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Graph braid groups and right angled Artin groups

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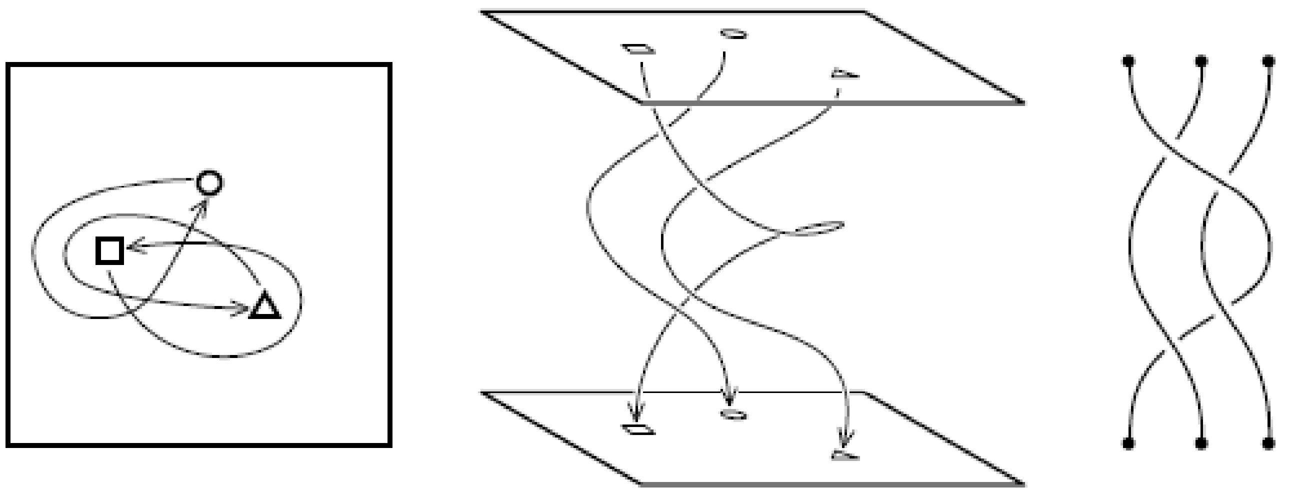
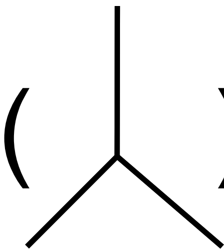
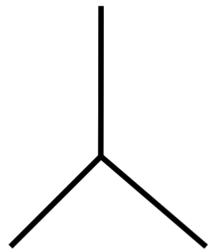
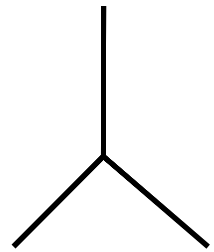
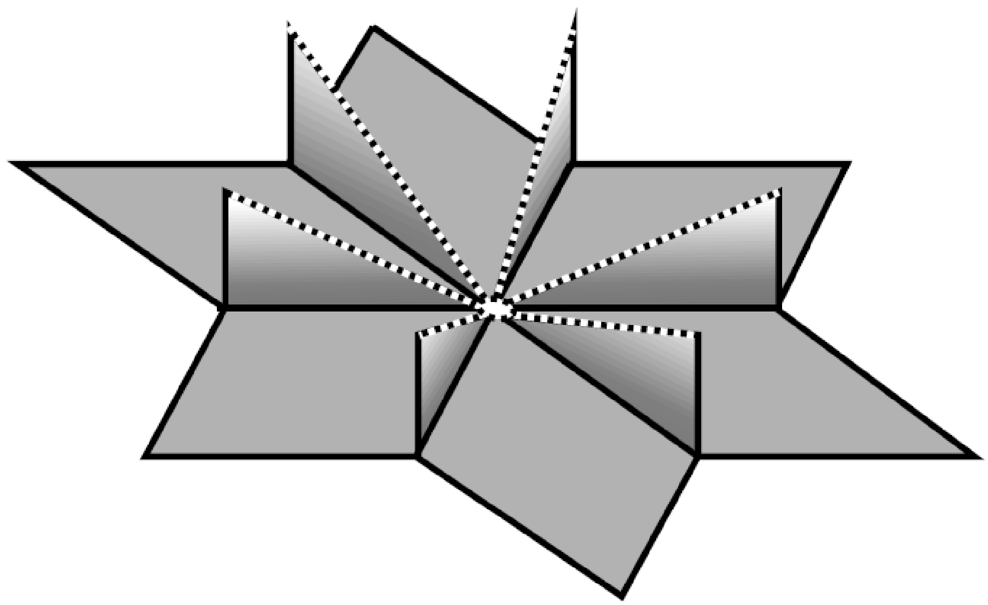
n Robots on $\mathbf{R}^2$ and n-Braids

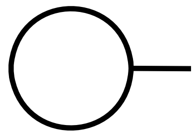
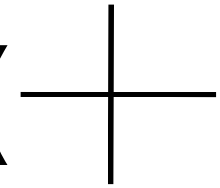


