Lens spaces and toroidal Dehn fillings

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\( \Delta(\text{LENS, TOR}) \leq ? \)

\( M \): a hyperbolic 3-manifold with \( \partial M \) a torus

\( M(\alpha) \): a lens space

\( M(\beta) \): a toroidal manifold

\( \Delta(\alpha, \beta) \leq ? \)
Known Results

**Theorem (Gordon).** Let $M$ be a hyperbolic 3-manifold with $\partial M$ a torus. If $\alpha, \beta$ are slopes on $\partial M$ such that $M(\alpha)$ is a lens space and $M(\beta)$ contains an incompressible torus, then $\Delta(\alpha, \beta) \leq 5$.  
($\Delta(\text{LENS}, \text{TOR}) \leq 5$)

**Theorem (Teragaito).** Let $M, M(\alpha), M(\beta)$ be as above. Suppose that $M(\alpha)$ contains a Klein bottle. Then $\Delta(\alpha, \beta) \leq 4$.  
($\Delta(L(4n, 2n \pm 1), \text{TOR}) \leq 4$)
Theorem (Lee). Let $M, M(\alpha), M(\beta)$ be as in Gordon’s Theorem. Then $\Delta(\alpha, \beta) \leq 4$.

Maximal observed distance: $\Delta(\text{LENS}, \text{TOR}) \leq 3$. 
(−2, 3, 7)-pretzel knot and exceptional surgery slopes

Exceptional slopes: 16, 17, 18, 37/2, 19, 20, 1/0
- $S^3$: 1/0
- Lens space: 18, 19 $\rightarrow K(18) \cong L(18, 5)$, $K(19) \cong L(19, 7)$
- Small Seifert fiber space: 17
- Toroidal manifold: 16, 37/2, 20
A sketch of the proof of the theorem

Assume for contradiction that $\Delta(\alpha, \beta) = 5$.

$M(\alpha) = M \cup V_\alpha$: Lens space
$M(\beta) = M \cup V_\beta$: Toroidal manifold

$\hat{P} \subset M(\alpha)$: a Heegaard surface
$\hat{T} \subset M(\beta)$: an incompressible torus
We may assume
\( \hat{P} \cap V_\alpha = u_1 \cup \ldots \cup u_p \) : meridian disks of \( V_\alpha \)
\( \hat{T} \cap V_\beta = v_1 \cup \ldots \cup v_t \) : meridian disks of \( V_\beta \)
(These meridian disks are numbered successively along \( V_\alpha \) or \( V_\beta \).)

\( \hat{T} \) is chosen so that \( t \) is minimal.
Let $P = \hat{P} \cap M$ and $T = \hat{T} \cap M$.
Gordon showed that $\hat{P}$ can be chosen so that the following conditions are satisfied.

- $P \pitchfork T$ and each component of $\partial P$ meets each component of $\partial T$ in $\Delta(\alpha, \beta) = 5$ points;
- no circle component of $P \cap T$ bounds a disk in $P$ (and in $T$);
- no arc component of $P \cap T$ is $\partial$-parallel in $P$ or $T$. 
The arc components of $P \cap T$ define two labelled graphs $G_P$ and $G_T$. 

$G_P$  

$G_T$
Orient $\partial P$ so that all components of $\partial P$ are homologous in $\partial V_\alpha = T \subset \partial M$. 
Give a sign to each edge of $G_P$.

Similarly for $G_T$.

**Parity Rule**

An edge is positive in one graph if and only if it is negative in the other.
Scharlemann cycles

Length 2

Length 3
Lemma. (1) Any family of parallel positive edges in $G_P$ contains at most $\frac{t}{2} + 1$ edges.

(2) Any family of parallel negative edges in $G_P$ contains at most $t$ edges.

Lemma. Each vertex of $G_P$ has at most $\frac{5t}{2}$ negative edge endpoints.
Reduced graphs

Let $G$ be $G_P$ or $G_T$.
Let $\bar{G}$ denote the reduced graph of $G$, i.e., $\bar{G}$ is obtained from $G$ by amalgamating each family of parallel edges into a single edge.
Lemma. Some vertex of $\overline{G}_P$ has valence at most 6.

