Topological types of 3-dimensional small covers

—A joint work with Li Yu

Zhi Lü

School of Mathematical Sciences
Fudan University, Shanghai

—The Fourth East Asian School of Knots and Related Topics
The University of Tokyo, Japan

January 22, 2008

Program

- Background
- Objective
- Main results
- An application to cobordism
- Six operations

Program

- Background
- Objective
- Main results
- An application to cobordism
- Six operations

Program

- Background
- Objective
- Main results
- An application to cobordism
- Six operations

Program

- Background
- Objective
- Main results
- An application to cobordism
- Six operations

Program

- Background
- Objective
- Main results
- An application to cobordism
- Six operations

Program

- Background
- Objective
- Main results
- An application to cobordism
- Six operations

§1 Background—Theory of small covers

Theory of small covers: introduced by M. Davis and T. Januszkiewicz, [Duke Math. J., 1991]

- **A small cover** M^n is a closed manifold M^n with an effective action of $(\mathbb{Z}_2)^n$ such that

  1) M^n is locally equiv. to the standard $(\mathbb{Z}_2)^n$-representation;

  2) its orbit space $M/(\mathbb{Z}_2)^n$ is a simple convex polytope.

A connection

Equivariant topology $\longleftrightarrow$ Combinatorics

§1 Background—Theory of small covers

Theory of small covers: introduced by M. Davis and T. Januszkiewicz, [Duke Math. J., 1991]

- **A small cover** M^n is a closed manifold M^n with an effective action of $(\mathbb{Z}_2)^n$ such that

1) M^n is locally equiv. to the standard $(\mathbb{Z}_2)^n$-representation;

2) its orbit space $M/(\mathbb{Z}_2)^n$ is a simple convex polytope.

A connection

Equivariant topology $\longleftrightarrow$ Combinatorics

§1 Background—Theory of small covers

Theory of small covers: introduced by M. Davis and T. Januszkiewicz, [Duke Math. J., 1991]

- **A small cover** M^n is a closed manifold M^n with an effective action of $(\mathbb{Z}_2)^n$ such that

  1) M^n is locally equiv. to the standard $(\mathbb{Z}_2)^n$-representation;

  2) its orbit space $M/(\mathbb{Z}_2)^n$ is a simple convex polytope.

A connection

Equivariant topology $\longleftrightarrow$ Combinatorics

§1 Background—Theory of small covers

Theory of small covers: introduced by M. Davis and T. Januszkiewicz, [Duke Math. J., 1991]

- **A small cover** M^n is a closed manifold M^n with an effective action of $(\mathbb{Z}_2)^n$ such that

  1) M^n is locally equiv. to the standard $(\mathbb{Z}_2)^n$-representation;

  2) its orbit space $M/(\mathbb{Z}_2)^n$ is a simple convex polytope.

A connection

Equivariant topology $\longleftrightarrow$ Combinatorics

§1 Background—Theory of small covers

Theory of small covers: introduced by M. Davis and T. Januszkiewicz, [Duke Math. J., 1991]

- **A small cover** M^n is a closed manifold M^n with an effective action of $(\mathbb{Z}_2)^n$ such that

  1) M^n is locally equiv. to the standard $(\mathbb{Z}_2)^n$-representation;

  2) its orbit space $M/(\mathbb{Z}_2)^n$ is a simple convex polytope.

A connection

Equivariant topology $\longleftrightarrow$ Combinatorics

§1 Background—Theory of small covers

Theory of small covers: introduced by M. Davis and T. Januszkiewicz, [Duke Math. J., 1991]

- **A small cover** M^n is a closed manifold M^n with an effective action of $(\mathbb{Z}_2)^n$ such that

  1) M^n is locally equiv. to the standard $(\mathbb{Z}_2)^n$-representation;

  2) its orbit space $M/(\mathbb{Z}_2)^n$ is a simple convex polytope.

A connection

Equivariant topology $\longleftrightarrow$ Combinatorics

Standard $(\mathbb{Z}_2)^n$-representation

$(\mathbb{Z}_2)^n \curvearrowright \mathbb{R}^n$ by

$$(x_1, \ldots, x_n) \longmapsto (g_1 x_1, \ldots, g_n x_n)$$

where $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $(g_1, \ldots, g_n) \in (\mathbb{Z}_2)^n$.

The orbit space is $\mathbb{R}^n_{\geq 0}$ (i.e., the positive cone in $\mathbb{R}^n$).

Convex polytopes

- **A convex polytope** P^n is a convex hull of finite points in $\mathbb{R}^n$.
- A convex polytope P^n is said to be **simple** if the number of codim-one faces (facets) meeting at each vertex is n.

Simple   Not simple

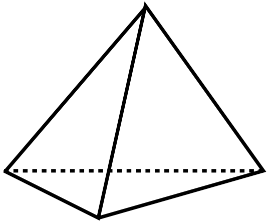
Convex polytopes

- **A convex polytope** P^n is a convex hull of finite points in $\mathbb{R}^n$.
- A convex polytope P^n is said to be **simple** if the number of codim-one faces (facets) meeting at each vertex is n.

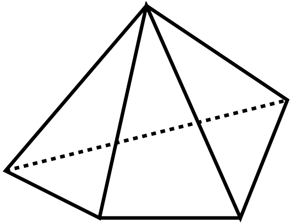
Simple Not simple

Convex polytopes

- **A convex polytope** P^n is a convex hull of finite points in $\mathbb{R}^n$.
- A convex polytope P^n is said to be **simple** if the number of codim-one faces (facets) meeting at each vertex is n.



Simple



Not simple

Examples for small covers

- $(\mathbb{Z}_2)^2 \curvearrowright \mathbb{R}P^2$ by $[x_0, x_1, x_2] \longmapsto [x_0, g_1x_1, g_2x_2]$

orbit space is

2–simplex

So $\mathbb{R}P^2$ is a small cover.

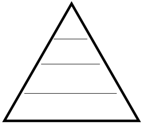
- $(\mathbb{Z}_2)^2 \curvearrowright S^2$ by $(x_0, x_1, x_2) \longmapsto (x_0, g_1x_1, g_2x_2)$

orbit space is (not a polytope)

So S^2 is not a small cover.

Examples for small covers

- $(\mathbb{Z}_2)^2 \curvearrowright \mathbb{R}P^2$ by $[x_0, x_1, x_2] \longmapsto [x_0, g_1x_1, g_2x_2]$

orbit space is 

2–simplex

So $\mathbb{R}P^2$ is a small cover.

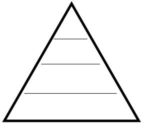
- $(\mathbb{Z}_2)^2 \curvearrowright S^2$ by $(x_0, x_1, x_2) \longmapsto (x_0, g_1x_1, g_2x_2)$

orbit space is (not a polytope)

So S^2 is not a small cover.

Examples for small covers

- $(\mathbb{Z}_2)^2 \curvearrowright \mathbb{R}P^2$ by $[x_0, x_1, x_2] \longmapsto [x_0, g_1x_1, g_2x_2]$

orbit space is 

2–simplex

So $\mathbb{R}P^2$ is a small cover.

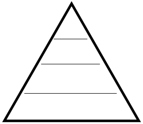
- $(\mathbb{Z}_2)^2 \curvearrowright S^2$ by $(x_0, x_1, x_2) \longmapsto (x_0, g_1x_1, g_2x_2)$

orbit space is (not a polytope)

So S^2 is not a small cover.

Examples for small covers

- $(\mathbb{Z}_2)^2 \curvearrowright \mathbb{R}P^2$ by $[x_0, x_1, x_2] \longmapsto [x_0, g_1 x_1, g_2 x_2]$

  orbit space is 

  2–simplex

  So $\mathbb{R}P^2$ is a small cover.

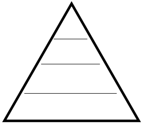
- $(\mathbb{Z}_2)^2 \curvearrowright S^2$ by $(x_0, x_1, x_2) \longmapsto (x_0, g_1 x_1, g_2 x_2)$

  orbit space is (not a polytope)

  So S^2 is not a small cover.

Examples for small covers

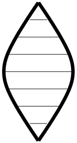
- $(\mathbb{Z}_2)^2 \curvearrowright \mathbb{R}P^2$ by $[x_0, x_1, x_2] \longmapsto [x_0, g_1x_1, g_2x_2]$

orbit space is 

2–simplex

So $\mathbb{R}P^2$ is a small cover.

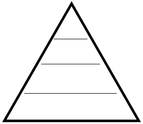
- $(\mathbb{Z}_2)^2 \curvearrowright S^2$ by $(x_0, x_1, x_2) \longmapsto (x_0, g_1x_1, g_2x_2)$

orbit space is  (not a polytope)

So S^2 is not a small cover.

Examples for small covers

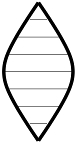
- $(\mathbb{Z}_2)^2 \curvearrowright \mathbb{R}P^2$ by $[x_0, x_1, x_2] \longmapsto [x_0, g_1x_1, g_2x_2]$

  orbit space is 

  2–simplex

  So $\mathbb{R}P^2$ is a small cover.