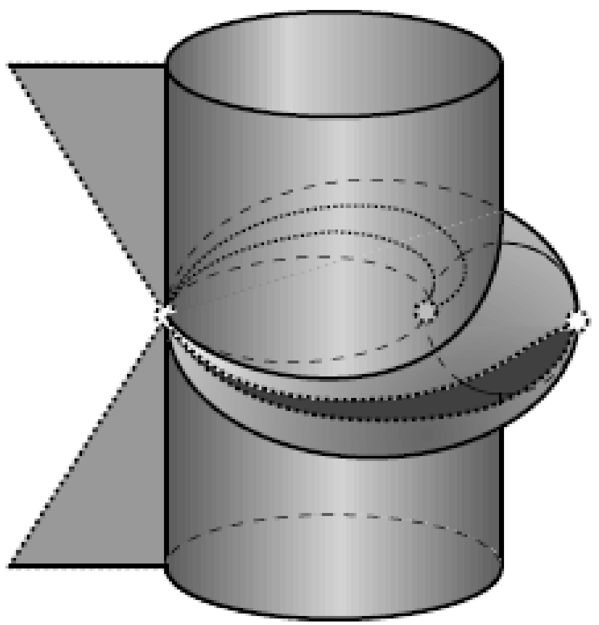
A motion of n robots is a one-parameter family of n-tuples of points in R^2 that are pair-wise distinct, that is, a path (or loop) in $(R^2)^n - \Delta$

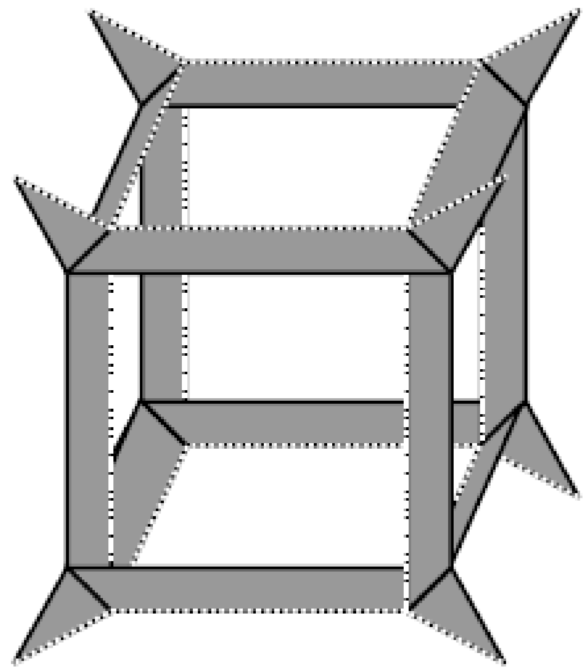
Configuration space of 2 robots

$C_2($  $)$ =  X  $- \ \Delta$

= 

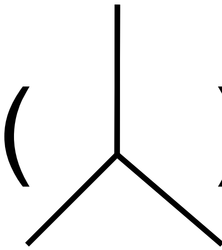
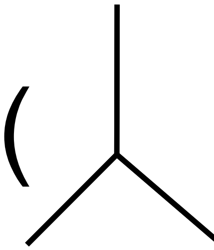
C_2() and C_2()

