Lens spaces and toroidal Dehn fillings

January 23, 2008

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Δ(**LENS**, **TOR**) $\leq$?

M : a hyperbolic 3-manifold with ∂M a torus

$M(\alpha)$: a lens space

$M(\beta)$: a toroidal manifold

$\Delta(\alpha, \beta) \leq$?

Known Results

Theorem (Gordon). *Let M be a hyperbolic 3-manifold with ∂M a torus. If α, β are slopes on ∂M such that $M(\alpha)$ is a lens space and $M(\beta)$ contains an incompressible torus, then $\Delta(\alpha, \beta) \leq 5$.*
(Δ(LENS, TOR) ≤ 5)

Theorem (Teragaito). *Let $M, M(\alpha), M(\beta)$ be as above. Suppose that $M(\alpha)$ contains a Klein bottle. Then $\Delta(\alpha, \beta) \leq 4$.*
($\Delta(L(4n, 2n \pm 1)$, TOR) ≤ 4)

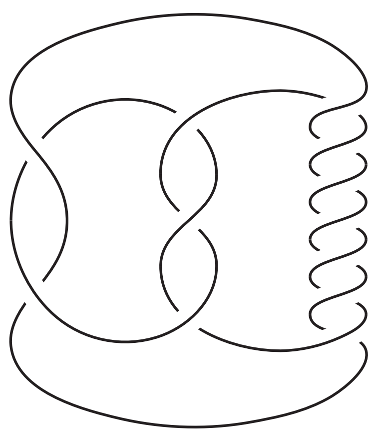
Theorem (Lee). *Let* $M, M(\alpha), M(\beta)$ *be as in Gordon's Theorem. Then* $\Delta(\alpha, \beta) \leq 4$*.*

Maximal observed distance: $\Delta(\text{LENS}, \text{TOR}) \leq 3$.

$(-2, 3, 7)$-pretzel knot and exceptional surgery slopes

Exceptional slopes : $16, 17, 18, 37/2, 19, 20, 1/0$

- S^3 : $1/0$
- Lens space : $18, 19 \dashrightarrow K(18) \cong L(18, 5), \quad K(19) \cong L(19, 7)$
- Small Seifert fiber space : 17
- Toroidal manifold : $16, 37/2, 20$



A sketch of the proof of the theorem

Assume for contradiction that $\Delta(\alpha, \beta) = 5$.

$M(\alpha) = M \cup V_\alpha$: Lens space
$M(\beta) = M \cup V_\beta$: Toroidal manifold

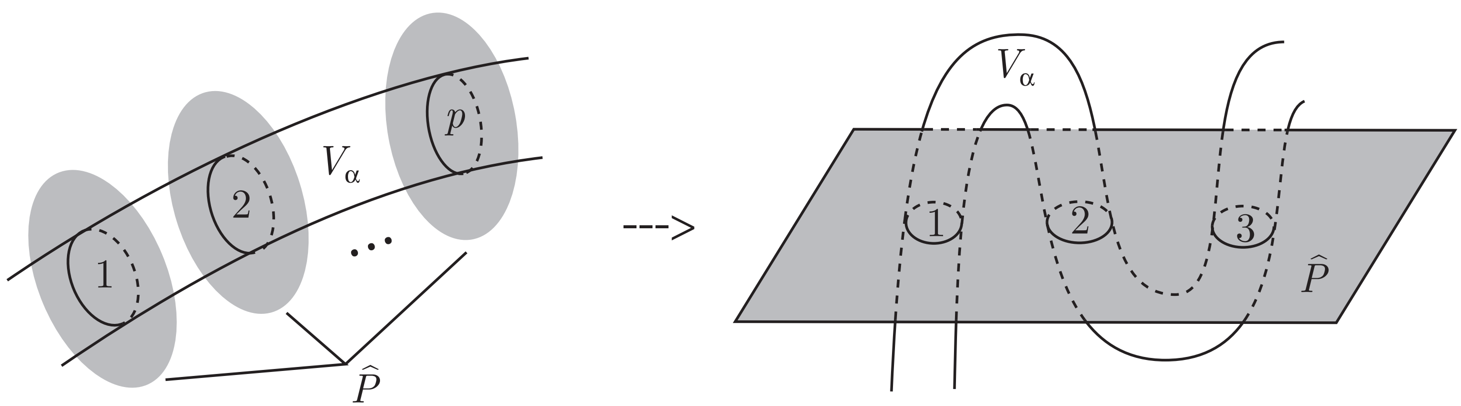
$\widehat{P} \subset M(\alpha)$: a Heegaard surface
$\widehat{T} \subset M(\beta)$: an incompressible torus

