Homotopy minimal periods for maps on the 3-nilmanifolds

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Outline

Abstract

One of the natural problems in dynamical systems is the study of the existence of periodic points of least period exactly n. Homotopically, a new concept, namely homotopy minimal periods,

$$\mathrm{HPer}(f) =: \bigcap_{g\simeq f} \{m \mid g^m(x) = x, g^q(x) \neq x, q < m \text{ for some } x \in X\}$$

was introduced by Alsedà, Baldwin, Llibre, Swanson and Szlenk in 1995. Since the homotopy minimal period is preserved under a small perturbation of a self-map f on a manifold X, the set $\mathrm{HPer}(f)$ of homotopy minimal periods of f describes the rigid part of dynamics of f.
In this talk, we illustrate a complete description of the homotopy minimal periods for all maps on the 3-dimensional nilmanifolds.

Given a self map $f : X \to X$. What can we say about:

$$\begin{aligned} Fix(f) &:= \{x \in X \mid x = f(x)\}, \\ P_n(f) &:= \{x \in X \mid x = f^n(x), \text{ but } x \neq f^k(x) \text{ for any } k < n\} \\ &= Fix(f^n) - \textstyle\bigcup_{k<n} Fix(f^k), \\ Per(f) &:= \{n \in N \mid P_n(f) \neq \emptyset\}. \end{aligned}$$

These are too sensitive. We seek for some stable characters for f. The set of homotopy minimal periods of f:

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where g ranges over all maps homotopic to f.

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Fixed point classes

Let $p\colon \tilde{X} \to X$ be the universal covering of X. Then

$$\mathrm{Fix}(f) = \cup_{\tilde{f}}\, p(\mathrm{Fix}(\tilde{f})),$$

where $\tilde{f}$ ranges over all liftings of f.

Proposition

For any two liftings $\tilde{f}$ and $\tilde{f}'$ of f,

- $p(\mathrm{Fix}(\tilde{f})) = p(\mathrm{Fix}(\tilde{f}'))$, *if* $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$ *for some* $\gamma \in \mathcal{D}(\tilde{X})$,
- $p(\mathrm{Fix}(\tilde{f})) \cap p(\mathrm{Fix}(\tilde{f}')) = \emptyset$, *otherwise.*

The subset $p(Fix(\tilde{f}))$ of fixed point set of f is said to be the fixed point class determined by $\tilde{f}$.

Nielsen number

Each fixed point class F has a well-defined index: $ind(f, F)$. The sum of the indices of all fixed point classes of f is just the Lefschetz number $L(f)$.

- $N(f)$: number of fixed point classes with non-zero indices,
- $R(f)$: number of fixed point classes.

Both are homotopy invariant.

Proposition

Any fixed point classes with non-zero index is non-empty. Any map homotopic to f has at least $N(f)$ fixed points.

Detect periodic point using Nielsen number

Proposition

Let $k|n$. Then any fixed point class of f^k is contained in the unique fixed point class of f^n.
If a fixed point class F of f^n does not contain any fixed point class of f^k with $k|n$, then $F \subset P_n(f)$.

Theorem

(Key Theorem [Alsedà L, Baldwin S, Llibre J (et. al) 1995]) If

$$\sum_{\frac{n}{k}:prime} R(f^k) < N(f^n),$$

then any self map homotopic to f has a periodic point of least period n, i.e. $n \in Hper(f)$.

An example

Let $f_d : S^1 \to S^1$ be a map on circle S^1 , which is defined by $f_d(e^{\theta i}) = (e^{d\theta i})$ (d is an integer).
Thus,

$$Fix(f_d) = \begin{cases} \{\frac{2m\pi i}{d-1} : i = 1, 2, \ldots, |d-1|\} & \text{if } d \neq 1, \\ S^1 & \text{if } d = 1, \end{cases}$$

In fact, if $d \neq 1$, then

$$R(f_d) = N(f_d) = |d-1|$$

If $|d| > 2$, then $\sum_{\frac{n}{k}:\text{prime}} R(f_d^k) < N(f_d^n) = |d^n - 1|$ for all n, thus $HPer(f_d) = N$.

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If $|d| > 2$, then $\sum_{\frac{n}{k}:\text{prime}} R(f_d^k) < N(f_d^n) = |d^n - 1|$ for all n, thus $HPer(f_d) = \mathbb{N}$.

How to prove $n \notin HPer(f)$

We have to construct a map g homotopic to f such that $P_n(g) = \emptyset$, .i.e. $n \notin Per(g)$.

3-dimensional nilmanifolds

A nilmanifold Γ/G,

- G: simply connected nilpotent Lie group,
- Γ: uniform lattices of G.