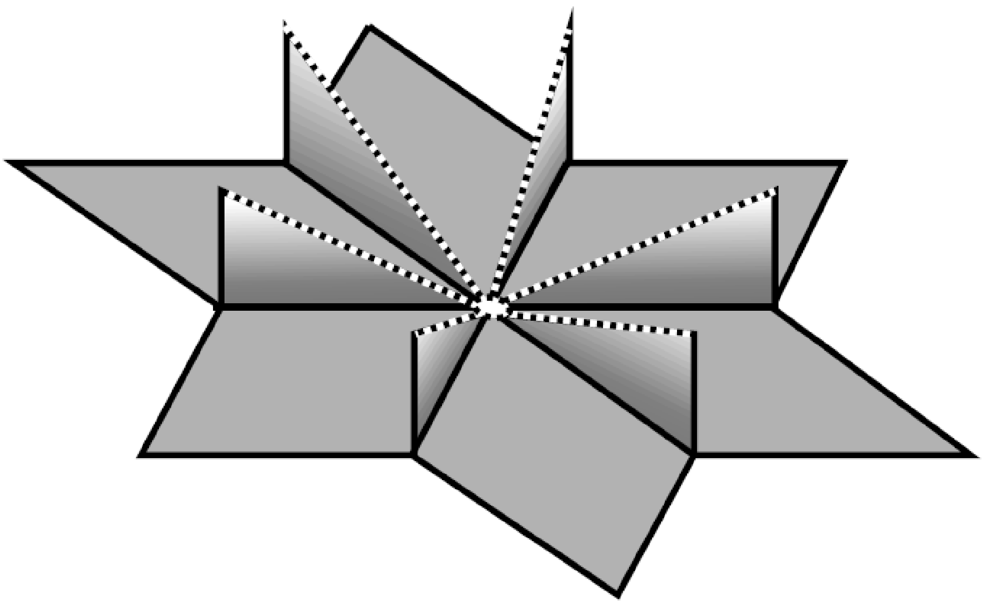


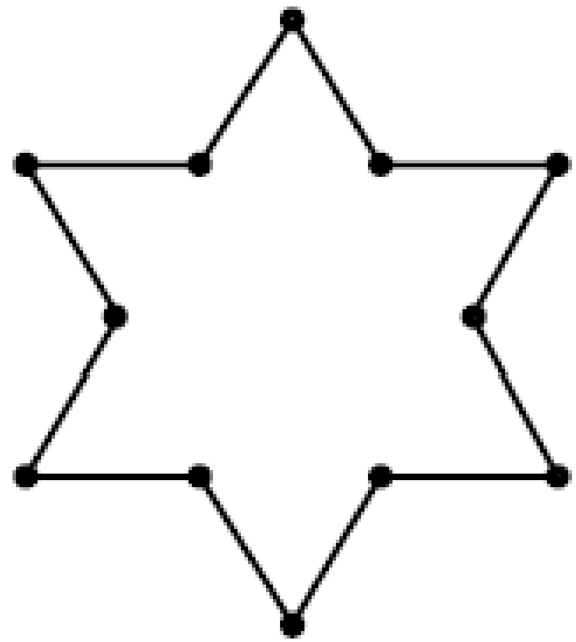


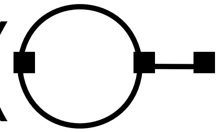
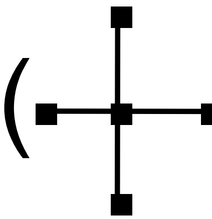
Discrete Configuration Space

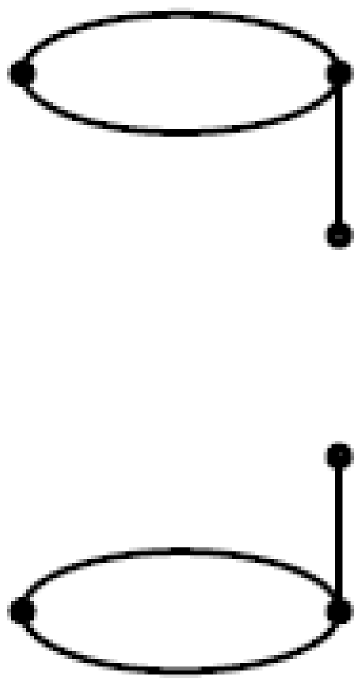
- $C_n(G) = G^n - \Delta$
  = all n-tuples of pair-wise distinct points of G
- Regard G as a 1-dimensional complex
  $D_n(G) = G^n$ – all cells touching Δ
  = all n-fold products of pair-wise disjoint 0- or 1-cells of G
- $D_n(G)$ is a nice space. In fact, it is a cube complex of non-positive curvature in the sense of Gromov.

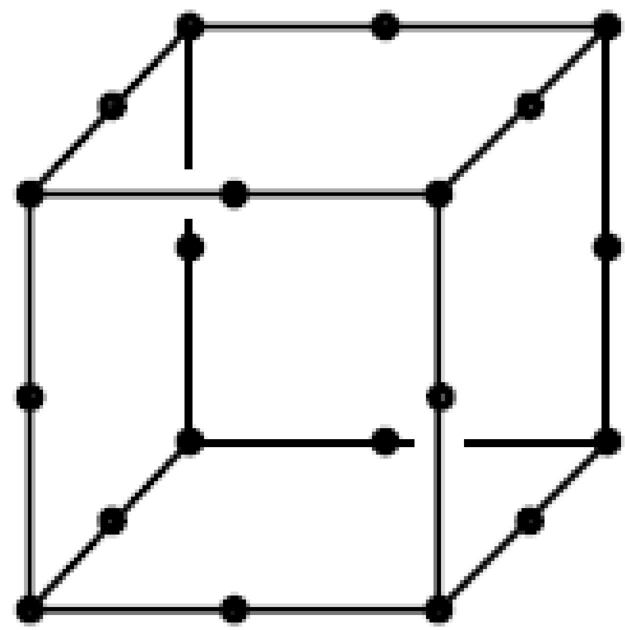
$C_2($  $)$ vs. $D_2($  $)$





$D_2($  $)$ and $D_2($  $)$



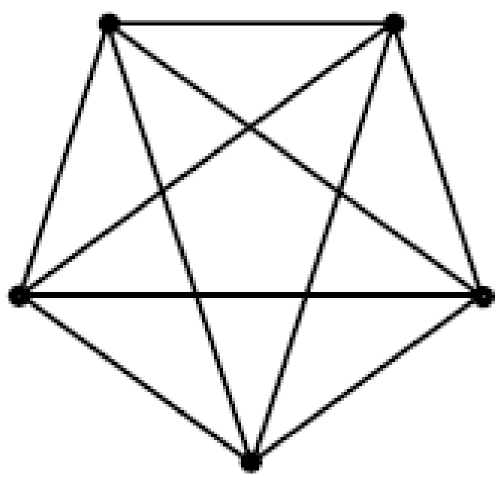


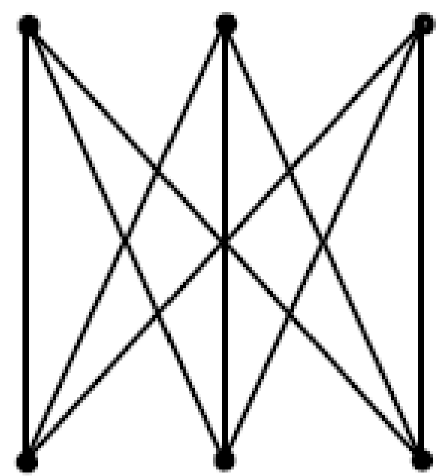
$C_n(G) \simeq D_n(G)$

[Abrams, 00] For any n > 1 and any graph G with at least n vertices, $C_n(G)$ deformation retracts to $D_n(G)$ if and only if

1. Each path between distinct vertices of valence not equal to two passes through at least $n - 1$ edges;
2. Each loop from a vertex to itself which cannot be shrunk to a point in G passes through at least $n + 1$ edges.