We may assume

$\widehat{P} \cap V_\alpha = u_1 \cup \ldots \cup u_p$: meridian disks of V_α

$\widehat{T} \cap V_\beta = v_1 \cup \ldots \cup v_t$: meridian disks of V_β

(These meridian disks are numbered successively along V_α or V_β.)

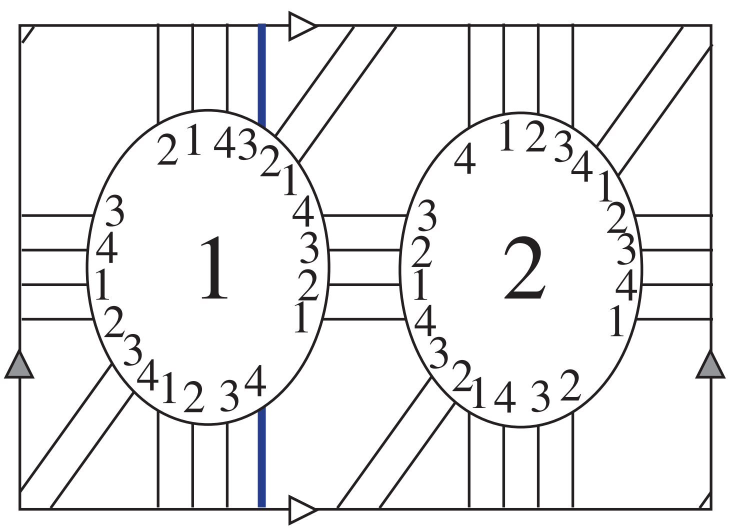


$\widehat{T}$ is chosen so that t is minimal.

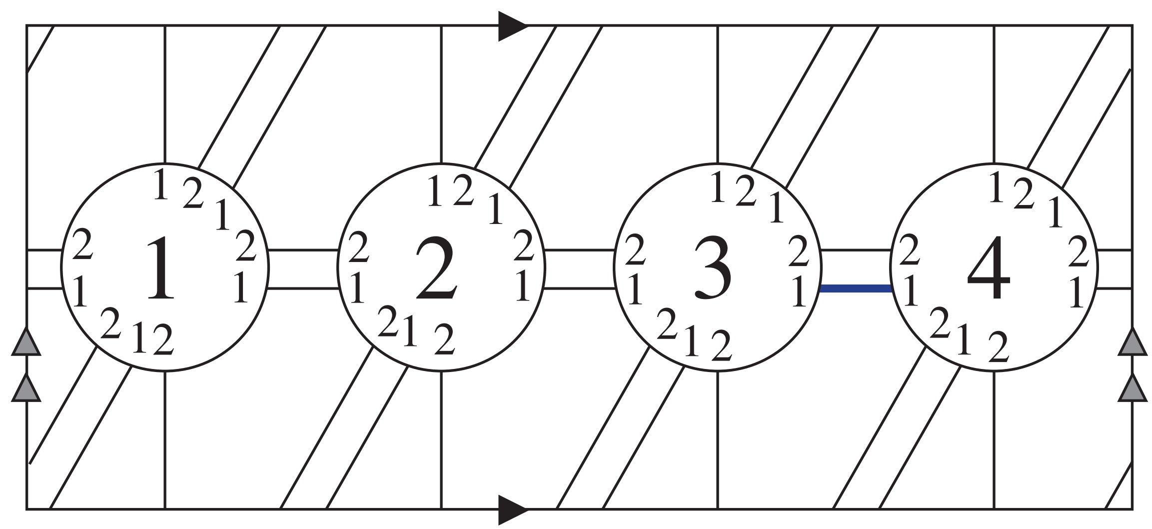
Let $P = \widehat{P} \cap M$ and $T = \widehat{T} \cap M$.
Gordon showed that $\widehat{P}$ can be chosen so that the following conditions are satisfied.