- $(\mathbb{Z}_2)^2 \curvearrowright S^2$ by $(x_0, x_1, x_2) \longmapsto (x_0, g_1x_1, g_2x_2)$

  orbit space is  (not a polytope)

  So S^2 is not a small cover.

Two key points for theory of small covers

$\pi : M^n \longrightarrow P^n$: a small cover over P^n.

–Algebraic topology

- **Equivariant cohomology:** $H^*_{(\mathbb{Z}_2)^n}(M^n; \mathbb{Z}_2) \cong R(P^n; \mathbb{Z}_2)$ where $R(P^n; \mathbb{Z}_2)$ is the Reisner-Stanley face ring of P^n:

$$R(P^n; \mathbb{Z}_2) = \mathbb{Z}_2[F_1, ..., F_l]/I$$

  $I = (F_{i_1} \cdots F_{i_r} | F_{i_1} \cap \cdots \cap F_{i_r} = \emptyset)$ is an ideal, and each F_i is a facet (ie., codim-one face) of P^n.
- **Mod 2 Betti numbers:** $(b_0, b_1, ..., b_n) = (h_0, h_1, ..., h_n)$ where $(h_0, h_1, ..., h_n)$ is the h-vector of P^n

Two key points for theory of small covers

$\pi : M^n \longrightarrow P^n$: a small cover over P^n.

–Algebraic topology

- **Equivariant cohomology:** $H^*_{(\mathbb{Z}_2)^n}(M^n; \mathbb{Z}_2) \cong R(P^n; \mathbb{Z}_2)$ where $R(P^n; \mathbb{Z}_2)$ is the Reisner-Stanley face ring of P^n:

$$R(P^n; \mathbb{Z}_2) = \mathbb{Z}_2[F_1, ..., F_l]/I$$

  $I = (F_{i_1} \cdots F_{i_r} | F_{i_1} \cap \cdots \cap F_{i_r} = \emptyset)$ is an ideal, and each F_i is a facet (ie., codim-one face) of P^n.
- **Mod 2 Betti numbers:** $(b_0, b_1, ..., b_n) = (h_0, h_1, ..., h_n)$ where $(h_0, h_1, ..., h_n)$ is the h-vector of P^n

Two key points for theory of small covers

$\pi : M^n \longrightarrow P^n$: a small cover over P^n.

–Algebraic topology

- **Equivariant cohomology:** $H^*_{(\mathbb{Z}_2)^n}(M^n;\mathbb{Z}_2) \cong R(P^n;\mathbb{Z}_2)$ where $R(P^n;\mathbb{Z}_2)$ is the Reisner-Stanley face ring of P^n:

$$R(P^n;\mathbb{Z}_2) = \mathbb{Z}_2[F_1, ..., F_l]/I$$

$I = (F_{i_1} \cdots F_{i_r} | F_{i_1} \cap \cdots \cap F_{i_r} = \emptyset)$ is an ideal, and each F_i is a facet (ie., codim-one face) of P^n.

- **Mod 2 Betti numbers:** $(b_0, b_1, ..., b_n) = (h_0, h_1, ..., h_n)$ where $(h_0, h_1, ..., h_n)$ is the h-vector of P^n

Two key points for theory of small covers

$\pi : M^n \longrightarrow P^n$: a small cover over P^n.

–Algebraic topology

- **Equivariant cohomology:** $H^*_{(\mathbb{Z}_2)^n}(M^n;\mathbb{Z}_2) \cong R(P^n;\mathbb{Z}_2)$ where $R(P^n;\mathbb{Z}_2)$ is the Reisner-Stanley face ring of P^n:

$$R(P^n;\mathbb{Z}_2) = \mathbb{Z}_2[F_1, ..., F_l]/I$$

  $I = (F_{i_1} \cdots F_{i_r} | F_{i_1} \cap \cdots \cap F_{i_r} = \emptyset)$ is an ideal, and each F_i is a facet (ie., codim-one face) of P^n.

- **Mod 2 Betti numbers:** $(b_0, b_1, ..., b_n) = (h_0, h_1, ..., h_n)$ where $(h_0, h_1, ..., h_n)$ is the h-vector of P^n

Two key points for theory of small covers

–Geometric topology

- **$(\mathbb{Z}_2)^n$-coloring:** Each small cover $\pi : M^n \longrightarrow P^n$ determines

$$\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$$

mapping n facets at each vertex to n linearly independent vectors, where $\mathcal{F}(P^n)$:=all facets of P^n.

- **Reconstruction:** Up to equivariant homeomorphism, M^n can be recovered by the pair (P^n, λ).

Two key points for theory of small covers

–Geometric topology

- **$(\mathbb{Z}_2)^n$-coloring:** Each small cover $\pi : M^n \longrightarrow P^n$ determines

$$\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$$

mapping n facets at each vertex to n linearly independent vectors, where $\mathcal{F}(P^n)$:=all facets of P^n.

- **Reconstruction:** Up to equivariant homeomorphism, M^n can be recovered by the pair (P^n, λ).

Two key points for theory of small covers

–Geometric topology

- **$(\mathbb{Z}_2)^n$-coloring:** Each small cover $\pi : M^n \longrightarrow P^n$ determines

$$\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$$

mapping n facets at each vertex to n linearly independent vectors, where $\mathcal{F}(P^n)$:=all facets of P^n.

- **Reconstruction:** Up to equivariant homeomorphism, M^n can be recovered by the pair (P^n, λ).

Two key points for theory of small covers

–Geometric topology

- **$(\mathbb{Z}_2)^n$-coloring:** Each small cover $\pi : M^n \longrightarrow P^n$ determines

$$\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$$

mapping n facets at each vertex to n linearly independent vectors, where $\mathcal{F}(P^n)$:=all facets of P^n.

- **Reconstruction:** Up to equivariant homeomorphism, M^n can be recovered by the pair (P^n, λ).

Reconstruction

Take a point $x \in \partial P^n$, then $\exists$ a l-face F^l of P^n s. t. x is in the relative interior of F^l, where $0 \leq l \leq n-1$. $\because$ P^n is simple $\therefore$ $\exists$ $n-l$ facets $F_1, ..., F_{n-l}$ s.t. $F^l = F_1 \cap \cdots \cap F_{n-l}$.

$G_{F^l} :=$ the rank-$(n-l)$ subgroup of $(\mathbb{Z}_2)^n$ determined by $\lambda(F_1), ..., \lambda(F_{n-l})$.

Define an equivalence relation $\sim$ on $P^n \times (\mathbb{Z}_2)^n$ as follows:

$$(x,g) \sim (y,h) \Longleftrightarrow \begin{cases} x = y \text{ and } g = h & \text{if } x \in \text{int} P^n \\ x = y \text{ and } gh^{-1} \in G_{F^l} & \text{if } x \in \text{int} F^l \subset \partial P^n. \end{cases}$$

The quotient space $P^n \times (\mathbb{Z}_2)^n / \sim$ denoted by $M(P^n, \lambda)$ recovers M^n up to equivariant homeomorphism.

Reconstruction

Take a point $x \in \partial P^n$, then $\exists$ a l-face F^l of P^n s. t. x is in the relative interior of F^l, where $0 \leq l \leq n-1$. $\because P^n$ is simple $\therefore \exists$ $n-l$ facets $F_1, ..., F_{n-l}$ s.t. $F^l = F_1 \cap \cdots \cap F_{n-l}$.

$G_{F^l} :=$ the rank-$(n-l)$ subgroup of $(\mathbb{Z}_2)^n$ determined by $\lambda(F_1), ..., \lambda(F_{n-l})$.

Define an equivalence relation $\sim$ on $P^n \times (\mathbb{Z}_2)^n$ as follows:

$$(x, g) \sim (y, h) \Longleftrightarrow \begin{cases} x = y \text{ and } g = h & \text{if } x \in \text{int} P^n \\ x = y \text{ and } gh^{-1} \in G_{F^l} & \text{if } x \in \text{int} F^l \subset \partial P^n. \end{cases}$$

The quotient space $P^n \times (\mathbb{Z}_2)^n / \sim$ denoted by $M(P^n, \lambda)$ recovers M^n up to equivariant homeomorphism.

Reconstruction

Take a point $x \in \partial P^n$, then $\exists$ a l-face F^l of P^n s. t. x is in the relative interior of F^l, where $0 \leq l \leq n-1$. $\because$ P^n is simple $\therefore$ $\exists$ $n-l$ facets $F_1, ..., F_{n-l}$ s.t. $F^l = F_1 \cap \cdots \cap F_{n-l}$.

$G_{F^l} :=$ the rank-$(n-l)$ subgroup of $(\mathbb{Z}_2)^n$ determined by $\lambda(F_1), ..., \lambda(F_{n-l})$.

Define an equivalence relation $\sim$ on $P^n \times (\mathbb{Z}_2)^n$ as follows:

$$(x,g) \sim (y,h) \Longleftrightarrow \begin{cases} x = y \text{ and } g = h & \text{if } x \in \text{int} P^n \\ x = y \text{ and } gh^{-1} \in G_{F^l} & \text{if } x \in \text{int} F^l \subset \partial P^n. \end{cases}$$

The quotient space $P^n \times (\mathbb{Z}_2)^n / \sim$ denoted by $M(P^n, \lambda)$ recovers M^n up to equivariant homeomorphism.

