Polynomial splittings of metabelian von Neumann rho–invariants of knots

Se-Goo Kim

Kyung Hee University, Korea

sgkim@khu.ac.kr

joint with

Taehee Kim

Konkuk University, Korea

January 21, 2008

The Fourth East Asian School of Knots and Related Topics

The University of Tokyo, Japan

Knot concordance

A knot K in S^3 is *slice* if $K = \partial D$ for a locally flat 2-disk D in B^4. Note that $S^3 = \partial B^4$.

Two knots K_1 and K_2 are *concordant* if $K_1 \# - K_2$ is slice. Here $-K$ denotes the mirror image of K with reversed orientation.

The *knot concordance group* $\mathcal{C}$:

Concordance is an equivalence relation and the concordance classes form an abelian group $\mathcal{C}$ under $\#$ operation.

Cochran, Orr, Teichner (1999) found a filtration of $\mathcal{C}$:

$$\cdots \subset \mathcal{F}_{(n.5)} \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(1.5)} \subset \mathcal{F}_{(1)} \subset \mathcal{F}_{(0.5)} \subset \mathcal{F}_{(0)} \subset \mathcal{C}$$

$\mathcal{F}_{(m)}$ = the set of m-solvable knots which is a subgroup of $\mathcal{C}$.

m-solvable knots

M_K = zero surgery on K in S^3.

For a group G, let $G^{(0)} = G$ and $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$, inductively.

Definition. K is called *n-solvable* if M_K bounds a (spin) 4-manifold W such that

- The inclusion $M_K \to W$ induces an isomorphism on H_1.
- The intersection form on $H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$ has a dual pair of self-annihilating submodules L_1 and L_2.
- The images of L_1 and L_2 in $H_2(W)$ generate $H_2(W)$.

Here, $\pi = \pi_1(W)$ and $H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$ is $H_2(\tilde{W}^{(n)})$ for $\pi/\pi^{(n)}$ cover $\tilde{W}^{(n)}$ of W. W is called an *n-solution* for K.

K is $(n.5)$-*solvable* if W admits $(n+1)$ self-annihilating submodule with (n)-dual.

Theorem 1 (COT 1999) *Slice knots are m-solvable and* $\mathcal{F}_{(m)} < \mathcal{F}_{(m.5)} < \mathcal{C}$ *for all* $m \in \frac{1}{2}\mathbb{Z}$.

K *is* 0*-solvable* $\Longleftrightarrow$ K *has vanishing Arf invariants.*

K *is* 0.5*-solvable* $\Longleftrightarrow$ K *is algebraically slice.*

All known metabelian concordance invariants for 1.5-solvable knots vanish.

Abelian concordance invariants:

Levine's Seifert form invariants.

Matabelian concordance invariants:

Casson-Gordon invariants, Casson-Gordon-Gilmer invariants
Kirk-Livingston's twisted Alexander invariants
Letsche's invariants, etc.

von Neumann ρ-invariants

Let M be a closed 3-manifold and Γ a group.

Given a homomorphism $\phi : \pi_1(M) \to \Gamma$, whenever $(M, \phi) = \partial(W, \varphi)$ for some compact oriented 4-manifold W and $\varphi : \pi_1(W) \to \Gamma$,

$$\rho(M, \phi) = \sigma^{(2)}_{\Gamma}(W, \phi) - \sigma_0(W) \in \mathbb{R}$$

where $\sigma^{(2)}_{\Gamma}$ is the von Neumann signature of the intersection form on $H_2(W; \mathbb{Z}\Gamma)$ and σ_0 is the ordinary signature. (Cheeger-Gromov 85)

For an n-solvable knot K, take $M = M_K$ and W an n-solution to get ρ an invariant of K.

Using ρ–invarinats, COT proved:

Theorem 2 (COT 1999 $(n = 2)$, Cochran–Teichner 2004) *$\mathcal{F}_{(n)}/\mathcal{F}_{(n.5)}$ has elements of infinite order.*

Metabelian von Neumann ρ-invariants

Let $\Lambda = \mathbb{Q}[t^{\pm 1}]$ and $\Gamma = \mathbb{Q}(t)/\Lambda \rtimes \mathbb{Z}$.