The unique 3-dimensional non-abelian connected simply connected nilpotent Lie group is the 3-dimensional Heisenberg group:

$$G = \left\{ \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \middle| x, y, z \in R \right\}.$$

Every uniform lattice of G is isomorphic to some Γ_k:

$$\Gamma_k = \left\{ \begin{bmatrix} 1 & n & \frac{\ell}{k} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \middle| m, n, \ell \in Z \right\}.$$

Homomorphisms on the lattices

Letting

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad z = \begin{bmatrix} 1 & 0 & -\frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Any homomorphism $\varphi : \Gamma_k \to \Gamma_k$ on the lattice of G is given as follows:

$$\varphi(x) = x^\alpha y^\gamma z^\mu, \ \varphi(y) = x^\beta y^\delta z^\nu, \ \varphi(z) = z^{\alpha\delta-\beta\gamma} \tag{1}$$

where $\alpha, \beta, \gamma, \delta, \mu, \nu \in F$.

Classification Theorem of the maps

Since any nilmanifold is $K(\pi, 1)$, map homotopy class is totally determined by the induced homomorphism in $\pi_(\Gamma_k \backslash G) = \Gamma_k$.

Theorem

([Jezierski J, Marzantowicz W 2003]) Any self map on $\Gamma_k \backslash G$ is homotopic to a map of the form $\varphi^{\Gamma_k}_{\alpha,\beta,\gamma,\delta,\mu,\nu} : \Gamma_k \backslash G \to \Gamma_k \backslash G$. Two maps $\varphi^{\Gamma_k}_{\alpha,\beta,\gamma,\delta,\mu,\nu}$ and $\varphi^{\Gamma_k}_{\alpha',\beta',\gamma',\delta',\mu',\nu'}$ on $\Gamma_k \backslash G$ are homotopic to each other if and only if $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, $\delta = \delta'$, and

$$\begin{bmatrix} \mu & \nu \end{bmatrix} - \begin{bmatrix} \mu' & \nu' \end{bmatrix} \in k \cdot \mathrm{Im} \left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right).$$

Main results

$\left(\det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \alpha+\delta\right)$	$\mathrm{HPer}(\varphi^{\Gamma_k}_{\alpha,\beta,\gamma,\delta,\mu,\nu})$
$(1,\ell)$	$\emptyset$
$(\ell,\ell+1)$	$\emptyset$
$(0,0),(0,-1)$	$\{1\}$
$(0,-2),(-2,0),(-2,2)$	$\mathbb{N}\setminus\{2\}$
$(-1,\ell)$ with $\ell\neq 0$	$\mathbb{N}\setminus 2\mathbb{N}$
$(\ell,-\ell-1)$ with $\ell\neq 0,\pm 1$	$\mathbb{N}\setminus 2\mathbb{N}$
Otherwise	$\mathbb{N}$

Our tools

1. ([Anosov D V 1985])

$$N(\varphi^{\Gamma_k}_{\alpha,\beta,\gamma,\delta,\mu,\nu}) = |1 - \det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}||\det(I - \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix})|$$

2. ([Jiang B J, Llibre J 1988], [Jezierski J 2002]) For any map f on nilmanifolds,

$$\mathrm{HPer}(f) = \{n \mid N(f^n) \neq 0, N(f^n) \neq N(f^{\frac{n}{q}}) \text{ for all prime } q \mid n, q \leq n\}$$

This topic begins with L. Block's work in seeking for an analogy of the Sarkovskii-like theorem in circle case.

[1] L. Block, Periods of periodic points of maps of the circle which have a fixed point. Proc. Amer. Math. Soc. 82 (1981), no. 3, 481–486.

[2] Block, Louis; Guckenheimer, John; Misiurewicz, Michał; Young, Lai Sang Periodic points and topological entropy of one-dimensional maps. Global theory of dynamical systems pp. 18–34, Lecture Notes in Math., 819, Springer, Berlin, 1980.

Later, the *n*-torus case was in consideration, a complete description for *HPer* was obtained in 2-torus.

[3] L. Alsedà, S. Baldwin, J. Llibre, R. Swanson and W. Szlenk, Minimal sets of periods for torus maps via Nielsen numbers, Pacific J. Math., **169** (1995), 1–32.

A description for *HPer* in 3-torus can be found in

[4] Jiang, Boju; Llibre, Jaume Minimal sets of periods for torus maps. Discrete Contin. Dynam. Systems 4 (1998), no. 2, 301–320.

Recent works:

[5] Jezierski, Jerzy; Marzantowicz, Wacław Homotopy minimal periods for maps of three-dimensional nilmanifolds. Pacific J. Math. 209 (2003), no. 1, 85–101.
[6] Jezierski, Jerzy; Kędra, Jarosław; Marzantowicz, Wacław Homotopy minimal periods for NR-solvmanifolds maps. Topology Appl. 144 (2004), no. 1-3, 29–49.
[7] Lee, Jong Bum; Zhao, Xuezhi Homotopy minimal periods for expanding maps on infra-nilmanifolds, Journal of the mathematical society of Japan, 59 (2007), No.1, 179 – 184.
[8] —-, Nielsen type numbers and homotopy minimal periods for maps on the 3-nilmanifolds, Sci. in China A, to appear.