Reconstruction

Take a point $x \in \partial P^n$, then $\exists$ a l-face F^l of P^n s. t. x is in the relative interior of F^l, where $0 \leq l \leq n-1$. $\because$ P^n is simple $\therefore$ $\exists$ $n-l$ facets $F_1, ..., F_{n-l}$ s.t. $F^l = F_1 \cap \cdots \cap F_{n-l}$.

G_{F^l}:= the rank-$(n-l)$ subgroup of $(\mathbb{Z}_2)^n$ determined by $\lambda(F_1), ..., \lambda(F_{n-l})$.

Define an equivalence relation $\sim$ on $P^n \times (\mathbb{Z}_2)^n$ as follows:

$$(x,g) \sim (y,h) \Longleftrightarrow \begin{cases} x = y \text{ and } g = h & \text{if } x \in \text{int} P^n \\ x = y \text{ and } gh^{-1} \in G_{F^l} & \text{if } x \in \text{int} F^l \subset \partial P^n. \end{cases}$$

The quotient space $P^n \times (\mathbb{Z}_2)^n / \sim$ denoted by $M(P^n, \lambda)$ recovers M^n up to equivariant homeomorphism.

Reconstruction

Take a point $x \in \partial P^n$, then $\exists$ a l-face F^l of P^n s. t. x is in the relative interior of F^l, where $0 \leq l \leq n-1$. $\because$ P^n is simple $\therefore$ $\exists$ $n-l$ facets $F_1, ..., F_{n-l}$ s.t. $F^l = F_1 \cap \cdots \cap F_{n-l}$.

G_{F^l}:= the rank-$(n-l)$ subgroup of $(\mathbb{Z}_2)^n$ determined by $\lambda(F_1), ..., \lambda(F_{n-l})$.

Define an equivalence relation $\sim$ on $P^n \times (\mathbb{Z}_2)^n$ as follows:

$$(x,g) \sim (y,h) \Longleftrightarrow \begin{cases} x = y \text{ and } g = h & \text{if } x \in \text{int} P^n \\ x = y \text{ and } gh^{-1} \in G_{F^l} & \text{if } x \in \text{int} F^l \subset \partial P^n. \end{cases}$$

The quotient space $P^n \times (\mathbb{Z}_2)^n / \sim$ denoted by $M(P^n, \lambda)$ recovers M^n up to equivariant homeomorphism.

Reconstruction

Take a point $x \in \partial P^n$, then $\exists$ a l-face F^l of P^n s. t. x is in the relative interior of F^l, where $0 \leq l \leq n-1$. $\because$ P^n is simple $\therefore$ $\exists$ $n-l$ facets $F_1, ..., F_{n-l}$ s.t. $F^l = F_1 \cap \cdots \cap F_{n-l}$.

G_{F^l}:= the rank-$(n-l)$ subgroup of $(\mathbb{Z}_2)^n$ determined by $\lambda(F_1), ..., \lambda(F_{n-l})$.

Define an equivalence relation $\sim$ on $P^n \times (\mathbb{Z}_2)^n$ as follows:

$$(x,g) \sim (y,h) \Longleftrightarrow \begin{cases} x = y \text{ and } g = h & \text{if } x \in \text{int} P^n \\ x = y \text{ and } gh^{-1} \in G_{F^l} & \text{if } x \in \text{int} F^l \subset \partial P^n. \end{cases}$$

The quotient space $P^n \times (\mathbb{Z}_2)^n / \sim$ denoted by $M(P^n, \lambda)$ recovers M^n up to equivariant homeomorphism.

An example of $(\mathbb{Z}_2)^n$-coloring

Example: $(\mathbb{Z}_2)^2 \curvearrowright \mathbb{R}P^2$ by $[x_0, x_1, x_2] \longmapsto [x_0, g_1 x_1, g_2 x_2]$.

Then its $(\mathbb{Z}_2)^2$-coloring λ is as follows:

where

$$\lambda(F_1) = (1,0) \leftrightsquigarrow \mathbb{Z}_2 \times \{0\} \leftrightsquigarrow \pi^{-1}(F_1) = \{[x_0, 0, x_2]\} \subset \mathbb{R}P^2$$

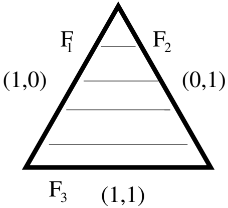
$$\lambda(F_2) = (0,1) \leftrightsquigarrow \{0\} \times \mathbb{Z}_2 \leftrightsquigarrow \pi^{-1}(F_2) = \{[x_0, x_1, 0]\} \subset \mathbb{R}P^2$$

$$\lambda(F_3) = (1,1) \leftrightsquigarrow \{(0,0),(1,1)\} \leftrightsquigarrow \pi^{-1}(F_3) = \{[0, x_1, x_2\} \subset \mathbb{R}P^2$$

An example of $(\mathbb{Z}_2)^n$-coloring

Example: $(\mathbb{Z}_2)^2 \curvearrowright \mathbb{R}P^2$ by $[x_0, x_1, x_2] \longmapsto [x_0, g_1x_1, g_2x_2]$.

Then its $(\mathbb{Z}_2)^2$-coloring λ is as follows:



where

$$\lambda(F_1) = (1,0) \leftrightsquigarrow \mathbb{Z}_2 \times \{0\} \leftrightsquigarrow \pi^{-1}(F_1) = \{[x_0, 0, x_2]\} \subset \mathbb{R}P^2$$

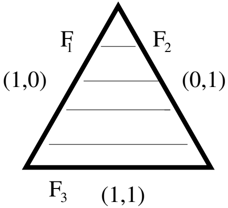
$$\lambda(F_2) = (0,1) \leftrightsquigarrow \{0\} \times \mathbb{Z}_2 \leftrightsquigarrow \pi^{-1}(F_2) = \{[x_0, x_1, 0]\} \subset \mathbb{R}P^2$$

$$\lambda(F_3) = (1,1) \leftrightsquigarrow \{(0,0),(1,1)\} \leftrightsquigarrow \pi^{-1}(F_3) = \{[0, x_1, x_2\} \subset \mathbb{R}P^2$$

An example of $(\mathbb{Z}_2)^n$-coloring

Example: $(\mathbb{Z}_2)^2 \curvearrowright \mathbb{R}P^2$ by $[x_0, x_1, x_2] \longmapsto [x_0, g_1 x_1, g_2 x_2]$.

Then its $(\mathbb{Z}_2)^2$-coloring λ is as follows:



where

$$\lambda(F_1) = (1,0) \leftrightsquigarrow \mathbb{Z}_2 \times \{0\} \leftrightsquigarrow \pi^{-1}(F_1) = \{[x_0, 0, x_2]\} \subset \mathbb{R}P^2$$

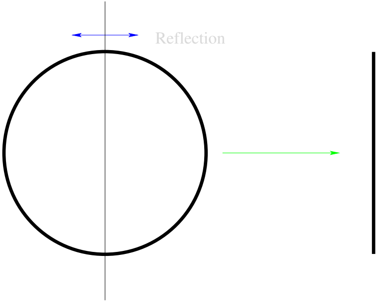
$$\lambda(F_2) = (0,1) \leftrightsquigarrow \{0\} \times \mathbb{Z}_2 \leftrightsquigarrow \pi^{-1}(F_2) = \{[x_0, x_1, 0]\} \subset \mathbb{R}P^2$$

$$\lambda(F_3) = (1,1) \leftrightsquigarrow \{(0,0), (1,1)\} \leftrightsquigarrow \pi^{-1}(F_3) = \{[0, x_1, x_2\} \subset \mathbb{R}P^2$$

An Example of Reconstruction

First, consider $\mathbb{Z}_2 \curvearrowright S^1$ by $(x_0, x_1) \longmapsto (x_0, gx_1)$

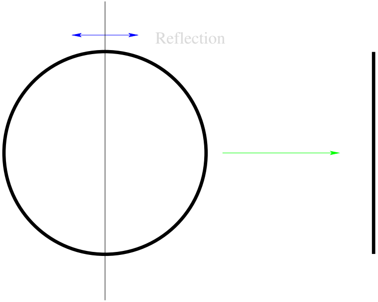
$\Longrightarrow$ orbit space $S^1/\mathbb{Z}_2$ is an interval (or a 1-simplex).



An Example of Reconstruction

First, consider $\mathbb{Z}_2 \curvearrowright S^1$ by $(x_0, x_1) \longmapsto (x_0, gx_1)$

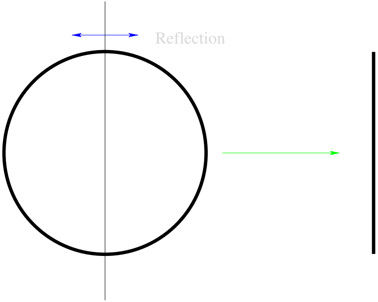
$\Longrightarrow$ orbit space $S^1/\mathbb{Z}_2$ is an interval (or a 1-simplex).