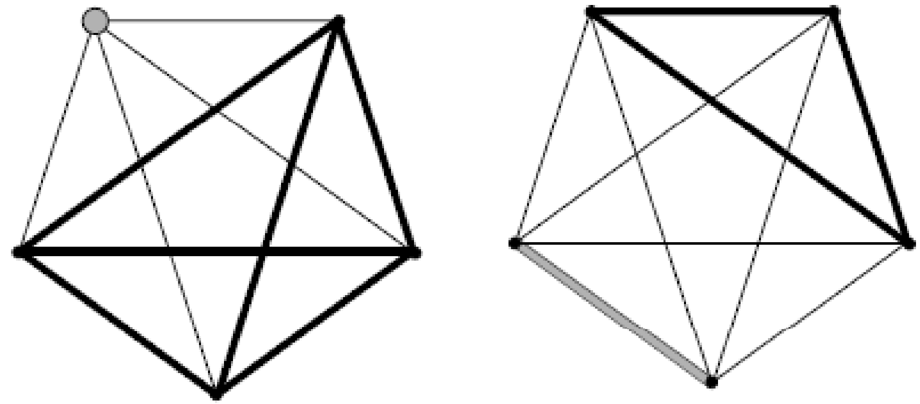
$C_2(K_5) \simeq D_2(K_5)$ $C_2(K_{3,3}) \simeq D_2(K_{3,3})$

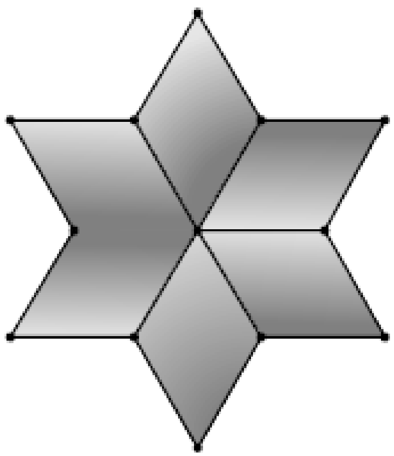




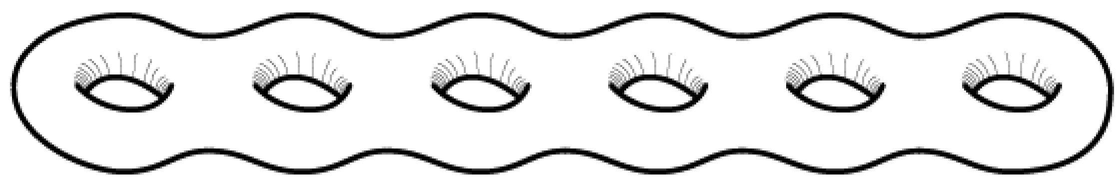
$D_2(K_5)$ = closed surface of genus 6

5 x 4 = 20 0-cells
5 x 6 x 2 = 60 1-cells
5 x 3 x 2 = 30 2-cells





$\chi(D_2(K_5)) = 20 – 60 + 30 = -10$
$g = 1 – (-10/2) = 6$



Similarly, $D_2(K_{3,3})$ is a closed surface of genus 37.

Pure braid groups and braid groups

- **[R. Ghrist, 99]** Given a graph G ($\approx S^1$) having v vertices of valence > 2, the space $C_n(G)$ deformation retracts to a subcomplex of dimension at most v. Furthermore, $C_n(G)$ and $UC_n(G) \equiv C_n(G)/\Sigma_n$ are $K(\pi, 1)$ spaces.
- Define the pure braid group and the braid group over G by

$$PB_n(G) \equiv \pi_1(C_n(G)),$$
$$B_n(G) \equiv \pi_1(UC_n(G)).$$

- Then they are torsion free.

Properties of graph braid groups

- **[A. Abrams, 00]** $D_n(G)$ and $UD_n(G) \equiv D_n(G)/\Sigma_n$ are cube complexes of non-positive curvature, that is, locally CAT(0) and have many useful consequences.
- $D_n(G)$ and $UD_n(G)$ are $K(\pi, 1)$ spaces.
- $PB_n(G)$ and $B_n(G)$ are torson free and have solvable word and conjugacy problems.

Right-angled Artin group

- A group is called a right-angled Artin group (RAAG) if it has a finite presentation whose defining relations are all commutators of generators.
- **[R. Charney and M. Davis, 94]** Given a simple graph Γ, A_Γ denotes the RAAG generated by $V(\Gamma)$ and related by a commutator of two ends of each edge. Then $K(A_\Gamma, 1)$ is obtained from $\Pi_{v\in V(\Gamma)}(S^1)_v$ by deleting all k-cells corresponding to $v_1, v_2, \ldots, v_k$ if they do not form a complete subgraph in Γ and so the cohomology $H^*(A_\Gamma;Z_2)$ forms an exterior face algebra of a flag complex.

Is any graph braid group a RAAG?