- $P \pitchfork T$ and each component of ∂P meets each component of ∂T in $\Delta(\alpha, \beta) = 5$ points;
- no circle component of $P \cap T$ bounds a disk in P (and in T);
- no arc component of $P \cap T$ is ∂-parallel in P or T.

The arc components of $P \cap T$ define two labelled graphs G_P and G_T.

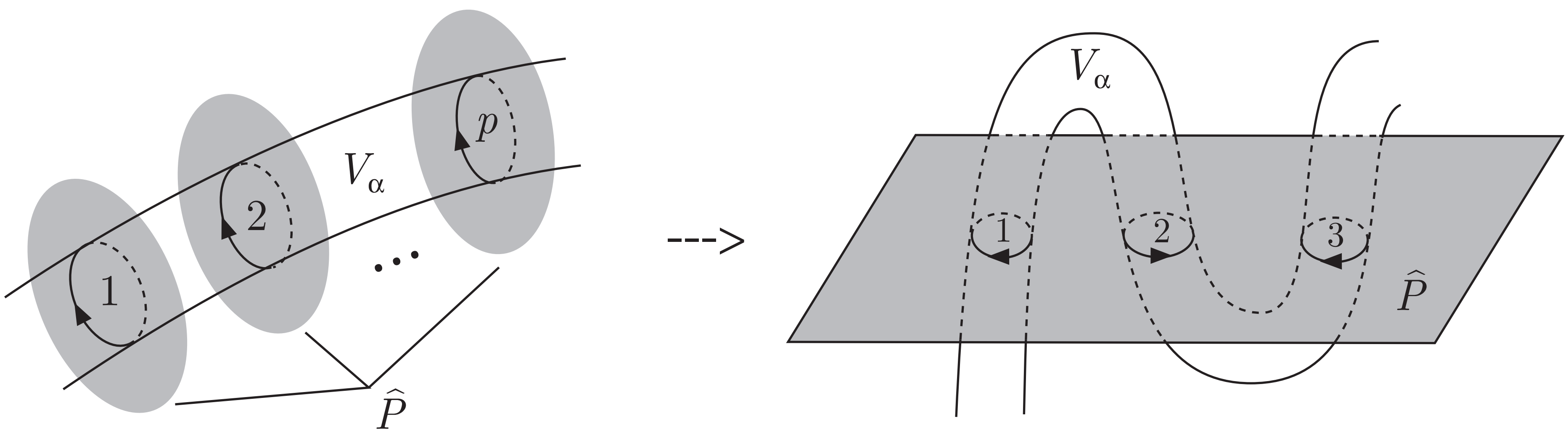


$\mathbf{G_P}$

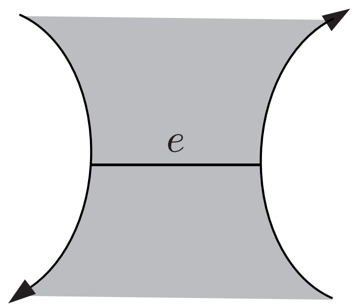


$\mathbf{G_T}$

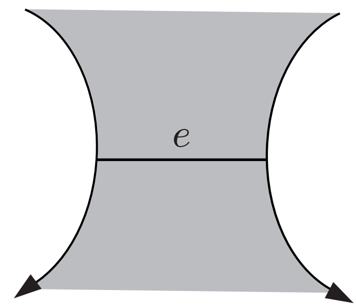
Orient ∂P so that all components of ∂P are homologous in $\partial V_\alpha = T \subset \partial M$.