An Example of Reconstruction

First, consider $\mathbb{Z}_2 \curvearrowright S^1$ by $(x_0, x_1) \longmapsto (x_0, gx_1)$

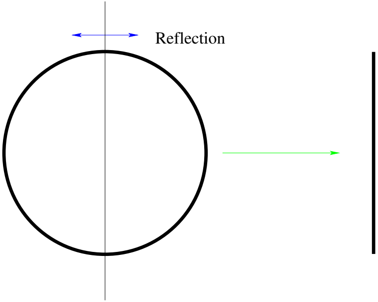
$\Longrightarrow$ orbit space $S^1/\mathbb{Z}_2$ is an interval (or a 1-simplex).



An Example of Reconstruction

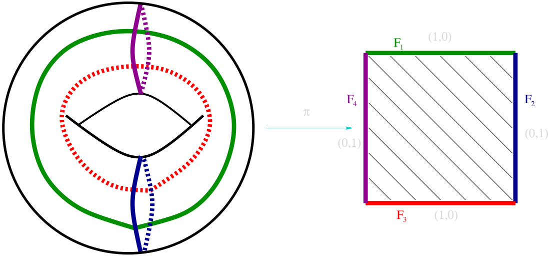
First, consider $\mathbb{Z}_2 \curvearrowright S^1$ by $(x_0, x_1) \longmapsto (x_0, gx_1)$

$\Longrightarrow$ orbit space $S^1/\mathbb{Z}_2$ is an interval (or a 1-simplex).



An Example of Reconstruction

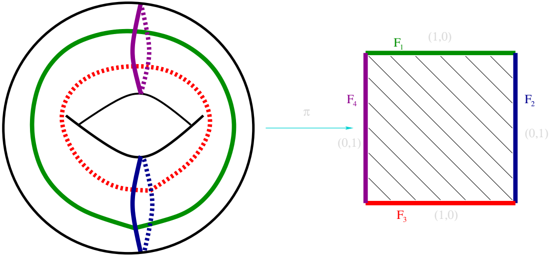
The product of two copies of the above action gives the following:
$(\mathbb{Z}_2)^2 \curvearrowright S^1 \times S^1 = T^2 \Rightarrow T^2/(\mathbb{Z}_2)^2$ is a square.



$\pi^{-1}(F_1)$ and $\pi^{-1}(F_3)$ are two circles in T^2 fixed by subgroup $\mathbb{Z}_2 \times \{0\} \leftrightsquigarrow (1,0)$
$\pi^{-1}(F_2)$ and $\pi^{-1}(F_4)$ are two circles in T^2 fixed by subgroup $\{0\} \times \mathbb{Z}_2 \leftrightsquigarrow (0,1)$

An Example of Reconstruction

The product of two copies of the above action gives the following: $(\mathbb{Z}_2)^2 \curvearrowright S^1 \times S^1 = T^2 \ \Rightarrow \ T^2/(\mathbb{Z}_2)^2$ is a square.

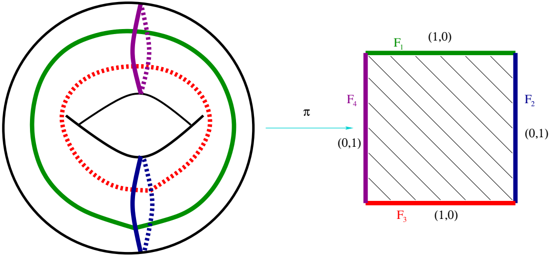


$\pi^{-1}(F_1)$ and $\pi^{-1}(F_3)$ are two circles in T^2 fixed by subgroup $\mathbb{Z}_2 \times \{0\} \leftrightsquigarrow (1,0)$

$\pi^{-1}(F_2)$ and $\pi^{-1}(F_4)$ are two circles in T^2 fixed by subgroup $\{0\} \times \mathbb{Z}_2 \leftrightsquigarrow (0,1)$

An Example of Reconstruction

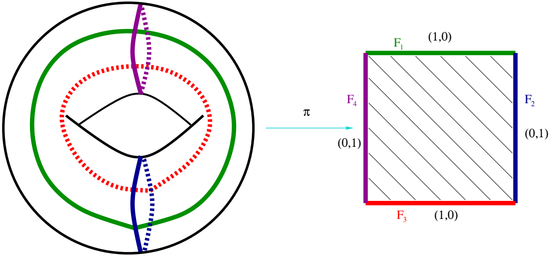
The product of two copies of the above action gives the following: $(\mathbb{Z}_2)^2 \curvearrowright S^1 \times S^1 = T^2 \ \Rightarrow \ T^2/(\mathbb{Z}_2)^2$ is a square.



$\pi^{-1}(F_1)$ and $\pi^{-1}(F_3)$ are two circles in T^2 fixed by subgroup $\mathbb{Z}_2 \times \{0\} \leftrightsquigarrow (1,0)$
$\pi^{-1}(F_2)$ and $\pi^{-1}(F_4)$ are two circles in T^2 fixed by subgroup $\{0\} \times \mathbb{Z}_2 \leftrightsquigarrow (0,1)$

An Example of Reconstruction

The product of two copies of the above action gives the following: $(\mathbb{Z}_2)^2 \curvearrowright S^1 \times S^1 = T^2 \ \Rightarrow \ T^2/(\mathbb{Z}_2)^2$ is a square.

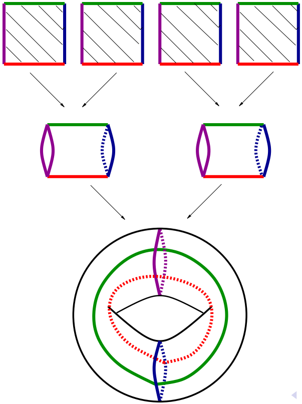


$\pi^{-1}(F_1)$ and $\pi^{-1}(F_3)$ are two circles in T^2 fixed by subgroup $\mathbb{Z}_2 \times \{0\} \leftrightsquigarrow (1,0)$
$\pi^{-1}(F_2)$ and $\pi^{-1}(F_4)$ are two circles in T^2 fixed by subgroup $\{0\} \times \mathbb{Z}_2 \leftrightsquigarrow (0,1)$

An Example of Reconstruction

Reconstruction process of T^2:

$(\mathbb{Z}_2)^2 \times P^2$ (i.e., four copies of P^2)



Reconstruction of small covers

We see that

Reconstruction of small covers

- Geometrically, $M(P^n, \lambda)$ is exactly obtained by gluing 2^n copies of P^n along their boundaries by using $(\mathbb{Z}_2)^n$-coloring λ.
- This reconstruction of small covers provides a way of studying closed manifolds by using $(\mathbb{Z}_2)^n$-colored polytopes.

Reconstruction of small covers

We see that

Reconstruction of small covers

- Geometrically, $M(P^n, \lambda)$ is exactly obtained by gluing 2^n copies of P^n along their boundaries by using $(\mathbb{Z}_2)^n$-coloring λ.
- This reconstruction of small covers provides a way of studying closed manifolds by using $(\mathbb{Z}_2)^n$-colored polytopes.

Reconstruction of small covers

We see that

Reconstruction of small covers

- Geometrically, $M(P^n, \lambda)$ is exactly obtained by gluing 2^n copies of P^n along their boundaries by using $(\mathbb{Z}_2)^n$-coloring λ.
- This reconstruction of small covers provides a way of studying closed manifolds by using $(\mathbb{Z}_2)^n$-colored polytopes.

3-colorable case–Izmestiev's work

In 2001, Izmestiev [Math. Notes **69** (2001)] studied a class of 3-dimensional small covers such that each λ of $(\mathbb{Z}_2)^3$-colorings on their orbit spaces is **3-colorable** (i.e., the image of λ contains only three linearly independent elements of $(\mathbb{Z}_2)^3$), and showed that

Izmestiev's Theorem

Each such small cover can be formed from finitely many **3-dimensional tori** with the canonical $(\mathbb{Z}_2)^3$-action under the operations of the equivariant connected sum and the equivariant Dehn surgery.

3-colorable case–Izmestiev's work

In 2001, Izmestiev [Math. Notes **69** (2001)] studied a class of 3-dimensional small covers such that each λ of $(\mathbb{Z}_2)^3$-colorings on their orbit spaces is **3-colorable** (i.e., the image of λ contains only three linearly independent elements of $(\mathbb{Z}_2)^3$), and showed that

Izmestiev's Theorem

Each such small cover can be formed from finitely many **3-dimensional tori** with the canonical $(\mathbb{Z}_2)^3$-action under the operations of the equivariant connected sum and the equivariant Dehn surgery.

3-colorable case–Izmestiev's work

In 2001, Izmestiev [Math. Notes **69** (2001)] studied a class of 3-dimensional small covers such that each λ of $(\mathbb{Z}_2)^3$-colorings on their orbit spaces is **3-colorable** (i.e., the image of λ contains only three linearly independent elements of $(\mathbb{Z}_2)^3$), and showed that

Izmestiev's Theorem

Each such small cover can be formed from finitely many **3-dimensional tori** with the canonical $(\mathbb{Z}_2)^3$-action under the operations of the equivariant connected sum and the equivariant Dehn surgery.