- R. Ghrist conjectured that every graph (pure) braid group is a RAAG.
- Ghrist and Abrams realized that K_5 and $K_{3,3}$ are counterexamples and revised the conjecture so that it holds only for planar graphs.
- In 2006, D. Farley and L. Sabalka found a counterexample for the revised conjecture.
- This talk will propose a necessary and sufficient condition for a graph to have a RAAG as its braid group for braid index ≥ 5.

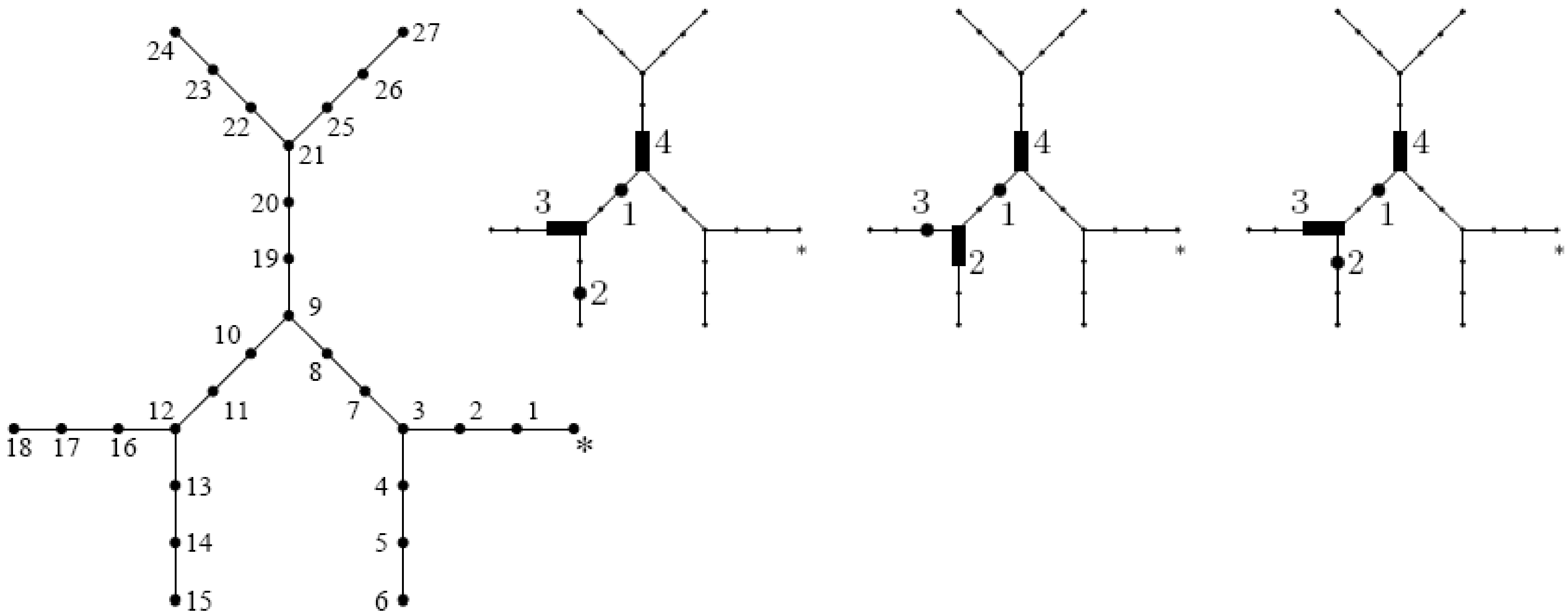
Embedding results

- **[J. Crisp and B. Wiest, 04]** The natural cubical map $\Phi : UD_n(G) \to K(A_\Gamma, 1)$ is a local isometry and therefore is π_1-injective, where $V(\Gamma) = E(G)$ and there is an edge in Γ for each pair of disjoint edges in G. Consequently, every graph braid group is a subgroup of a RAAG.
- **[J. Crisp and B. Wiest, 04]** A surface group $\pi_1(S)$ is embedded in a RAAG iff $S \neq P^2, P^2\#P^2, P^2\#P^2\#P^2$
- **[L. Sabalka, 05]** Every RAAG is realized by a graph 2-braid group.

Discrete Morse theory

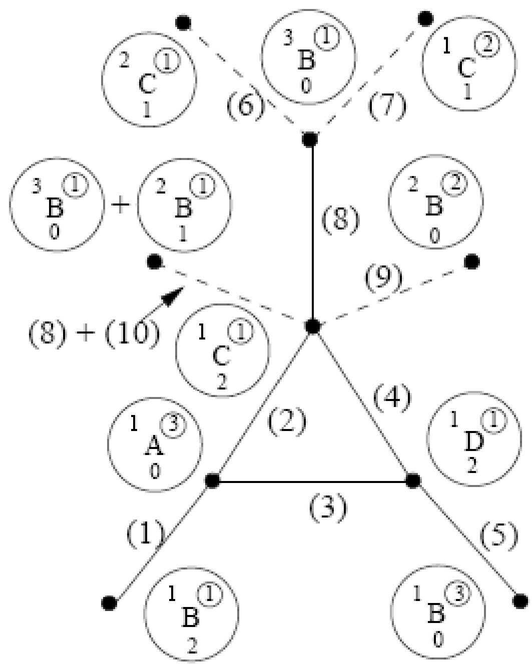
- **[D. Farley and L. Sabalka, 04]** Using discrete Morse theory [Forman, 98], $UD_n(G)$ can be systematically collapsed to yield a minimal cube complex $M_n(G)$, called the Morse complex, that is simple enough to compute π_1. In fact, they showed how to compute tree braid groups and found a counterexample for Ghrist's modified conjecture.