Give a sign to each edge of G_P.



positive



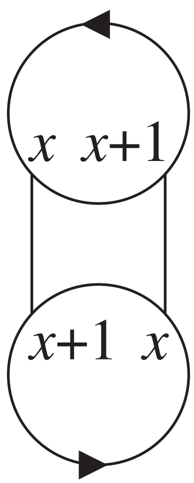
negative

Similarly for G_T.

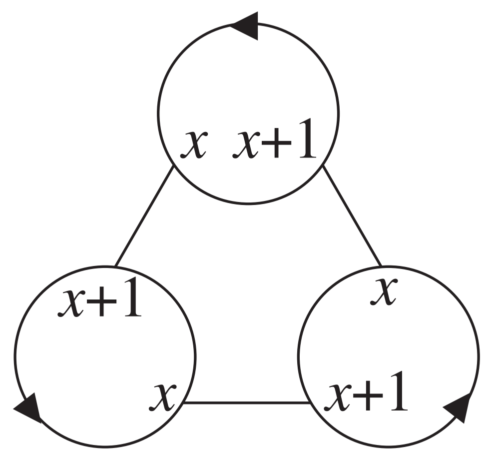
Parity Rule

An edge is positive in one graph if and only if it is negative in the other.

Scharlemann cycles



Length 2



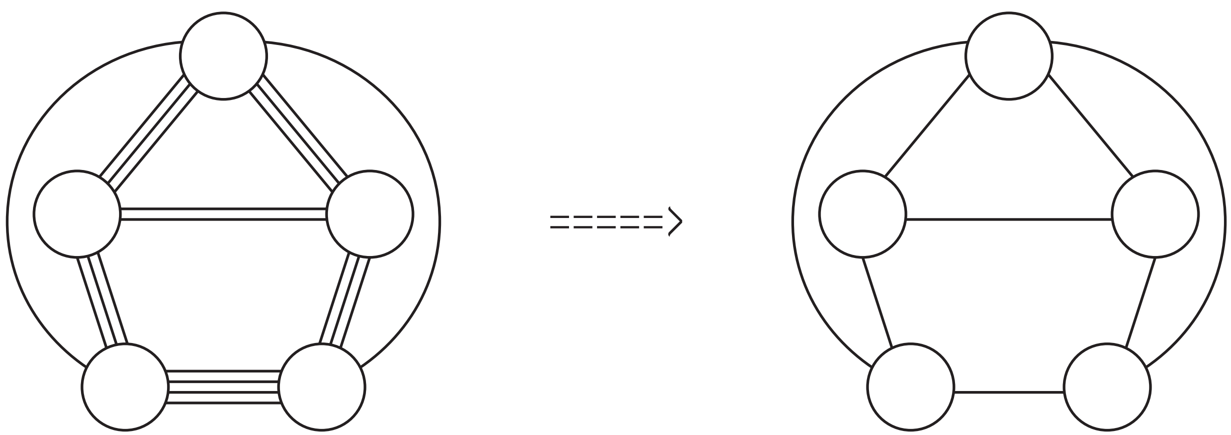
Length 3

Lemma. *(1) Any family of parallel* **positive** *edges in* G_P *contains at most* $\frac{t}{2}+1$ *edges.*
(2) Any family of parallel **negative** *edges in* G_P *contains at most* t *edges.*

Lemma. *Each vertex of* G_P *has at most* $\frac{5t}{2}$ **negative** *edge endpoints.*

Reduced graphs

Let G be G_P or G_T.
Let $\overline{G}$ denote the reduced graph of G, i.e., $\overline{G}$ is obtained from G by amalgamating each family of parallel edges into a single edge.