Objective of this talk

We shall consider **all possible 3-dimensional small covers**.

Our objective

To determine the (equivariant) topological types of such a class of 3-dimensional manifolds.

RK: Four Color Theorem makes sure that **each simple convex 3-polytope always admits $(\mathbb{Z}_2)^3$-colorings**. Thus, **all simple convex 3-polytopes** with $(\mathbb{Z}_2)^3$-colorings can recover **all 3-dimensional small covers**.

Objective of this talk

We shall consider **all possible 3-dimensional small covers**.

Our objective

To determine the (equivariant) topological types of such a class of 3-dimensional manifolds.

RK: Four Color Theorem makes sure that **each simple convex 3-polytope always admits $(\mathbb{Z}_2)^3$-colorings**. Thus, **all simple convex 3-polytopes** with $(\mathbb{Z}_2)^3$-colorings can recover **all 3-dimensional small covers**.

Objective of this talk

We shall consider **all possible 3-dimensional small covers**.

Our objective

To determine the (equivariant) topological types of such a class of 3-dimensional manifolds.

RK: Four Color Theorem makes sure that **each simple convex 3-polytope always admits $(\mathbb{Z}_2)^3$-colorings**. Thus, **all simple convex 3-polytopes** with $(\mathbb{Z}_2)^3$-colorings can recover **all 3-dimensional small covers**.

Objective of this talk

We shall consider **all possible 3-dimensional small covers**.

Our objective

To determine the (equivariant) topological types of such a class of 3-dimensional manifolds.

RK: Four Color Theorem makes sure that **each simple convex 3-polytope always admits $(\mathbb{Z}_2)^3$-colorings**. Thus, **all simple convex 3-polytopes** with $(\mathbb{Z}_2)^3$-colorings can recover **all 3-dimensional small covers**.

A one-to-one correspondence

$\mathcal{P}$:= the set of all pairs (P^3, λ) where P^3 is a simple convex 3-polytope and λ is a $(\mathbb{Z}_2)^3$-coloring on it.

$\mathcal{M}$:= the set of all 3-dimensional small covers.

A one-to-one correspondence

$$\mathcal{P} \longleftrightarrow \mathcal{M}$$

$$(P^3, \lambda) \longmapsto M(P^3, \lambda)$$

A one-to-one correspondence

$\mathcal{P}$:= the set of all pairs (P^3, λ) where P^3 is a simple convex 3-polytope and λ is a $(\mathbb{Z}_2)^3$-coloring on it.

$\mathcal{M}$:= the set of all 3-dimensional small covers.

A one-to-one correspondence

$$\mathcal{P} \longleftrightarrow \mathcal{M}$$

$$(P^3, \lambda) \longmapsto M(P^3, \lambda)$$

A one-to-one correspondence

$\mathcal{P}$:= the set of all pairs (P^3, λ) where P^3 is a simple convex 3-polytope and λ is a $(\mathbb{Z}_2)^3$-coloring on it.

$\mathcal{M}$:= the set of all 3-dimensional small covers.

A one-to-one correspondence

$$\mathcal{P} \longleftrightarrow \mathcal{M}$$

$$(P^3, \lambda) \longmapsto M(P^3, \lambda)$$

$\mathrm{GL}(3, \mathbb{Z}_2)$-equivalence

Define a natural action of $\mathrm{GL}(3, \mathbb{Z}_2)$ on $\mathcal{P}$ by

$$(P^3, \lambda) \longmapsto (P^3, \sigma \circ \lambda) \text{ where } \sigma \in \mathrm{GL}(3, \mathbb{Z}_2)$$

This action is free, and induces an action of $\mathrm{GL}(3, \mathbb{Z}_2)$ on $\mathcal{M}$ by mapping $M(P^3, \lambda)$ to $M(P^3, \sigma \circ \lambda)$.

Both $M(P^3, \lambda)$ and $M(P^3, \sigma \circ \lambda)$ are σ-equivariantly homeomorphic

All elements of each equivalence class of $\mathcal{P}/\mathrm{GL}(3, \mathbb{Z}_2)$ (resp. $\mathcal{M}/\mathrm{GL}(3, \mathbb{Z}_2)$) are said to be **$\mathrm{GL}(3, \mathbb{Z}_2)$-equivalent**.

$\mathrm{GL}(3, \mathbb{Z}_2)$-equivalence

Define a natural action of $\mathrm{GL}(3, \mathbb{Z}_2)$ on $\mathcal{P}$ by

$$(P^3, \lambda) \longmapsto (P^3, \sigma \circ \lambda) \text{ where } \sigma \in \mathrm{GL}(3, \mathbb{Z}_2)$$

This action is free, and induces an action of $\mathrm{GL}(3, \mathbb{Z}_2)$ on $\mathcal{M}$ by mapping $M(P^3, \lambda)$ to $M(P^3, \sigma \circ \lambda)$.

Both $M(P^3, \lambda)$ and $M(P^3, \sigma \circ \lambda)$ are σ-equivariantly homeomorphic

All elements of each equivalence class of $\mathcal{P}/\mathrm{GL}(3, \mathbb{Z}_2)$ (resp. $\mathcal{M}/\mathrm{GL}(3, \mathbb{Z}_2)$) are said to be **$\mathrm{GL}(3, \mathbb{Z}_2)$-equivalent**.

$\mathrm{GL}(3,\mathbb{Z}_2)$-equivalence

Define a natural action of $\mathrm{GL}(3,\mathbb{Z}_2)$ on $\mathcal{P}$ by

$$(P^3,\lambda) \longmapsto (P^3,\sigma\circ\lambda) \text{ where } \sigma\in \mathrm{GL}(3,\mathbb{Z}_2)$$

This action is free, and induces an action of $\mathrm{GL}(3,\mathbb{Z}_2)$ on $\mathcal{M}$ by mapping $M(P^3,\lambda)$ to $M(P^3,\sigma\circ\lambda)$.

Both $M(P^3,\lambda)$ and $M(P^3,\sigma\circ\lambda)$ are σ-equivariantly homeomorphic

All elements of each equivalence class of $\mathcal{P}/\mathrm{GL}(3,\mathbb{Z}_2)$ (resp. $\mathcal{M}/\mathrm{GL}(3,\mathbb{Z}_2)$) are said to be **$\mathrm{GL}(3,\mathbb{Z}_2)$-equivalent**.

$\mathrm{GL}(3,\mathbb{Z}_2)$-equivalence

Define a natural action of $\mathrm{GL}(3,\mathbb{Z}_2)$ on $\mathcal{P}$ by

$$(P^3,\lambda) \longmapsto (P^3,\sigma\circ\lambda) \text{ where } \sigma\in \mathrm{GL}(3,\mathbb{Z}_2)$$

This action is free, and induces an action of $\mathrm{GL}(3,\mathbb{Z}_2)$ on $\mathcal{M}$ by mapping $M(P^3,\lambda)$ to $M(P^3,\sigma\circ\lambda)$.

Both $M(P^3,\lambda)$ and $M(P^3,\sigma\circ\lambda)$ are σ-equivariantly homeomorphic

All elements of each equivalence class of $\mathcal{P}/\mathrm{GL}(3,\mathbb{Z}_2)$ (resp. $\mathcal{M}/\mathrm{GL}(3,\mathbb{Z}_2)$) are said to be **$\mathrm{GL}(3,\mathbb{Z}_2)$-equivalent**.

$\mathrm{GL}(3, \mathbb{Z}_2)$-equivalence

Define a natural action of $\mathrm{GL}(3, \mathbb{Z}_2)$ on $\mathcal{P}$ by

$$(P^3, \lambda) \longmapsto (P^3, \sigma \circ \lambda) \text{ where } \sigma \in \mathrm{GL}(3, \mathbb{Z}_2)$$

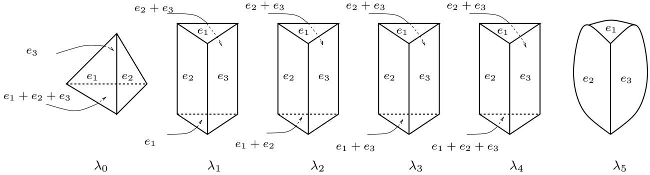
This action is free, and induces an action of $\mathrm{GL}(3, \mathbb{Z}_2)$ on $\mathcal{M}$ by mapping $M(P^3, \lambda)$ to $M(P^3, \sigma \circ \lambda)$.

Both $M(P^3, \lambda)$ and $M(P^3, \sigma \circ \lambda)$ are σ-equivariantly homeomorphic

All elements of each equivalence class of $\mathcal{P}/\mathrm{GL}(3, \mathbb{Z}_2)$ (resp. $\mathcal{M}/\mathrm{GL}(3, \mathbb{Z}_2)$) are said to be **$\mathrm{GL}(3, \mathbb{Z}_2)$-equivalent**.