Let $B\ell : H_1(M_K;\Lambda) \otimes H_1(M_K;\Lambda) \to \mathbb{Q}(t)/\Lambda$ be the Blanchfield pairing of K.

A submodule P of $H_1(M_K;\Lambda)$ is said to be *self-annihilating* if $P = P^\perp$ with respect to $B\ell$.

Given $x \in H_1(M_K;\Lambda)$, define $\phi_x : \pi_1(M_K) \to \Gamma$ by

$$\phi_x(a) = (B\ell(x, a\mu^{-\epsilon(a)}), \epsilon(a)) \quad \text{for } a \in \pi_1(M_K),$$

where μ is a meridian of K and $\epsilon : \pi_1(M_K) \to \mathbb{Z}$ is the abelianization.

Theorem 3 (COT 1999) *If K is 1.5-solvable, there exists a self-annihilating submodule P of $H_1(M_K;\Lambda)$ such that $\rho(K, \phi_x) = 0$ for all $x \in P$.*

Polynomial splittings

$\Delta_K(t) = \det(A - tA^t) =$ Alexander polynomial of K.

Theorem 4 (Levine 1969) *Suppose that K_1 and K_2 have coprime Alexander polynomials. If $K_1 \# K_2$ is algebraically slice, so are both K_1 and K_2.*

Theorem 5 (S. Kim 2001) *Suppose that K_1 and K_2 have coprime Alexander polynomials in* integer coefficients. *If $K_1 \# K_2$ have vanishing Casson-Gordon-Gilmer invariants, so do both K_1 and K_2.*

COT used von Neumann ρ-invariants to detect n-solvable knots not being $n.5$-solvable.

Q. Do splitting properties hold for ρ-invariants?

We find a positive answer for ρ–invariants associated with certain metabelian representations.

Splitting of metabelian ρ–invariants

We say K has *vanishing ρ–invariants* if there is a self-annihilating submodule P of $H_1(M_K;\Lambda)$ such that $\rho(K,\phi_x) = 0$ for all $x \in P$.

Theorem 6 (Main Theorem) *Suppose K_1 and K_2 have coprime Alexander polynomials. If $K_1 \# K_2$ has vanishing ρ–invariants, so do both K_1 and K_2.*

Application:

Theorem 7 (T. Kim 2002) *There are infinitely many knots having nonvanishing ρ–invariants but vanishing Casson–Gordon invariants with coprime Alexander polynomials.*

Corollary 8 *The knots in the previous theorem are all linearly independent in $\mathcal{F}_{(1)}/\mathcal{F}_{(1.5)}$.*

This gives a new method of determining linear independence of knots in $\mathcal{C}$.

Sketch of proof

Let P be a self-annihilating submodule in $H_1(M_{K_1 \# K_2}; \Lambda)$ for which ρ vanish. Note $H_1(M_{K_1 \# K_2}; \Lambda) = H_1(M_{K_1}; \Lambda) \oplus H_1(M_{K_2}; \Lambda)$.

The projection of P into $H_1(M_{K_i}; \Lambda)$, P_i, is a self-annihilating submodule for which ρ vanish.

That is, we can show

1. $P = P_1 \oplus P_2$, which follows from action of Alexander polynomial on $H_1(M_K; \Lambda)$ and coprimeness of Alexander polynomials.
2. Each P_i is self-annihilating, which follows from nonsigularities of Blanchfield pairings.
3. For $x \in P_i$, $\rho(K_1, \phi_x) = 0$, which follows from the splitting property of ρ under connected sum.

We are working on the following question:

Q. Do similar splitting properties hold in all $\mathcal{F}_{(m)}$ for $m > 1$?

We have to use "coprimeness" of higher–order Alexander polynomials corresponding to $\mathcal{F}_{(n)}$.

Thank you very much.