Morse complex of T_0



The Morse complex $M_4(T_0)$ obtained by collapsing $UD_4(T_0)$ along the given gradient vector field has one 0-cell + twenty four 1-cells + six 2-cells

Ring structure of $H^*(M_4(T_0) ; \mathbf{Z}_2)$



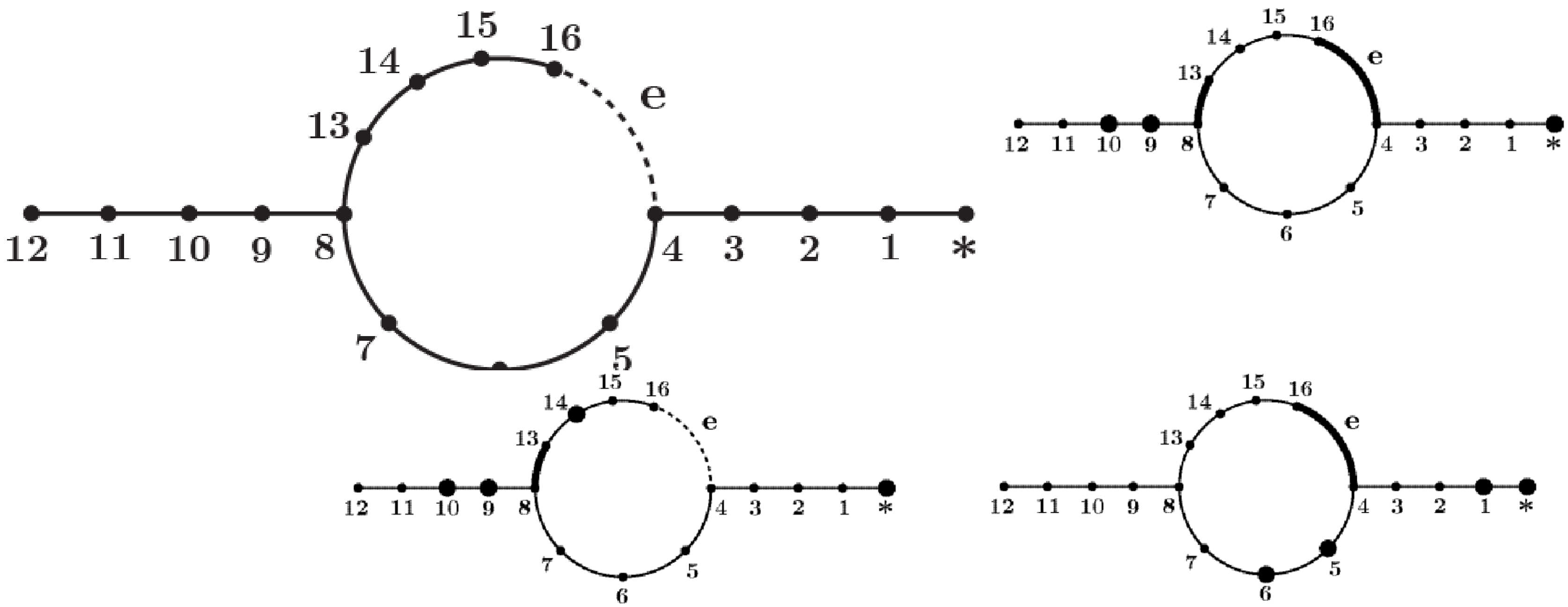
$H^*(M_4(T_0); \mathbf{Z}_2)$ is an exterior face algebra of a simplicial complex that is not flag.

If $B_n(T_0)$ were a RAAG, $H^*(A_\Gamma; \mathbf{Z}_2)$ is an exterior face algebra of a flag complex $K(A_\Gamma, 1)$ and so $\Phi^* : H^*(K(A_\Gamma, 1); \mathbf{Z}_2) \to H^*(B_n(T_0) ; \mathbf{Z}_2)$ has a nontrivial kernel generated by homogeneous degree 1 elements where $\Phi : UD_n(T_0) \to K(A_\Gamma, 1)$ is the natural cubical map considered earlier.

Results in [D. Farley and L. Sabalka, 06]

- **Lemma.** The kernel of Φ_* can not be generated by homogeneous degree 1 and degree 2 elements and consequently $B_n(T_0)$ is not a RAAG.
- **Theorem.** For a tree T, the braid group B_nT is a RAAG if and only if T is linear (i.e. T contains no T_0) or $n \leq 3$.
- We generalize this results to arbitrary graphs for $n \geq 5$.