Combinatorial version of main result

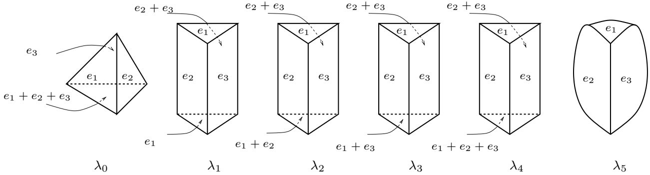
- To introduce six operations $\sharp^v, \sharp^e, \sharp^{eve}, \natural, \sharp^\triangle, \sharp^{\copyright}$ on $\mathcal{P}$.
- Under these six operations, up to $\mathrm{GL}(3,\mathbb{Z}_2)$-equivalence we find five basic pairs (Δ^3, λ_0), $(P^3(3), \lambda_1)$, $(P^3(3), \lambda_2)$, $(P^3(3), \lambda_3)$, $(P^3(3), \lambda_4)$ of $\mathcal{P}$ and an exceptional pair $(\emptyset, \lambda_5)$, where Δ^3 is a 3-simplex, $P^3(3)$ is a 3-sided prism, and $\lambda_i, i = 0, 1, ..., 5$, are shown as follows:



where $\{e_1, e_2, e_3\}$ is the standard basis of $(\mathbb{Z}_2)^3$.

Combinatorial version of main result

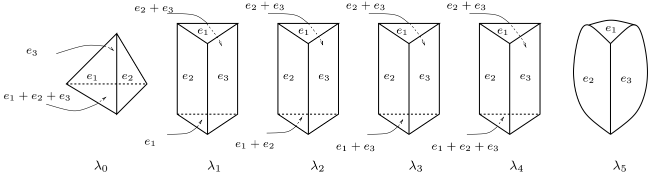
- To introduce six operations $\sharp^v, \sharp^e, \sharp^{eve}, \natural, \sharp^{\triangle}, \sharp^{©}$ on $\mathcal{P}$.
- Under these six operations, up to $\mathrm{GL}(3, \mathbb{Z}_2)$-equivalence we find five basic pairs (Δ^3, λ_0), $(P^3(3), \lambda_1)$, $(P^3(3), \lambda_2)$, $(P^3(3), \lambda_3)$, $(P^3(3), \lambda_4)$ of $\mathcal{P}$ and an exceptional pair $(\emptyset, \lambda_5)$, where Δ^3 is a 3-simplex, $P^3(3)$ is a 3-sided prism, and $\lambda_i, i = 0, 1, ..., 5$, are shown as follows:



where $\{e_1, e_2, e_3\}$ is the standard basis of $(\mathbb{Z}_2)^3$.

Combinatorial version of main result

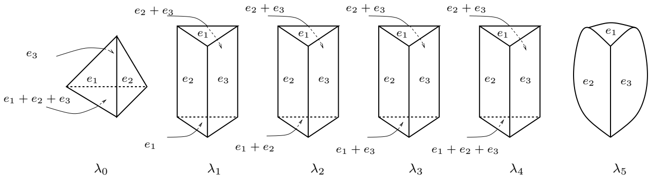
- To introduce six operations $\sharp^v, \sharp^e, \sharp^{eve}, \natural, \sharp^{\triangle}, \sharp^{©}$ on $\mathcal{P}$.
- Under these six operations, up to $\mathrm{GL}(3,\mathbb{Z}_2)$-equivalence we find five basic pairs (Δ^3, λ_0), $(P^3(3), \lambda_1)$, $(P^3(3), \lambda_2)$, $(P^3(3), \lambda_3)$, $(P^3(3), \lambda_4)$ of $\mathcal{P}$ and an exceptional pair $(\oslash, \lambda_5)$, where Δ^3 is a 3-simplex, $P^3(3)$ is a 3-sided prism, and $\lambda_i, i = 0, 1, ..., 5$, are shown as follows:



where $\{e_1, e_2, e_3\}$ is the standard basis of $(\mathbb{Z}_2)^3$.

Combinatorial version of main result

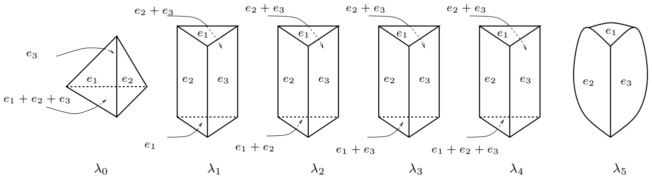
- To introduce six operations $\sharp^v, \sharp^e, \sharp^{eve}, \natural, \sharp^{\triangle}, \sharp^{©}$ on $\mathcal{P}$.
- Under these six operations, up to $\mathrm{GL}(3,\mathbb{Z}_2)$-equivalence we find five basic pairs (Δ^3,λ_0), $(P^3(3),\lambda_1)$, $(P^3(3),\lambda_2)$, $(P^3(3),\lambda_3)$, $(P^3(3),\lambda_4)$ of $\mathcal{P}$ and an exceptional pair $(\oslash,\lambda_5)$, where Δ^3 is a 3-simplex, $P^3(3)$ is a 3-sided prism, and $\lambda_i, i=0,1,...,5$, are shown as follows:



where $\{e_1,e_2,e_3\}$ is the standard basis of $(\mathbb{Z}_2)^3$.

Combinatorial version of main result

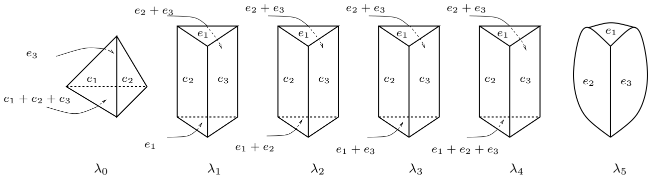
- To introduce six operations $\sharp^v, \sharp^e, \sharp^{eve}, \natural, \sharp^{\triangle}, \sharp^{©}$ on $\mathcal{P}$.
- Under these six operations, up to $\mathrm{GL}(3, \mathbb{Z}_2)$-equivalence we find five basic pairs (Δ^3, λ_0), $(P^3(3), \lambda_1)$, $(P^3(3), \lambda_2)$, $(P^3(3), \lambda_3)$, $(P^3(3), \lambda_4)$ of $\mathcal{P}$ and an exceptional pair $(\oslash, \lambda_5)$, where Δ^3 is a 3-simplex, $P^3(3)$ is a 3-sided prism, and $\lambda_i, i = 0, 1, ..., 5$, are shown as follows:



where $\{e_1, e_2, e_3\}$ is the standard basis of $(\mathbb{Z}_2)^3$.

Combinatorial version of main result

- To introduce six operations $\sharp^v, \sharp^e, \sharp^{eve}, \natural, \sharp^{\triangle}, \sharp^{©}$ on $\mathcal{P}$.
- Under these six operations, up to $\mathrm{GL}(3, \mathbb{Z}_2)$-equivalence we find five basic pairs (Δ^3, λ_0), $(P^3(3), \lambda_1)$, $(P^3(3), \lambda_2)$, $(P^3(3), \lambda_3)$, $(P^3(3), \lambda_4)$ of $\mathcal{P}$ and an exceptional pair $(\oslash, \lambda_5)$, where Δ^3 is a 3-simplex, $P^3(3)$ is a 3-sided prism, and $\lambda_i, i = 0, 1, ..., 5$, are shown as follows:



where $\{e_1, e_2, e_3\}$ is the standard basis of $(\mathbb{Z}_2)^3$.

Combinatorial version of main result

Combinatorial version of main result

Each pair (P^3, λ) of $\mathcal{P}$ is an expression of $(\Delta^3, \sigma \circ \lambda_0)$, $(P^3(3), \sigma \circ \lambda_1)$, $(P^3(3), \sigma \circ \lambda_2)$, $(P^3(3), \sigma \circ \lambda_3)$, $(P^3(3), \sigma \circ \lambda_4)$, $(\oslash, \sigma \circ \lambda_5)$, $\sigma \in \mathrm{GL}(3, \mathbb{Z}_2)$, under six operations $\sharp^v$, $\sharp^e$, $\sharp^{eve}$,$\natural$, $\sharp^{\triangle}$, $\sharp^{©}$.

Topological version of main result

By the reconstruction of small covers,

$$\sharp^v, \sharp^e, \sharp^{eve}, \natural, \sharp^{\triangle}, \sharp^{©} \text{ on } \mathcal{P} \longleftrightarrow \widetilde{\sharp^v}, \widetilde{\sharp^e}, \widetilde{\sharp^{eve}}, \widetilde{\natural}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}} \text{ on } \mathcal{M}$$

Six operations on $\mathcal{M}$

$\widetilde{\sharp^v}$ is the equivariant connected sum, and $\widetilde{\natural}$ is the equivariant Dehn surgery, and other four operations $\widetilde{\sharp^e}, \widetilde{\sharp^{eve}}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}}$ can be understood as the generalized equivariant connected sums.

Algebraic system

$M(\Delta^3, \sigma\circ\lambda_0)$ and $M(P^3(3), \sigma\circ\lambda_i)(i=1,...,4), \sigma\in \mathrm{GL}(3,\mathbb{Z}_2)$, $M(\oslash, \sigma\circ\lambda_5)$, give all elementary generators of the algebraic system $\langle \mathcal{M}; \widetilde{\sharp^v}, \widetilde{\sharp^e}, \widetilde{\sharp^{eve}}, \widetilde{\natural}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}}\rangle$.