Lemma. *Some vertex of* $\overline{G}_P$ *has valence at most 6.*

Proof. Suppose that all vertices of $\overline{G}_P$ have valence at least 7.
Let V, E, F be the number of vertices, edges, and disk faces of $\overline{G}_P$, respectively. Then

$$V - E + F \geq V - E + \sum_{f:\text{face}} \chi(f) = \chi(\widehat{P}) = 0,$$
$$2E \geq 3F, \text{ and}$$
$$2E \geq 7V.$$

Hence $2E \geq 3F \geq 3(E - V)$, which gives

$$3V \geq E.$$

This is a contradiction. □

We can choose a vertex u_x of $\overline{G}_P$ which has valence at most 6. Let k be the number of positive edges of $\overline{G}_P$ incident to u_x. Then we have

$$5t \leq k(\frac{t}{2} + 1) + (6 - k)t.$$

Solving this, we obtain

$$k \leq \frac{t}{t - (\frac{t}{2} + 1)} = 2 + \frac{4}{t - 2}.$$

If $t \geq 5$, then $k \leq 3$ and there are at least $5t - 3 \cdot (\frac{t}{2} + 1)$ negative edges of G_P incident to u_x. Hence we have

$$5t - 3 \cdot (\frac{t}{2} + 1) \leq \frac{5t}{2},$$

which gives $t \leq 3$. This is a contradiction.

So, $t = 1, 2, 3,$ or 4.

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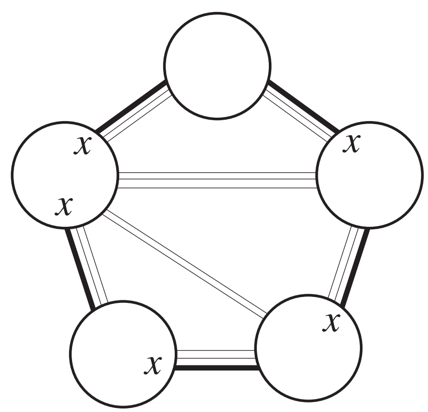
x-faces

x-edge : an edge of G having label x at its one endpoint
$G^+(x)$: the subgraph of G consisting of all positive x-edges
x-face : a disk face of $G^+(x)$



Theorem (Hayashi and Motegi). *Any x-face contains a Scharlemann cycle of G.*

Lemma. *Each vertex of G_P has at most $\frac{5t}{2}$ negative edge endpoints.*

Proof. There are at most four labels of Scharlemann cycles in G_T.

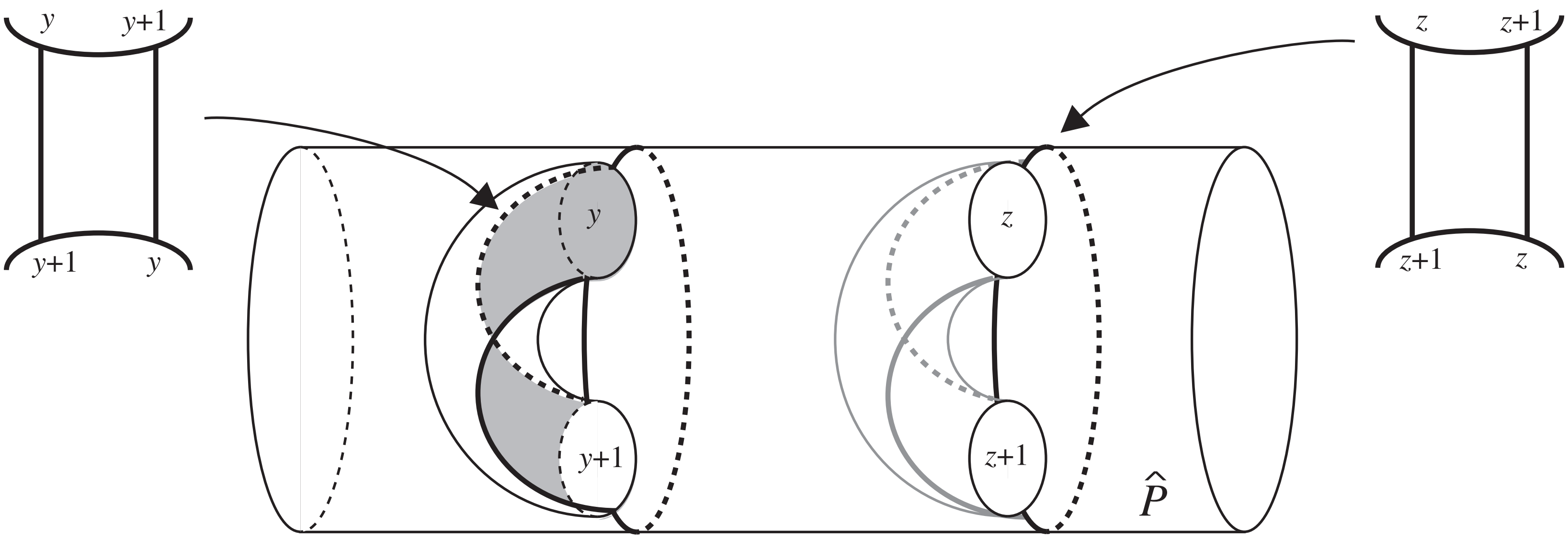
Case(I) Assume that G_T has at most three labels of Scharlemann cycles. ....