Morse complex of S_0



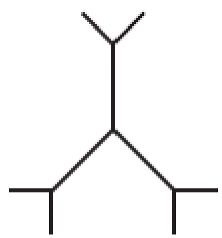
The Morse complex $M_5(S_0)$ has
one 0-cell + fifteen 1-cells + ten 2-cells

B_5S_0 is not a RAAG

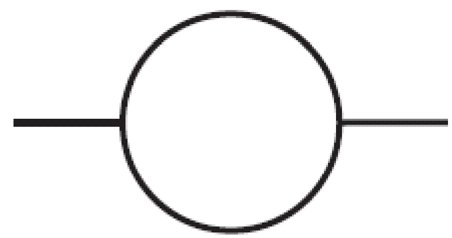
- Let $B_nG = F/R$ for a free group F and its normal subgroup R. If G is a commutator-relator group, that is, $R \subset [F,F]$, then the inclusion induces
$$\Psi : H_2(B_nG) = R/[F,R] \to [F,F]/[F,[F,F]]$$
which is, in fact, the dual of cup product.
- If B_nG is a RAAG, this homomorphism Ψ is injective.
- Rank(Im Ψ) = 3 obtained from a presentation of B_5S_0 . Rank($H_2(B_5S_0)$) = 4 obtained from the Morse complex $M_5(S_0)$. Thus B_5S_0 is not a RAAG.

A complete description for $n \geq 5$

- A graph G contains another graph H if a subdivsion of H is a subgraph of a subdivision of G.
- **Main Theorem.** For a graph G and $n \geq 5$, the n-braid group B_nG is a RAAG if and only if G contains neither T_0 nor S_0.



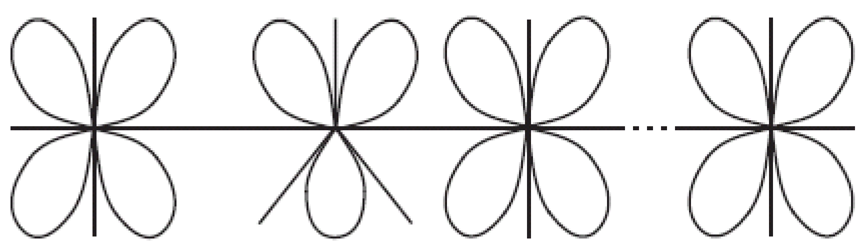
T_0



S_0

One half of Main Theorem

- **Theorem.** If a graph G contains neither T_0 nor S_0 then the n-braid group B_nG is a RAAG for $n \geq 5$.
- **Proof.** A graph G containing neither T_0 nor S_0 must be a "linear star-bouquet". Then compute a presentation of B_nG directly from a Morse complex $M_n(G)$ to show it is a RAAG.



Breakup of the converse

- **Theorem A.** If a graph G contain T_0 but does not contain S_0 then B_nG is not a RAAG for $n \geq 5$.
- **Theorem B.** If a planar graph G contains S_0 then B_nG is not a RAAG for $n \geq 5$.
- **Theorem C.** If a graph G is non-planar then B_nG is not a RAAG.
- We remark that classes of graphs in Theorem A and Theorem B are further divided into smaller classes due to some technical difficulties.

Exterior face algebra of a flag complex

- **Proposition.** Let K_1 and K_2 be finite simplicial complexes. If $\varphi : \Lambda(K_1) \to \Lambda(K_2)$ is a degree-preserving epimorphism, K_1 is a flag complex , and $Ker(\varphi)$ is generated by homogeneous elements of degree 1 and 2, then K_2 is also a flag complex where $\Lambda(K)$ denotes the exterior face algebra of K over $\mathbf{Z}_2$.
- We use the contraposition of this proposition.

Graphs without S_0

- Suppose B_nH is not a RAAG and G contains H but not S_0. Then one can show that the inclusion $i : H \to G$ induces a degree-preserving epimorphism
$$i^* : H^*(UD_nG; \mathbf{Z}_2) \to H^*(UD_nH; \mathbf{Z}_2)$$
and Ker(i^*) is generated by homogeneous elements of degree 1 and 2.
- Since $H^*(UD_nH; \mathbf{Z}_2) = H^*(B_nH; \mathbf{Z}_2)$ is an exterior face algebra of a non-flag complex, so is $H^*(UD_nG; \mathbf{Z}_2)$ and therefore B_nG is not a RAAG.

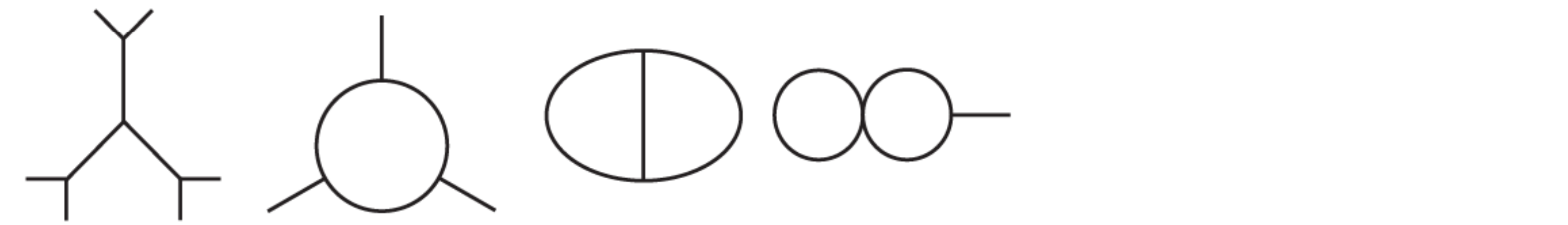
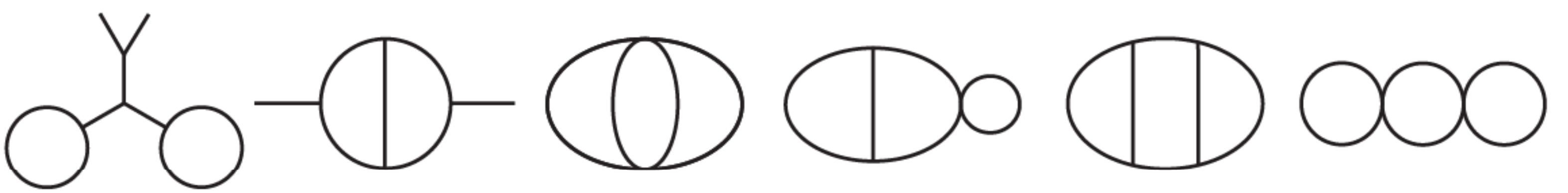
Preview of Hyo Won Park's talk

