diagram in this talk is reasonably good (being a regular link diagram).

Given a link diagram, there is a standard algorithm to get a Wirtinger presentation of its fundamental group.

**Theorem 1.** Let $K_i$, $i = 1, 2$, be an oriented tame knot in the oriented 3-sphere, $(G_i, \mu_i, \lambda_i)$ be the peripheral system and $M_i$ the knot manifold of $K_i$. Then:

1. $K_1$ is isotopic to $K_2$ if and only if there is an isomorphism of triples $(G_1, \mu_1, \lambda_1) \to (G_2, \mu_2, \lambda_2)$.
2. $K_1$ and $K_2$ are equivalent (resp. isotopic) if and only if $M_1$ is homeomorphic to $M_2$ (resp. by an orientation preserving homeomorphism).
Theorem 2. (1) (The Whitten rigidity theorem) Prime knots in the 3-sphere with isomorphic groups have homeomorphic knot manifolds, i.e. they are equivalent.  
(2) If two knots have isomorphic groups, then either both knots are prime or both are composite.

W. Whitten,  
Knot complements and groups, Topology 26 (1987), 41-44.

However, fundamental group can't distinguish:  
1. Any knot and its mirror image.  
2. Some different composite knots, like $3_1 \# 3$ and $3_1 \# 3_1$  
3. Sometimes two different links might have homeomorphic complements.
On the other hand, a Wirtinger presentation tells the orientation of knot, and can distinguish:
1. knot and its mirror image.
2. different composite knots with same fundamental group, like $3_1 \# 3_1$ and $3_1 \# 3_1$
3. different links with homeomorphic complements (stronger than the peripheral system).

**Theorem 3.** If two link diagrams have same Wirtinger presentation, then they are the diagrams of the same link.

Given a Wirtinger presentation, we shall describe an algorithm to draw its diagram.

Hence, if one has a “Wirtinger type/like presentation”, we can determine whether or not it is a real Wirtinger presentation for some link.
\[ yx = xz \]
\[ y = xz x^{-1} \]

Mirror images destroy Wirtinger relations.

Note: Using Wirtinger relations, one can say in figure 1
\[ y \oplus \text{is on the right side} \] \[ z \ominus \text{is on the left side of} \ x. \]
Let's consider the relation:

link diagram $\Rightarrow$ Wirtinger presentation

First of all, a link diagram does not gives a unique Wirtinger presentation of its fundamental group. One has to mark it with a fixed starting point and a direction on each component.

Then, we can get a unique Wirtinger presentation of the fundamental group. In such a presentation, we denote the generators on corresponding to a component $k_i$ to be $x^i_1, x^i_2, \cdots, x^i_{n_i}$, according to the fixed orientation of this component. If a link diagram has $n$ crossing points, $k$ components, the Wirtinger presentation has $n$ relations, but only $n - 1$ of them are necessary. When we write a Wirtinger presentation, we write all the $n$ relations in the presentation. Hence any marked link diagram gives rise to a unique Wirtinger presentation.
One the other hand, different diagrams might share same Wirtinger presentation. For example: the following two diagrams has the same Wirtinger presentation.

\[ s_2 \]

However, the difference is superficial. When we view the two diagrams in $S^2$, they are isotopic. So the question is when does a Wirtinger presentation determine a unique diagram in $S^2$?
Examples that different diagrams with same Wirtinger presentation.

1. knot diagrams

2. link diagrams

They are the only cases that cause problems!
Generalized Reidemeister move-II (GR2)

If a diagram can't be isotopic to another diagram with less crossing number by applying GR2 once, we call it a GR2-reduced diagram.

**Theorem 4.** A GR2-reduced knot diagram is uniquely determined by its Wirtinger presentation.

**Cor.** A minimal crossing knot diagram is uniquely determined by its Wirtinger presentation.
To understand the proof, let’s first warm up by a baby case:

**Theorem 5.** An marked alternating (or almost alternating) link diagram is uniquely determined by its Wirtinger presentation.

Note: Not only the diagram, but also the marking is also unique. However, in the case of a general link diagram, there might be different ways to mark the link diagram and get the same Wirtinger presentation.

The idea of the proof:

Step 1. From the Wirtinger presentation, one can get a unique oriented graph.

At each vertex, one can get a orientation from the Wirtinger relation.

Step 2. There is a unique way to embed the graph in $S^2$ and preserve the orientation at each vertex.
knot diagram

marked oriented graph

This oriented circle uniquely embeds in $S^2$

Each step - unique embedding in $S^2$!

In a disk, there is only one way to connect two boundary points.
Each vertex is of valency 4.
**Theorem:** A GR2-reduced knot diagram is uniquely determined by its Wirtinger presentation.

What do we know from a Wirtinger presentation $W = \langle X \mid R \rangle$?

1. One can get a graph $D$, which is not unique.
2. A trick: contract each overcrossing edges, one get a new graph $C(D)$, which is uniquely determined by $W$.

The graph inherits the Wirtinger label (generator $x_i$ on each edge), and orientation from $D$ and $W$.

**Lemma 1.** 1. An embedding of $D \to S^2$ as a link diagram gives rise to an unique embedding $C(D) \to S^2$.
2. An embedding $C(D) \to S^2$ gives rise to an unique embedding $D \to S^2$.

Hence we pass to the problem how to embed the graph $C(D) \to S^2$, with the label and orientation.
can't be determined in the Graph.

same graph with

1. undercrossing/overcrossing
2. pairing
3. orientation of each component
4. label on each edge.

Now some vertex has valency > 4, we need GR2-reduced.
If $O \in GR_2$-reduced, it can't have the following case in $C(O)$.

A bubble connected to a vertex $A$.

bubble is not occurring.
Theorem 6. Any GR2 reduced knot diagram is uniquely determined by its Wirtinger presentation.

Theorem 7. Any Wirtinger presentation uniquely determines a knot type.

Theorem 8. Any Wirtinger presentation uniquely determines a link type.