Topological version of main result

By the reconstruction of small covers,

$$\sharp^v, \sharp^e, \sharp^{eve}, \natural, \sharp^{\triangle}, \sharp^{©} \text{ on } \mathcal{P} \longleftrightarrow \widetilde{\sharp^v}, \widetilde{\sharp^e}, \widetilde{\sharp^{eve}}, \widetilde{\natural}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}} \text{ on } \mathcal{M}$$

Six operations on $\mathcal{M}$

$\widetilde{\sharp^v}$ is the equivariant connected sum, and $\widetilde{\natural}$ is the equivariant Dehn surgery, and other four operations $\widetilde{\sharp^e}, \widetilde{\sharp^{eve}}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}}$ can be understood as the generalized equivariant connected sums.

Algebraic system

$M(\Delta^3, \sigma \circ \lambda_0)$ and $M(P^3(3), \sigma \circ \lambda_i)(i = 1, ..., 4), \sigma \in \mathrm{GL}(3, \mathbb{Z}_2)$, $M(\oslash, \sigma \circ \lambda_5)$, give all elementary generators of the algebraic system $\langle \mathcal{M}; \widetilde{\sharp^v}, \widetilde{\sharp^e}, \widetilde{\sharp^{eve}}, \widetilde{\natural}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}} \rangle$.

Topological version of main result

By the reconstruction of small covers,

$$\sharp^{v}, \sharp^{e}, \sharp^{eve}, \natural, \sharp^{\triangle}, \sharp^{©} \text{ on } \mathcal{P} \longleftrightarrow \widetilde{\sharp^{v}}, \widetilde{\sharp^{e}}, \widetilde{\sharp^{eve}}, \widetilde{\natural}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}} \text{ on } \mathcal{M}$$

Six operations on $\mathcal{M}$

$\widetilde{\sharp^{v}}$ is the equivariant connected sum, and $\widetilde{\natural}$ is the equivariant Dehn surgery, and other four operations $\widetilde{\sharp^{e}}, \widetilde{\sharp^{eve}}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}}$ can be understood as the generalized equivariant connected sums.

Algebraic system

$M(\Delta^3, \sigma \circ \lambda_0)$ and $M(P^3(3), \sigma \circ \lambda_i)(i = 1, ..., 4), \sigma \in \mathrm{GL}(3, \mathbb{Z}_2)$, $M(\oslash, \sigma \circ \lambda_5)$, give all elementary generators of the algebraic system $\langle \mathcal{M}; \widetilde{\sharp^{v}}, \widetilde{\sharp^{e}}, \widetilde{\sharp^{eve}}, \widetilde{\natural}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}} \rangle$.

Topological version of main result

By the reconstruction of small covers,

$$\sharp^{v}, \sharp^{e}, \sharp^{eve}, \natural, \sharp^{\triangle}, \sharp^{©} \text{ on } \mathcal{P} \longleftrightarrow \widetilde{\sharp^{v}}, \widetilde{\sharp^{e}}, \widetilde{\sharp^{eve}}, \widetilde{\natural}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}} \text{ on } \mathcal{M}$$

Six operations on $\mathcal{M}$

$\widetilde{\sharp^{v}}$ is the equivariant connected sum, and $\widetilde{\natural}$ is the equivariant Dehn surgery, and other four operations $\widetilde{\sharp^{e}}, \widetilde{\sharp^{eve}}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}}$ can be understood as the generalized equivariant connected sums.

Algebraic system

$M(\Delta^3, \sigma\circ\lambda_0)$ and $M(P^3(3), \sigma\circ\lambda_i)(i=1,...,4), \sigma\in \mathrm{GL}(3,\mathbb{Z}_2)$, $M(\oslash, \sigma\circ\lambda_5)$, give all elementary generators of the algebraic system $\langle \mathcal{M}; \widetilde{\sharp^{v}}, \widetilde{\sharp^{e}}, \widetilde{\sharp^{eve}}, \widetilde{\natural}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}}\rangle$.

Topological version of main result

On the other hand, we shall show that

Generators

$$M(\Delta^3, \lambda_0) \approx \mathbb{R}P^3 \text{ with the canonical linear } (\mathbb{Z}_2)^3\text{-action}$$

$$M(P^3(3), \lambda_i)(i = 1, ..., 4) \approx S^1 \times \mathbb{R}P^2 \text{ with four differ. } (\mathbb{Z}_2)^3\text{-actions}$$

$$M(\oslash, \sigma \circ \lambda_5) \approx S^3 \text{ with standard } (\mathbb{Z}_2)^3\text{-action}$$

Topological version of our main result

Each 3-dimensional small cover can be obtained from $S^3, \mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$-actions by using six operations $\widetilde{\sharp^v}, \widetilde{\sharp^e}, \widetilde{\sharp^{eve}}, \widetilde{\natural}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}}$.

Topological version of main result

On the other hand, we shall show that

Generators

$$M(\Delta^3, \lambda_0) \approx \mathbb{R}P^3 \text{ with the canonical linear } (\mathbb{Z}_2)^3\text{-action}$$

$$M(P^3(3), \lambda_i)(i = 1, ..., 4) \approx S^1 \times \mathbb{R}P^2 \text{ with four differ. } (\mathbb{Z}_2)^3\text{-actions}$$

$$M(\oslash, \sigma \circ \lambda_5) \approx S^3 \text{ with standard } (\mathbb{Z}_2)^3\text{-action}$$

Topological version of our main result

Each 3-dimensional small cover can be obtained from $S^3, \mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$-actions by using six operations $\widetilde{\sharp^v}, \widetilde{\sharp^e}, \widetilde{\sharp^{eve}}, \widetilde{\natural}, \widetilde{\sharp^\triangle}, \widetilde{\sharp^©}$.

Topological version of main result

On the other hand, we shall show that

Generators

$$M(\Delta^3, \lambda_0) \approx \mathbb{R}P^3 \text{ with the canonical linear } (\mathbb{Z}_2)^3\text{-action}$$

$$M(P^3(3), \lambda_i)(i = 1, ..., 4) \approx S^1 \times \mathbb{R}P^2 \text{ with four differ. } (\mathbb{Z}_2)^3\text{-actions}$$

$$M(\oslash, \sigma \circ \lambda_5) \approx S^3 \text{ with standard } (\mathbb{Z}_2)^3\text{-action}$$

Topological version of our main result

Each 3-dimensional small cover can be obtained from $S^3, \mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$-actions by using six operations $\widetilde{\sharp^v}, \widetilde{\sharp^e}, \widetilde{\sharp^{eve}}, \widetilde{\natural}, \widetilde{\sharp^{\triangle}}, \widetilde{\sharp^{©}}$.

Remark

Remark

Main result is an **equivariant analogue** of a well-known result as follows: "Each closed 3-manifold can be obtained from a 3-sphere by using a finite number of Dehn surgeries". (See, [Lickorish, Ann. of Math. **76** (1962); Proc. Camb. Phil. Soc. **59** (1963)] or [Kirby, Invent. Math. **45** (1978)])

Remark

Remark

Main result is an **equivariant analogue** of a well-known result as follows: "Each closed 3-manifold can be obtained from a 3-sphere by using a finite number of Dehn surgeries". (See, [Lickorish, Ann. of Math. **76** (1962); Proc. Camb. Phil. Soc. **59** (1963)] or [Kirby, Invent. Math. **45** (1978)])

Application to cobordism

$\widehat{\mathcal{M}}$:= the set of equivariant unoriented cobordism classes of all 3-manifolds in $\mathcal{M}$.

Fact

$\widehat{\mathcal{M}}$ forms an abelian group under disjoint union, so it is also a vector space over $\mathbb{Z}_2$.

Theorem (Equivariant cobordism classification)

$\widehat{\mathcal{M}}$ is generated by classes of $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$-actions.

Application to cobordism

$\widehat{\mathcal{M}}$:= the set of equivariant unoriented cobordism classes of all 3-manifolds in $\mathcal{M}$.

Fact

$\widehat{\mathcal{M}}$ forms an abelian group under disjoint union, so it is also a vector space over $\mathbb{Z}_2$.

Theorem (Equivariant cobordism classification)

$\widehat{\mathcal{M}}$ is generated by classes of $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$-actions.

Application to cobordism

$\widehat{\mathcal{M}} :=$ the set of equivariant unoriented cobordism classes of all 3-manifolds in $\mathcal{M}$.

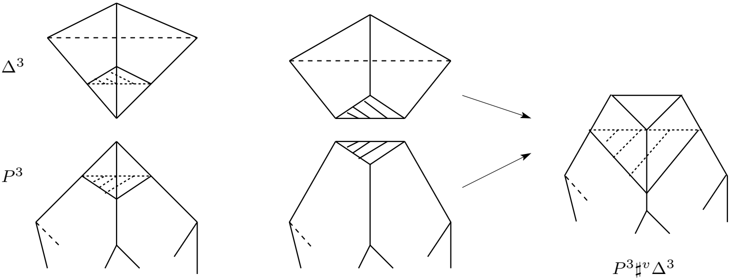
Fact

$\widehat{\mathcal{M}}$ forms an abelian group under disjoint union, so it is also a vector space over $\mathbb{Z}_2$.