Proof. Suppose that all vertices of $\overline{G}_P$ have valence at least 7. Let $V, E, F$ be the number of vertices, edges, and disk faces of $\overline{G}_P$, respectively. Then

$$V - E + F \geq V - E + \sum_{f: \text{face}} \chi(f) = \chi(\hat{P}) = 0,$$

$$2E \geq 3F,$$

and

$$2E \geq 7V.$$

Hence $2E \geq 3F \geq 3(E - V)$, which gives

$$3V \geq E.$$

This is a contradiction. \qed
We can choose a vertex $u_x$ of $\overline{G}_P$ which has valence at most 6. Let $k$ be the number of positive edges of $\overline{G}_P$ incident to $u_x$. Then we have

$$5t \leq k\left(\frac{t}{2} + 1\right) + (6 - k)t.$$  

Solving this, we obtain

$$k \leq \frac{t}{t - \left(\frac{t}{2} + 1\right)} = 2 + \frac{4}{t - 2}.$$  

If $t \geq 5$, then $k \leq 3$ and there are at least $5t - 3 \cdot \left(\frac{t}{2} + 1\right)$ negative edges of $G_P$ incident to $u_x$. Hence we have

$$5t - 3 \cdot \left(\frac{t}{2} + 1\right) \leq \frac{5t}{2},$$  

which gives $t \leq 3$. This is a contradiction.
So, $t = 1, 2, 3, \text{ or } 4.$
**$x$-faces**

$x$-edge: an edge of $G$ having label $x$ at its one endpoint

$G^+(x)$: the subgraph of $G$ consisting of all positive $x$-edges

$x$-face: a disk face of $G^+(x)$

\[ \text{Theorem (Hayashi and Motegi). Any } x\text{-face contains a Scharlemann cycle of } G. \]
Lemma. Each vertex of $G_P$ has at most $\frac{5t}{2}$ negative edge endpoints.

Proof. There are at most four labels of Scharlemann cycles in $G_T$.

Case (I) Assume that $G_T$ has at most three labels of Scharlemann cycles. ..........

Case (II) Assume that $G_T$ has four labels of Scharlemann cycles.

\begin{align*}
\text{Scharlemann cycles} = \begin{cases} (y, y + 1)-\text{Scharlemann cycles}, & \text{or} \\ (z, z + 1)-\text{Scharlemann cycles}, & \end{cases}
\end{align*}

where $\{y, y + 1\} \cap \{z, z + 1\} = \emptyset$. 
Both kinds of Scharlemann cycles cannot have length 2.

\[ \Delta(L(4n, 2n \pm 1), \text{TOR}) \leq 4 \]
We may assume that \((y, y + 1)\)-Scharlemann cycles have length at least 3. Then in \(G_T\),

\[
\begin{cases}
\# \text{ of } (y, y + 1)\text{-Scharlemann cycles} \leq \frac{t}{2} \\
\# \text{ of } (z, z + 1)\text{-Scharlemann cycles} \leq t.
\end{cases}
\]

Therefore the total number of Scharlemann cycles in \(G_T\) \(\leq \frac{3t}{2}\).
Now assume: some vertex $u_x$ of $G_P$ has more than $\frac{5t}{2}$ negative edge endpoints.

\[-\neg\neg\neg \Rightarrow G_T^+(x) \text{ has more than } \frac{5t}{2} \text{ edges.}\]

\[-\neg\neg\neg \Rightarrow G_T^+(x) \text{ has more than } \frac{3t}{2} \text{ disk faces.}\]

\[-\neg\neg\neg \Rightarrow \text{There are more than } \frac{3t}{2} \text{ } x\text{-faces in } G_T.\]

\[-\neg\neg\neg \Rightarrow \text{There are more than } \frac{3t}{2} \text{ Scharlemann cycles in } G_T.\]

\[-\neg\neg\neg \Rightarrow \text{A contradiction.} \quad \Box\]