Case (II) Assume that G_T has four labels of Scharlemann cycles.

$$\text{Scharlemann cycles} = \begin{cases} (y, y+1)\text{-Scharlemann cycles}, & \text{or} \\ (z, z+1)\text{-Scharlemann cycles}, & \end{cases}$$

where $\{y, y+1\} \cap \{z, z+1\} = \emptyset$.

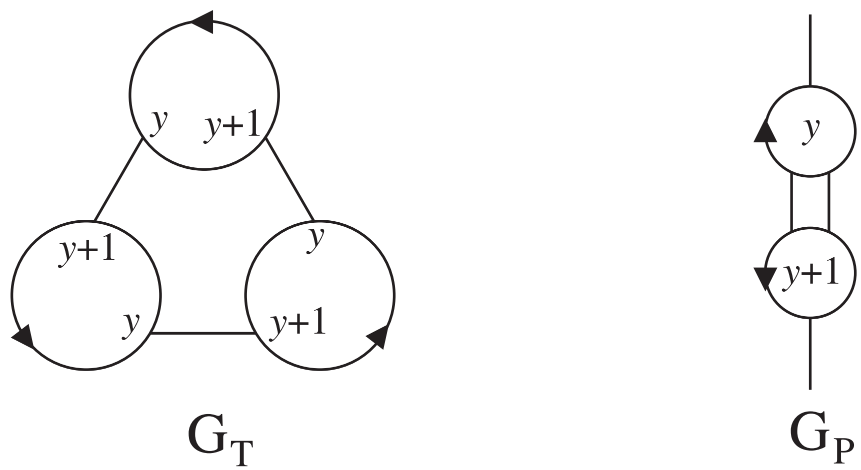
Both kinds of Scharlemann cycles cannot have length 2.



$(\because \Delta(L(4n, 2n \pm 1), \mathsf{TOR}) \leq 4)$

We may assume that $(y, y+1)$-Scharlemann cycles have length at least 3. Then in G_T,

$$\begin{cases} \#\text{ of }(y,y+1)\text{-Scharlemann cycles} \leq \dfrac{t}{2} & \text{and} \\ \#\text{ of }(z,z+1)\text{-Scharlemann cycles} \leq t. \end{cases}$$



Therefore the total number of Scharlemann cycles in $G_T \leq \frac{3t}{2}$.

Now assume : some vertex u_x of G_P has more than $\frac{5t}{2}$ negative edge endpoints.

$--->$ $G_T^+(x)$ has more than $\frac{5t}{2}$ edges.

$--->$ $G_T^+(x)$ has more than $\frac{3t}{2}$ disk faces.

$--->$ There are more than $\frac{3t}{2}$ x-faces in G_T.

$--->$ There are more than $\frac{3t}{2}$ Scharlemann cycles in G_T.

$--->$ A contradiction. $\square$