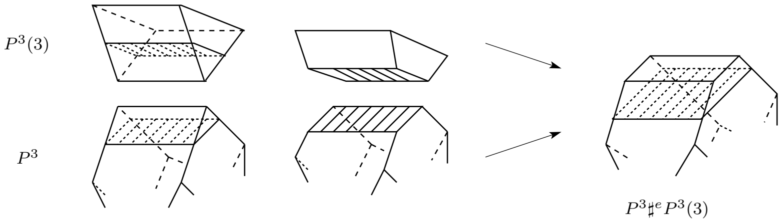
Theorem (Equivariant cobordism classification)

$\widehat{\mathcal{M}}$ is generated by classes of $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$-actions.

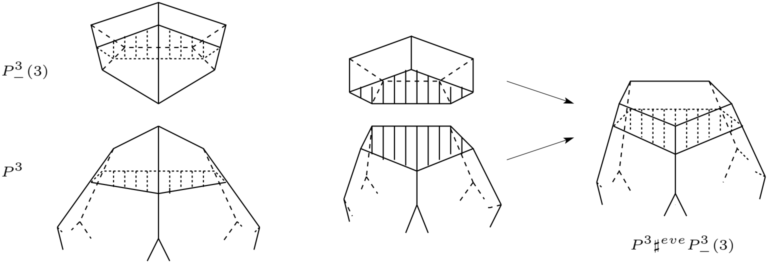
Operation $\sharp^V$ on $\mathcal{P}$



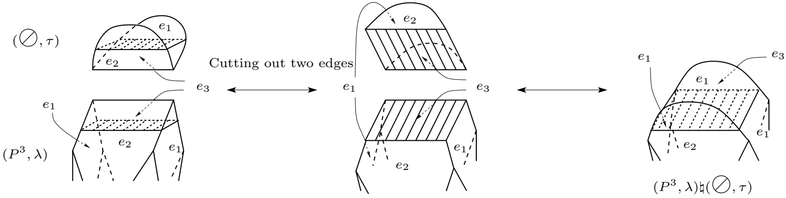
Operation $\sharp^e$ on $\mathcal{P}$



Operation $\sharp^{eve}$ on $\mathcal{P}$

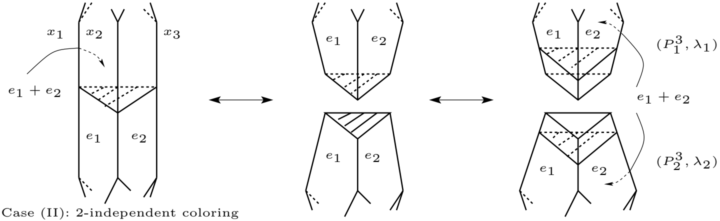


Operation $\natural$ on $\mathcal{P}$



Note that two neighboring facets marked by e_2 and e_3 are needed to be **big**.

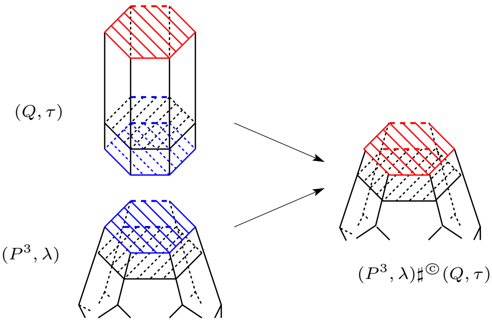
Operation $\sharp^\triangle$ on $\mathcal{P}$



Case (II): 2-independent coloring

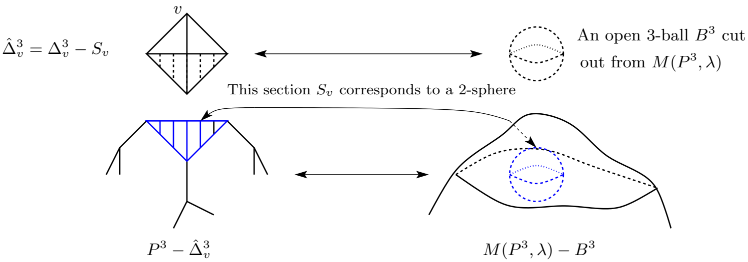
$(P^3, \lambda) = (P_1^3, \lambda_1)\sharp^\triangle(P_2^3, \lambda_2)$

Operation $\sharp^{©}$ on $\mathcal{P}$



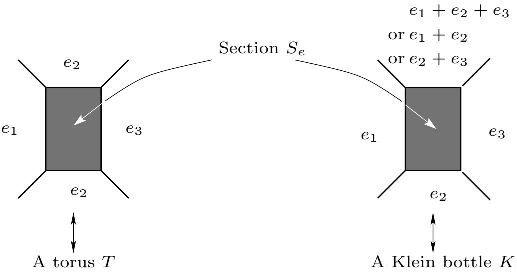
Section corresponding to operation $\sharp^v$

Cutting out a vertex gives a triangular section S_v, which corresponds to a 2-sphere.



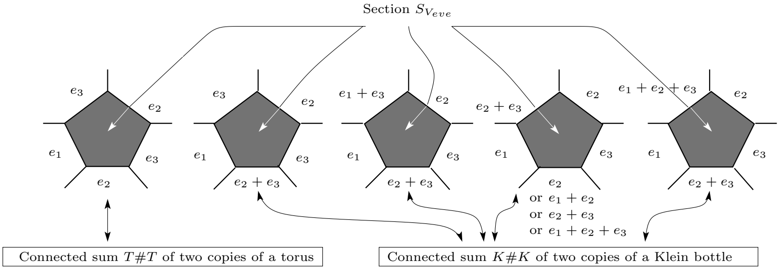
Sections corresponding to operations $\sharp^e$ and $\natural$

Cutting out an edge gives a 4-polygon section S_e, which corresponds to a 2-dimensional torus T or a Klein bottle K



Sections corresponding to operation $\sharp^{eve}$

Cutting out a V_{eve} gives a 5-polygon section $S_{V_{eve}}$, which corresponds to a $T\#T$ or a $K\#K$

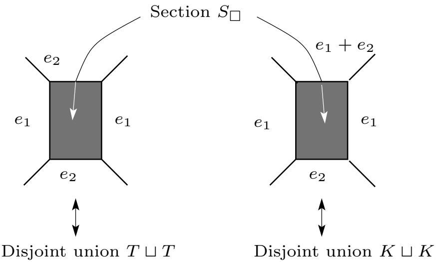


Sections corresponding to operations $\sharp^{\triangle}$ and $\sharp^{\copyright}$

Cutting out a 2-independent triangular facet gives a triangular section $S_{\triangle}$, which corresponds to a disjoint union $\mathbb{R}P^2 \sqcup \mathbb{R}P^2$

Sections corresponding to operation $\sharp^{©}$

Cutting out a 2-independent 4-polygon facet gives a 5-polygon section $S_\square$, which corresponds to a $T \sqcup T$ or a $K \sqcup K$



Sections corresponding to operation $\sharp^{©}$

Cutting out a 2-independent 5-polygon facet gives a 5-polygon section $S_{\maltese}$, which corresponds to a disjoint union $(\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2) \sqcup (\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2)$

Compatibility of six operations with disjoint union

Let $[M(P_1^3,\lambda_1)]$ and $[M(P_2^3,\lambda_2)]$ be two classes in $\widehat{\mathcal{M}}$. Then

$$[M(P_1^3,\lambda_1)\widetilde{\sharp^v}M(P_2^3,\lambda_2)] = [M(P_1^3,\lambda_1)] + [M(P_2^3,\lambda_2)]$$

$$[M(P^3,\lambda)\widetilde{\sharp^e}M(P^3(3),\tau)] = [M(P^3,\lambda)] + [M(P^3(3),\tau)]$$

$$[M(P^3,\lambda)\widetilde{\sharp^{eve}}M(P_-^3(3),\tau)] = [M(P^3,\lambda)] + [M(P_-^3(3),\tau)]$$

$$[M(P^3,\lambda)\widetilde{\natural}M(\oslash,\tau)] = [M(P^3,\lambda)]$$

$$[M(P^3,\lambda)\widetilde{\sharp^{©}}M(P^3(i),\tau)] = [M(P^3,\lambda)] + [M(P^3(i),\tau)], i = 3,4,5.$$

$$[M(P_1^3,\lambda_1)]\widetilde{\sharp^{\triangle}}[M(P_2^3,\lambda_2)]$$

$$= \begin{cases} [M(P_1^3,\lambda_1)] + [M(P_2^3,\lambda_2)] \\ \text{or } [M(P_1^3,\lambda_1)] + [M(P_2^3,\lambda_2)] + [M(P^3(3),\lambda_1\sharp^{\triangle}\lambda_2)]. \end{cases}$$