- **Theorem B.** If a planar graph G contains S_0 then B_nG is not a RAAG for $n \geq 5$.
- **Proof.** Prove and use the monomorphism
$$\Psi : H_2(B_nG) = R/[F,R] \rightarrow [F,F]/[F,[F,F]]$$
in another way.
- **Theorem C.** If a graph G is non-planar then B_nG is not a right-angled Artin group.
- **Proof.** For a non-planar graph G, $H_1(B_nG)$ can be shown to have a torsion but the homology groups of a RAAG is torsion free.

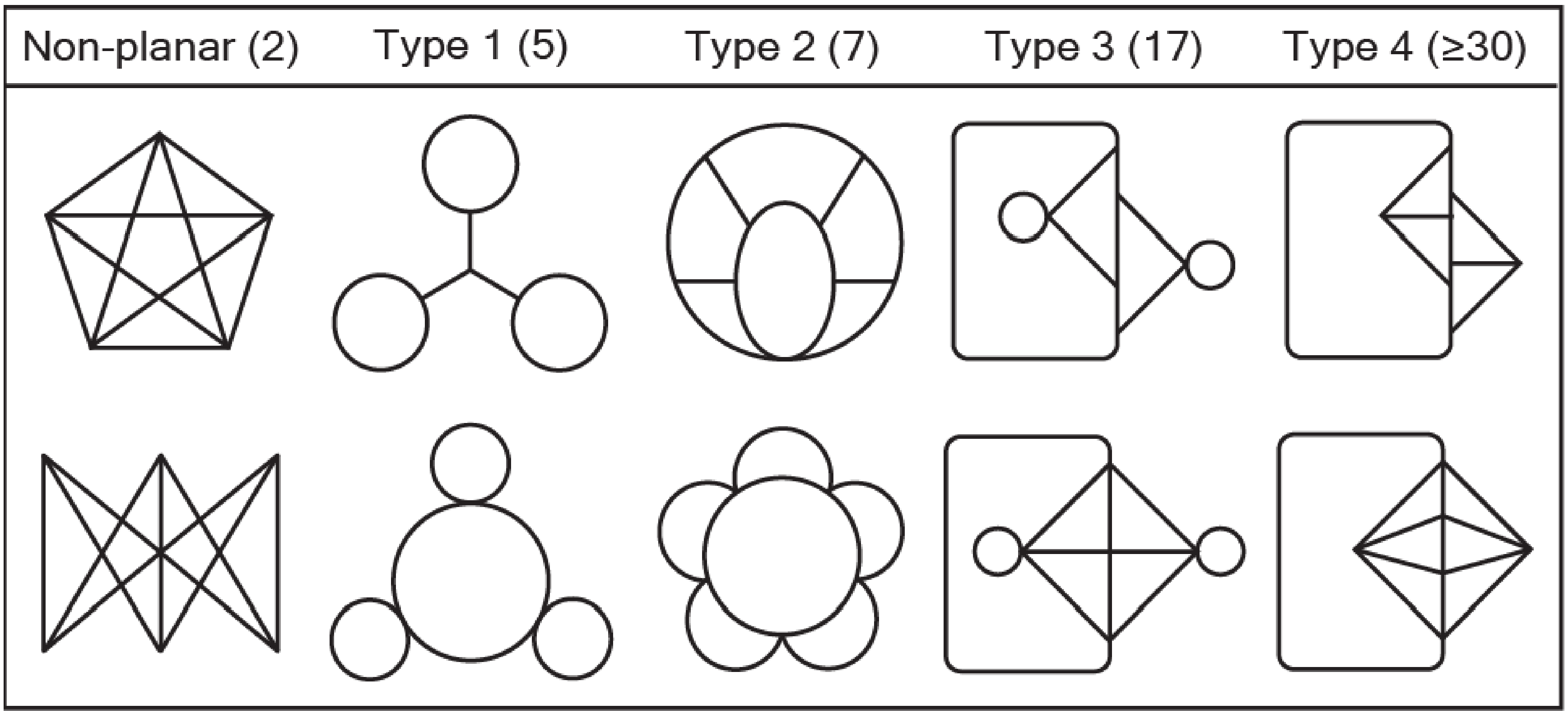
Remark for $n \leq 4$

- An n-nucleus is a minimal graph G such that B_nG is not a RAAG.
- We have just observed that there are two n-nuclei T_0 and S_0 for $n \geq 5$.
- There are four 4-nuclei, six 3-nuclei, and more than sixty 2-nuclei. We conjecture that for any n, B_nG is a right-angled Artin group if and only if G contains no n-nuclei.

4-Nuclei and 3-nuclei

n=4	
n=3	

2-Nuclei



THANK YOU